Bartholdi zeta functions of some graphs

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Abstract

We give a decomposition formula for the Bartholdi zeta function of a graph $G$ which is partitioned into some irregular coverings. As a corollary, we obtain a decomposition formula for the Bartholdi zeta function of $G$ which is partitioned into some regular coverings.

Keywords: Zeta function; Graph covering; L-function

1. Introduction

Graphs treated here are finite.

Let $G$ be a graph and $\mathbf{A}(G)$ its adjacency matrix. Then the characteristic polynomial $\Phi(G; \lambda)$ of $G$ is defined by $\Phi(G; \lambda) = \det(\lambda I - A(G))$. Characteristic polynomials for various graphs were obtained (see \cite{3,4}). Schwenk \cite{17} studied relations between the characteristic polynomials of some related graphs.

Characteristic polynomials for graph coverings were obtained in \cite{6,14}. A decomposition formula for the characteristic polynomial of a regular covering of a graph was given by Mizuno and Sato \cite{14}. Feng et al. \cite{6} generalized the above result on a regular covering of a graph to its irregular covering.

Characteristic polynomials for branched covering of graphs or digraphs were computed in \cite{5,13}. Lee and Kim \cite{13} computed the characteristic polynomial of a graph having a semi-free action, whose free part is an abelian covering graph. Deng and Wu \cite{5} established an explicit formula for the characteristic polynomial of a digraph having a semi-free action.

Zeta functions of graphs are closely related to characteristic polynomials of graphs, because zeta functions of graphs have determinant expressions.

Zeta functions of graphs started from zeta functions of regular graphs by Ihara \cite{10}. In \cite{10}, he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph $G$ associated with a unitary representation of the fundamental group of $G$ was developed by Sunada \cite{20,21}. Hashimoto \cite{9} treated multivariable zeta functions of

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bipartite graphs. Bass [2] generalized Ihara’s result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial.

Stark and Terras [19] gave an elementary proof of Bass’ Theorem, and discussed three different zeta functions of any graph. Furthermore, various proofs of Bass’ Theorem were given by Foata and Zeilberger [7], Kotani and Sunada [11]. A determinant expression for the zeta function of a regular covering of a graph was given in [15].

Let \( G = (V(G), E(G)) \) be a connected graph (possibly multiple edges and loops) with the set \( V(G) \) of vertices and the set \( E(G) \) of unoriented edges \( uv \) joining two vertices \( u \) and \( v \). For \( uv \in E(G) \), an arc \((u, v)\) is the oriented edge from \( u \) to \( v \). Set \( D(G) = \{(u, v), (v, u) \mid uv \in E(G)\} \). For \( e = (u, v) \in D(G) \), let \( u = o(e) \) and \( v = t(e) \). Furthermore, let \( e^{-1} = (v, u) \) be the inverse of \( e = (u, v) \).

A path \( P \) of length \( n \) in \( G \) is a sequence \( P = (e_1, \ldots, e_n) \) of \( n \) arcs such that \( e_i \in D(G), t(e_i) = o(e_{i+1}) (1 \leq i \leq n-1) \), where indices are treated mod \( n \). Set \( |P| = n \), \( o(P) = o(e_1) \) and \( t(P) = t(e_n) \). Also, \( P \) is called a \((o(P), t(P))\)-path. We say that a path \( P = (e_1, \ldots, e_n) \) has a backtracking if \( e_{i+1} = e_i \) for some \( i (1 \leq i \leq n-1) \). A \((u, v)\)-path is called a \( v \)-cycle (or \( v \)-closed path) if \( v = u \). The inverse cycle of a cycle \( C = (e_1, \ldots, e_n) \) is the cycle \( C^{-1} = (e_n^{-1}, \ldots, e_1^{-1}) \).

We introduce an equivalence relation between cycles. Such two cycles \( C_1 = (e_1, \ldots, e_m) \) and \( C_2 = (f_1, \ldots, f_m) \) are called equivalent if there exists \( k \) such that \( f_j = e_{j+k} \) for all \( j \). The inverse cycle of \( C \) is in general not equivalent to \( C \). Let \([C]\) be the equivalence class which contains a cycle \( C \). Let \( B' \) be the cycle obtained by going \( r \) times around a cycle \( B \). Such a cycle is called a power of \( B \). A cycle \( C \) is reduced if \( C \) has no backtracking. Furthermore, a cycle \( C \) is prime if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph \( G \) corresponds to a unique conjugacy class of the fundamental group \( \pi_1(G, v) \) of \( G \) at a vertex \( v \) of \( G \).

The (Ihara) zeta function of a graph \( G \) is defined to be a function of \( u \in \mathbb{C} \) with \(|u|\) sufficiently small, by

\[
Z(G, u) = Z_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},
\]

where \([C]\) runs over all equivalence classes of prime, reduced cycles of \( G \) (see [11]). Bass [2] presented a determinant expression for the zeta function of a connected graph \( G \):

\[
Z(G, u)^{-1} = (1 - u^2)^r^{-1} \det(I - uA(G) + u^2(D - I)),
\]

where \( r \) and \( A(G) \) are the Betti number and the adjacency matrix of \( G \), respectively, and \( D = D_G = (d_{ij}) \) is the diagonal matrix with \( d_{ij} = \deg v_i \) where \( V(G) = \{v_1, \ldots, v_n\} \).

Bartholdi [1] introduced the Bartholdi zeta function of a graph as a generalization of the (Ihara) zeta function of a graph, and gave a determinant expression of the Bartholdi zeta function of a graph.

Let \( G \) be a connected graph. For each \( u, v \in V(G) \), let \([u, v]\) be the set of all \((u, v)\)-paths in \( G \). We say that a path \( P = (e_1, \ldots, e_n) \) has a bump at \( t(e_i) \) if \( e_{i+1} = e_i^{-1} (1 \leq i \leq n) \). The bump count \( bc(P) \) of a path \( P \) is the number of bumps in \( P \). Furthermore, the cyclic bump count \( cbc(\pi) \) of a cycle \( \pi = (\pi_1, \ldots, \pi_n) \) is

\[
cbc(\pi) = |\{i = 1, \ldots, n \mid \pi_i = \pi_{i+1}^{-1}\}|,
\]

where \( \pi_{n+1} = \pi_1 \). Then the Bartholdi zeta function of \( G \) is defined to be a function of \( u, t \in \mathbb{C} \) with \(|u|, |t|\) sufficiently small, by

\[
\zeta_G(u, t) = \zeta(G, u, t) = \prod_{[C]} (1 - u^{cb(C)} t^{|C|})^{-1},
\]

where \([C]\) runs over all equivalence classes of prime cycles of \( G \) (see [1]). If \( u = 0 \), then the Bartholdi zeta function of \( G \) is the (Ihara) zeta function of \( G \).

Theorem 1 (Bartholdi). Let \( G \) be a connected graph with \( n \) vertices and \( m \) unoriented edges. Then the reciprocal of the Bartholdi zeta function of \( G \) is given by

\[
\zeta(G, u, t)^{-1} = (1 - (1 - u)^2 t^2)^{m-n} \det(I - tA(G) + (1-u)(D - (1-u)I)t^2).
\]

In the case of \( u = 0 \), Theorem 1 implies Bass’ Theorem.
Decomposition formulas for the Bartholdi zeta functions of a regular or an irregular covering of a graph were given in [12, 16]. Mizuno and Sato [16] presented a determinant expression for the Bartholdi zeta function of a regular covering of a graph. Kwak et al. [12] computed the Bartholdi zeta function of a general covering of a graph.

In this paper, we consider a generalization of a graph having a semi-free action, and compute its Bartholdi zeta function. In Section 2, we consider a graph divided into some coverings, and give its adjacency matrix. In Section 3, we give a decomposition formula of the Bartholdi zeta function of a graph divided into coverings. In Section 4, we present a determinant expression of the Bartholdi zeta function of a graph divided into some regular coverings of graphs. In Section 5, we discuss the Bartholdi zeta function of a branched covering of a graph.

2. Coverings of graphs

A graph $H$ is called a covering of a graph $G$ with projection $\pi : H \rightarrow G$ if there is a surjection $\pi : V(H) \rightarrow V(G)$ such that $\pi|_{\{v\}} : N(v) \rightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi : H \rightarrow G$ is an $n$-fold covering of $G$ if $\pi$ is $n$-to-one. For a group $\Gamma$ of automorphisms of $G$ acting on $H$, the orbit graph (quotient graph) $H/\Gamma$ is a graph whose vertices are the $\Gamma$-orbits on $V(G)$, with two vertices adjacent in $H/\Gamma$ if and only if their representatives are adjacent in $H$. A covering $\pi : H \rightarrow G$ is said to be regular if there is a subgroup $B$ of the automorphism group $Aut\ H$ of $H$ acting freely on $H$ such that the quotient graph $H/B$ is isomorphic to $G$.

Let $G$ be a graph and $S_\pi$ the symmetric group on the set $N = \{1, 2, \ldots, n\}$. Then a mapping $\pi : D(G) \rightarrow S_n$ is called a permutation voltage assignment if $\pi(v, u) = (\pi(u), \pi(v))^{-1}$ for each $(u, v) \in E(G)$. The pair $(G, \pi)$ is called a permutation voltage graph. The derived graph $G^\pi$ of the permutation voltage graph $(G, \pi)$ is defined as follows: $V(G^\pi) = V(G) \times N$ and $((u, h), (v, k)) \in E(G^\pi)$ if and only if $(u, v) \in E(G)$ and $k = \pi(u, v)(h)$. The natural projection $\pi^\pi : G^\pi \rightarrow G$ is defined by $\pi^\pi(u, h, v, k) = u$. The graph $G^\pi$ is called a derived covering graph of $G$ with voltages in $S_n$ or an $n$-covering of $G$. Note that the $n$-covering of $G$ is an $n$-fold covering of $G$. Furthermore, every $n$-fold covering of a graph $G$ is an $n$-covering $G^\pi$ of $G$ for some permutation voltage assignment $\pi : D(G) \rightarrow S_n$ (see [8]).

**Theorem 2** (Gross and Tucker). Let $\pi : G \rightarrow G$ be an $n$-fold covering of a connected digraph $G$. Then there exists a permutation voltage assignment $\pi : D(G) \rightarrow S_n$ such that the $n$-covering $G^\pi$ is isomorphic to $G$.

We consider a graph divided into covering.

Let $K$ be a connected graph and $K_1, K_2, \ldots, K_n$ subgraphs of $K$ such that $V(K) = V(K_1) \cup V(K_2) \cup \cdots \cup V(K_n)$, $V(K_i) \cap V(K_j) = \emptyset (i \neq j)$. Then $K$ is denoted by $K = K_1 \sqcup \cdots \sqcup K_n$. Furthermore, let $N_i = \{1, 2, \ldots, m_i\}$ and $S_{m_i}$ be the symmetric group on the set $N_i$ for each $i = 1, 2, \ldots, n$.

Now, let $G$ be a connected graph satisfying the following conditions. There are $n$ subgraphs $G_1, G_2, \ldots, G_n$ of $G$ such that

1. $V(G) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$, $V(G_i) \cap V(G_j) = \emptyset (i \neq j)$.
2. $G_i$ is a $m_i$-covering of $K_i$ for each $i = 1, 2, \ldots, n$, i.e., $G_i = K_i^{x_i}$, where $x_i : D(K_i) \rightarrow S_{m_i}$ is a permutation voltage assignment.
3. For any $1 \leq i \neq j \leq n$, let $(V(G_i), V(G_j)) = \{(u, h), (v, k)|u \in V(K_i), v \in V(K_j), (u, v) \in D(K), 1 \leq h \leq m_i, 1 \leq k \leq m_j\}$, where $(B, \tilde{B}) = \{(u, v) \in D(G)|u \in B, v \in \tilde{B}\} = V(G) - B$ for any subset $B \subset V(G)$.

Then $G$ is denoted by $G = K_1^{x_1} \ast \cdots \ast K_n^{x_n}$.

Next, we consider the adjacency matrix of the above graph $G$.

Let $V(K_i) = \{v_{1i}, \ldots, v_{ni}\}$. Furthermore, let

$$A(K) = \begin{bmatrix}
A(K_1) & F_{12} & \cdots & F_{1n} \\
F_{12} & A(K_2) & \cdots & F_{2n} \\
& & \ddots & \vdots \\
F_{1n} & F_{2n} & \cdots & A(K_n)
\end{bmatrix},$$

where $F_{ij}$ is a $v_i \times v_j$ matrix for any $1 \leq i < j \leq n$. Arrange arcs of $G$ in $m_1 + \cdots + m_n$ blocks: $(v_{11}, 1), \ldots, (v_{11}, 1); \ldots; (v_{11}, m_1), \ldots, (v_{11}, 1); \ldots; (v_{n1}, 1), \ldots, (v_{n1}, 1); \ldots; (v_{nn}, 1), \ldots, (v_{nn}, 1)$. We consider the adjacency matrix.
A(G) under this order. By the hypothesis 3 of G, we have

$$A(G) = \begin{bmatrix}
A(G_1) & J_{m_1,m_2} \otimes F_{12} & \cdots & J_{m_1,m_n} \otimes F_{1n} \\
J_{m_2,m_1} \otimes F_{12} & A(G_2) & \cdots & J_{m_2,m_n} \otimes F_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
J_{m_n,m_1} \otimes F_{1n} & J_{m_n,m_2} \otimes F_{2n} & \cdots & A(G_n)
\end{bmatrix},$$

where $J_{ab}$ is the $a \times b$ matrix with all one entries.

By the hypothesis 2 of G and Theorem 2, there exists a permutation voltage assignment $\varphi_i : D(K_i) \rightarrow S_{m_i}$ such that the $n$-covering $K_i^{\varphi_i}$ is isomorphic to $G_i$ for each $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, let $\Gamma_i = \langle \{ \varphi_i(e) | e \in D(K_i) \} \rangle$ be the subgroup of $S_{m_i}$ generated by $\{ \varphi_i(e) | e \in D(K_i) \}$ for $h \in \Gamma_i(i = 1, \ldots, n)$, the matrix $P_h = (p_{ij}^{(h)})$ is defined as follows:

$$p_{ij}^{(h)} = \begin{cases} 1 & \text{if } h(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $i = 1, \ldots, n$. If $(u, v) \in D(K_i)$ and $\varphi_i(u, v) = h \in \Gamma_i$, then $b = \varphi_i(u, v)(a) = h(a)$, i.e., $((u, a), (v, b)) \in D(K_i^{\varphi_i})$. Thus, we have

$$A(G_i) = A(K_i^{\varphi_i}) = \sum_{h \in \Gamma_i} P_h \otimes A_{i_h},$$

where the matrix $A_{i,h} = (a_{i(u,v)}^{(i,h)})$ is given by

$$a_{i(u,v)}^{(i,h)} := \begin{cases} |\{ e \in D(K_i) | e = (u, v), \varphi_i(e) = h \}| & \text{if there exists an arc } (u, v) \text{ in } K_i, \\ 0 & \text{otherwise.} \end{cases}$$

The Kronecker product $A \otimes B$ of matrices $A$ and $B$ is considered as the matrix $A$ having the element $a_{ij}$ replaced by the matrix $a_{ij}B$.

**Proposition 1.**

$$A(G) = \begin{bmatrix}
\sum_{h \in \Gamma_i} P_h \otimes A_{1,h} & \cdots & J_{m_1,m_n} \otimes F_{1n} \\
\vdots & \ddots & \vdots \\
J_{m_n,m_1} \otimes F_{1n} & \cdots & \sum_{h \in \Gamma_n} P_h \otimes A_{n,h}
\end{bmatrix}.$$

3. Bartholdi zeta functions of graphs divided into graph coverings

Let $K = K_1 \sqcup \cdots \sqcup K_n$ be a connected graph and $N_i = \{1, 2, \ldots, m_i\}$ for each $i = 1, \ldots, n$. Furthermore, let $G = K_1^{\varphi_1} \star \cdots \star K_n^{\varphi_n}$, where $\varphi_i : D(K_i) \rightarrow S_{m_i}$ is a permutation voltage assignment. For any $1 \leq i \neq j \leq n$, let $(V(K_i^{\varphi_i}), V(K_j^{\varphi_j})) = \{(u, v), (v, k) | u \in V(K_i), v \in V(K_j), (u, v), (v, k) \in D(K_i), 1 \leq k \leq m_i, 1 \leq k \leq m_j\}$.

Let $V(K_i) = \{v_{ij}, \ldots, v_{ij} | 1 \leq i \leq n\}$. For each $i = 1, \ldots, n$, let $\Gamma_i = \langle \{ \varphi_i(e) | e \in D(K_i) \} \rangle$. Next, for any $i = 1, \ldots, n$, let $\rho_{ij} = 1, \rho_{i2}, \ldots, \rho_{ik}$ be the inequivalent irreducible representations of $V(\Gamma_i)$, and $f_{ij}$ the degree of $\rho_{ij}$ for each $j = 1, \ldots, k_i$, where $f_{i1} = 1$. Furthermore, let $P_i : \Gamma_i \rightarrow GL(m_i, C)$ be a permutation representation of $\Gamma_i$ such that $P_i(g) = P_g$, and $m_{ij}$ the multiplicity of $\rho_{ij}$ in $P_i$ for each $j = 1, \ldots, k_i$, that is, $P_i$ is equivalent to a representation of $m_{i1}1 + m_{i2}\rho_{i2} + \cdots + m_{ik}\rho_{ik}$. Assume that $K_i^{\varphi_i}$ is connected. Then we have $m_{i1} = 1$ for each $i = 1, \ldots, n$ (see [18]).

Let $M_1 \oplus \cdots \oplus M_n$ be the block diagonal sum of square matrices $M_1, \ldots, M_n$. If $M_1 = M_2 = \cdots = M_n = M$, then we write $s \circ M = M_1 \oplus \cdots \oplus M_n$.

By the fact of group representation, there exists a nonsingular matrix $U_i$ such that

$$U_i^{-1} P_i(h) U_i = (1 \oplus m_{i1} \circ \rho_{i1}(h) \oplus \cdots \oplus m_{ik} \circ \rho_{ik}(h))$$

for each $i = 1, \ldots, n$ and $h \in \Gamma_i$ (see [18]).
Theorem 3. Let $G$ be a connected graph with $n$ vertices and $m$ edges, $K = K_1 \sqcup \cdots \sqcup K_n$ for each $i = 1, \ldots, n$. Furthermore, let $G = K_1^{q_1} \ast \cdots \ast K_n^{q_n}$, where $\zeta_i : D(K_i) \rightarrow S_m(1 \leq i \leq n)$ is a permutation voltage assignment. For any $1 \leq i \neq j \leq n$, let $(V(K_i), V(K_j)) = \{(u, v)|u \in V(K_i), v \in V(K_j), (u, v) \in D(K), 1 \leq h \leq m, 1 \leq k \leq m_j\}$. Let $\varepsilon_i = |E(K_i)|$ and $\varepsilon_i = |V(K_i)|$ for each $i = 1, \ldots, n$. Set $v_0 = v_1 + \cdots + v_n$. Furthermore, let $\rho_{i1} = 1, \rho_{i2}, \ldots, \rho_{ik_i}$ be the inequivalent irreducible representations of $\Gamma_i(1 \leq i \leq n)$, and $f_{ij}$ the degree of $\rho_{ij}$ for each $j = 1, \ldots, k_i$, where $\sum_{j} f_{ij} = 1$. Let $P_i : \Gamma_i \rightarrow GL(m_i, \mathbb{C})(1 \leq i \leq n)$ be a permutation representation of $\Gamma_i$ such that $P_i(g) = P_i$. Assume that $G_i = K_i^{q_i}$ is connected and $P_i = 1 + m_{i2}p_{i2} + \cdots + m_{ik_i}p_{ik_i}$.

Then the reciprocal of the Bartholdi zeta function of $G$ is

$$\zeta(G, u, t)^{-1} = (1 - (1 - u)^2t^2)^{-\varepsilon_i}m_1^{q_1} - \cdots - m_n^{q_n}\det(I_{v_0} - tA' + (1 - u)t^2(D_K' - (1 - u)I_{v_0}))$$

where

$$A' = \begin{bmatrix} A(K_1) & \cdots & m_n F_{1n} \\ \vdots & \ddots & \vdots \\ m_1 F_{1n} & \cdots & A(K_n) \end{bmatrix}.$$ 

Proof. By Theorem 1, we have

$$\zeta(G, u, t)^{-1} = (1 - (1 - u)^2t^2)^{-v} \det(I_v - tA(G) + (1 - u)(D_G - (1 - u)I_v)t^2).$$

Note that $v = v_1m_1 + \cdots + v_nm_n$. By Proposition 1, we have

$$A(G_i) = \sum_{h \in \Gamma_i} P_h \otimes A_{i,h}.$$ 

At first, we have

$$D_G = (I_m \otimes D_{K_1}') + \cdots + (I_{m_n} \otimes D_{K_n}').$$

Let

$$A(K) = \begin{bmatrix} A(K_1) & F_{12} & \cdots & F_{1n} \\ iF_{12} & A(K_2) & \cdots & F_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ iF_{1n} & iF_{2n} & \cdots & A(K_n) \end{bmatrix}.$$
Therefore, it follows that for each \( i = 1, \ldots, n \). Again, by Proposition 1, we have

\[
\begin{pmatrix}
 f(G_1) & \cdots & -tJ_{m_1m_n} \otimes F_{1n} \\
\vdots & \ddots & \vdots \\
-tJ_{m_nm_1} \otimes F_{1n} & \cdots & f(G_n)
\end{pmatrix}
\]

For each \( i = 1, \ldots, n \), let \( \Gamma_i = \{ (x_i(e)|e \in D(K_i)) \} \), let \( \rho_{i1} = 1, \rho_{i2}, \ldots, \rho_{ik_i} \) be the inequivalent irreducible representations of \( \Gamma_i \), and let \( f_{ij} \) be the degree of \( \rho_{ij} \) for each \( j = 1, \ldots, k_i \), where \( f_{i1} = 1 \). Furthermore, let \( \rho_i : \Gamma_i \rightarrow GL(m_i, \mathbb{C}) \) be a permutation representation of \( \Gamma_i \) such that \( \rho_i(g) = \rho_{ij} \) for each \( i = 1, \ldots, n \). By Lemma 1, there exists a nonsingular matrix \( U_i \) such that

\[
\tilde{j}_m U_i = (\sqrt{m_i}0 \cdots 0), \quad U_i^{-1} \tilde{j}_m = (\sqrt{m_i}0 \cdots 0)
\]

and

\[
U_i^{-1} \rho_i(h) U_i = (1) \oplus m_{i2} \circ \rho_{i2}(h) \oplus \cdots \oplus m_{ik_i} \circ \rho_{ik_i}(h)
\]

for each \( h \in \Gamma_i \).

Putting \( Y_i = (U_i^{-1} \otimes I_{v_i}) A(G_i)(U_i \otimes I_{v_i}) \), we have

\[
Y_i = \sum_{h \in \Gamma_i} \{ (1) \oplus m_{i2} \circ \rho_{i2}(h) \oplus \cdots \oplus m_{ik_i} \circ \rho_{ik_i}(h) \} \otimes A_{i,h}.
\]

Note that \( A(K_i) = \sum_{h \in \Gamma_i} A_{i,h}, i = 1, \ldots, n \).

Next, let

\[
X = \begin{bmatrix}
U_1 \otimes I_{v_1} & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & U_n \otimes I_{v_n}
\end{bmatrix}
\]

Therefore, it follows that

\[
X^{-1} f(G) X = \begin{pmatrix}
(U_1^{-1} \otimes I_{v_1}) f(G_1)(U_1 \otimes I_{v_1}) & \cdots & -tU_1^{-1}J_{m_1m_n} U_n \otimes F_{1n} \\
\vdots & \ddots & \vdots \\
-tU_n^{-1}J_{m_nm_1} U_1 \otimes F_{1n} & \cdots & (U_n^{-1} \otimes I_{v_n}) f(G_n)(U_n \otimes I_{v_n})
\end{pmatrix}
\]

Since \( 1 + m_{i2}f_{ij} + \cdots + m_{ik_i}f_{ik_i} = m_i \), we have

\[
(U_i^{-1} \otimes I_{v_i}) f(G_i)(U_i \otimes I_{v_i}) = I_{v_i} - t \sum_{h \in \Gamma_i} U_i^{-1} \rho_i(h) U_i \otimes A_{i,h} + (1 - u)(I_{pm_i} \otimes (D'_{K_i} - (1 - u)I_{v_i})) t^2
\]

\[
= \oplus_{j=1}^{k_i} \left( m_{ij} \circ \left( I_{v_i}f_{ij} - t \sum_{h \in \Gamma_i} \rho_{ij}(h) \otimes A_{i,h} + (1 - u)(I_{fi_j} \otimes (D'_{K_i} - (1 - u)I_{v_i})) t^2 \right) \right).
\]

Furthermore, we have

\[
U_i^{-1}J_{m_i} U_j \otimes F_{ij} = \sqrt{m_i} \sqrt{m_j} \begin{bmatrix}
F_{ij} & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{bmatrix}
\]
Therefore, it follows that
\[
X^{-1} f(G)X \\
\sim \left[ \begin{array}{cccc}
    f(K_1) & \cdots & -t\sqrt{m_1}m_{1n}F_{1n} \\
    \vdots & \ddots & \vdots \\
    -t\sqrt{m_1}m_{1n}F_{1n} & \cdots & f(K_n)
\end{array} \right] + (1-u)(I_{f_{ij}} \otimes (D'_{p_{ij}} - (1-u)I_{v_i}))t^2(1 \leq i \leq n).
\]

But, let
\[
Y = \left[ \begin{array}{cccc}
    \sqrt{m_1}I_{v_1} & 0 & \cdots & 0 \\
    0 & \ddots & \cdots & \vdots \\
    \vdots & \cdots & \ddots & 0 \\
    0 & \cdots & 0 & \sqrt{m_n}I_{v_n}
\end{array} \right].
\]

Then we have
\[
Y^{-1} \left[ \begin{array}{cccc}
    f(K_1) & \cdots & -t\sqrt{m_1}m_{1n}F_{1n} \\
    \vdots & \ddots & \vdots \\
    -t\sqrt{m_1}m_{1n}F_{1n} & \cdots & f(K_n)
\end{array} \right] Y = \left[ \begin{array}{cccc}
    f(K_1) & \cdots & -t\sqrt{m_1}m_{1n}F_{1n} \\
    \vdots & \ddots & \vdots \\
    -t\sqrt{m_1}m_{1n}F_{1n} & \cdots & f(K_n)
\end{array} \right].
\]

Therefore, it follows that
\[
\zeta(G, u, t)^{-1} = (1 - (1-u)^2t^2)^{\epsilon-e_1m_1-\cdots-e_nm_n} \det(I_{v_0} - tA' + (1- u)t^2(D'_{p_{ij}} - (1-u)I_{v_i}))
\times \prod_{i=1}^{n} \prod_{j=2}^{k_i} (1- (1-u)^2t^2)^{(v_i-v_i)}f_{ij} \det \left( I_{v_i} - t \sum_{h \in \Gamma_i} \rho_{ij}(h) \otimes A_{i,h} \right)
\times (1-u)^2(I_{f_{ij}} \otimes (D'_{p_{ij}} - (1-u)I_{v_i}))
\]

where
\[
A' = \left[ \begin{array}{cccc}
    A(K_1) & \cdots & m_{1n}F_{1n} \\
    \vdots & \ddots & \vdots \\
    m_{1n}F_{1n} & \cdots & A(K_n)
\end{array} \right].
\]

4. Bartholdi zeta functions of graphs divided into regular coverings

Let $G$ be a graph and $\Gamma$ a finite group. Then a mapping $\varkappa : D(G) \rightarrow \Gamma$ is called an ordinary voltage assignment if $\varkappa(u, v) = \varkappa(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair $(G, \varkappa)$ is called an ordinary voltage graph. The derived graph $G^\varkappa$ of the ordinary voltage graph $(G, \varkappa)$ is defined as follows: $V(G^\varkappa) = V(G) \times \Gamma$ and $((u, h), (v, k)) \in D(G^\varkappa)$ if and only if $(u, v) \in D(G)$ and $k = h\varkappa(u, v)$. The natural projection $\pi_\varkappa : G^\varkappa \rightarrow G$ is defined by $\pi_\varkappa(u, h) = u$. The graph $G^\varkappa$ is called a derived graph covering of $G$ with voltages in $\Gamma$ or a $\Gamma$-covering of $G$. The natural projection $\pi_\varkappa$ commutes with the right multiplication action of the $\varkappa(e), e \in D(G)$ and the left action of $\Gamma$ on the fibers: $g(u, h) = (u, gh), g \in \Gamma$, which is free and transitive. Thus, the $\Gamma$-covering $G^\varkappa$ is a $|\Gamma|$-fold regular covering of $G$ with covering transformation group $\Gamma$. Furthermore, every regular covering of a graph $G$ is a $\Gamma$-covering of $G$ for some group $\Gamma$ (see [8]).

If $\varkappa(e), e \in D(G)$ is considered as a permutation of $S_\Gamma$ by the right multiplication $\varkappa(e)(g) = g\varkappa(e), g \in \Gamma$, then the $\Gamma$-covering $G^\varkappa$ of $G$ is considered as a $|\Gamma|$-covering of $G$. The group $\{\varkappa(e) : e \in D(G)\}$ coincides with $\Gamma$. 


Furthermore, the permutation representation \( P : \Gamma_i \rightarrow GL(|\Gamma|, C) \) of \( \Gamma \) is the right regular representation of \( \Gamma \). If \( \rho_1 = 1, \rho_2, \ldots, \rho_k \) are inequivalent irreducible representations of \( \Gamma \), then the multiplicity \( m_i \) of \( \rho_i \) for \( P \) is equal to its degree \( f_i \) for each \( i = 1, \ldots, k \).

Let \( K = K_1 \sqcup \cdots \sqcup K_n \) be a connected graph and \( \Gamma_i \) a finite set for each \( i = 1, \ldots, n \). Furthermore, let \( G = K_{2^1} \sqcup \cdots \sqcup K_{2^n} \), where \( x_i : D(K_i) \rightarrow \Gamma_i(1 \leq i \leq n) \) is an ordinary voltage assignment. For any \( 1 \leq i \neq j \leq n \), let \((V(K_i^j), V(K_j^i)) = \{(u, h), (v, k)) | u \in V(K_i), v \in V(K_j), (u, v) \in D(K), h \in \Gamma_i, k \in \Gamma_j \} \).

Similar to the proof of Theorem 3, we obtain a decomposition formula for the Bartholdi zeta function of the above graph \( G \).

**Theorem 4.** Let \( G \) have \( v \) vertices and \( e \) edges, \( e_i = |E(K_i)| \) and \( v_i = |V(K_i)| \) for each \( i = 1, \ldots, n \). Set \( v_0 = v_1 + \cdots + v_n \). Furthermore, let \( \rho_{i1} = 1, \rho_{i2}, \ldots, \rho_{ik} \) be the inequivalent irreducible representations of \( \Gamma_i(1 \leq i \leq n) \), and \( f_{ij} \) the degree of \( \rho_{ij} \). For each \( j = 1, \ldots, k_i \), where \( f_{i1} = 1 \). For each \( i = 1, \ldots, n \), the matrix \( D'_{K_i} = (d_{jj}) \) is a \( v_i \times v_i \) diagonal matrix with \( d_{jj} = \deg_G(v_{ij}, 1_{G_i}) (1 \leq j \leq v_i) \), where \( 1_{G_i} \) is the unit of \( \Gamma_i \).

Then the reciprocal of the Bartholdi zeta function of \( G \) is

\[
\zeta(G, u, t)^{-1} = (1 - (1 - u)^2) e_{\varepsilon_1[I_1]} - e_{\varepsilon_n[I_n]} \det(I_{v_0} - tA' + (1 - u)^2(D'_K - (1 - u)I_{v_0}))
\]

\[
\times \prod_{i=1}^n \prod_{j=2}^{k_i} \left\{ (1 - (1 - u)^2)^{(v_i - v_j)} f_{ij} \det(I_{v_i} - t \sum_{h \in \Gamma_i} \rho_{ih}(h) \otimes A_{i,h}) 
\right. \\
\left. + (1 - u)^2(I_{f_{ij}} \otimes (D'_{K_i} - (1 - u)I_{v_i})) \right\}^{f_{ij}},
\]

where

\[
A' = \begin{bmatrix}
A(K_1) & \cdots & |I_{v_0}|F_{1_{v_0}} \\
\vdots & \ddots & \vdots \\
|I_n|F_{1_{v_n}} & \cdots & A(K_n)
\end{bmatrix}
\]

and \( D'_K = D'_{K_1} \oplus \cdots \oplus D'_{K_n} \).

5. Branched coverings of graphs

We consider a branched covering of a graph. Let \( G \) be a connected graph. Then a graph \( H \) is called a branched covering of \( G \) with projection \( \pi : H \rightarrow G \) if there are a surjection \( \pi : V(H) \rightarrow V(G) \) and a subset \( B \subset V(G) \) such that \( \pi|_{\{v' \in V(H) : \pi(v') = v \}} : N(v') \rightarrow N(v) \) is a bijection for all vertices \( v \in V(D) - B \) and \( v' \in \pi^{-1}(v) \) (see [8]). Note that \( \pi|_{H - \pi^{-1}(B)} : H - \pi^{-1}(B) \rightarrow G - B \) is a covering. The set \( B \) is called a branch set of \( \pi : H \rightarrow G \). Furthermore, the branch set \( B \) is called be of index 1 if \( |\pi^{-1}(v)| = 1 \) for each \( v \in B \). The projection \( \pi : H \rightarrow G \) is an \( n \)-fold branched covering of \( G \) if \( \pi|_{H - \pi^{-1}(B)} \) is \( n \)-to-one.

Let \( G \) be a connected graph, \( B \subset V(G) \), \( S_n \) the symmetric group on \( N = \{1, 2, \ldots, n\} \) and \( \alpha : D(G - B) \rightarrow S_n \) a permutation voltage assignment. The subdigraph \( \langle B \rangle_G \) of \( G \) induced by \( B \) is a graph with vertex set \( B \) and arc set \( \{(u, v) \in D(G) | u \in B, v \in B \} \). Then a branched \( n \)-covering \( G_B^x \) with branch set \( B \subset V(G) \) of index 1 is defined as follows: \( V(G_B^x) = (V(G - B) \times N) \cup B \) and \( D(G_B^x) = D(\langle B \rangle_G) \cup \{(u, (v, i)) | (u, v) \in (B, B), i \in N\} \cup \{(u, i, v) | (u, v) \in (B, B), i \in N\} \cup \{(u, i, j) | (u, v) \in (B, B), j = \alpha(u, v)(i)\} \).

**Theorem 5.** Let \( \pi : \tilde{G} \rightarrow G \) is a \( n \)-fold branched covering of a connected graph \( G \) which has the branch set \( B \) with index 1. Then there exists a permutation voltage assignment \( \alpha : D(G - B) \rightarrow S_n \) such that the \( n \)-covering \( G_B^x \) is isomorphic to \( \tilde{G} \).

**Proof.** Since \( \tilde{G} - B \) is an \( n \)-fold covering of \( G - B \), there exists a permutation voltage assignment \( \alpha : D(G - B) \rightarrow S_n \) such that \( (G - B)^x \cong (\tilde{G} - B) \). For \( u \in V(G - B) \), let \( \pi_u^{-1} = \{u_1, u_2, \ldots, u_n\} \).
Let $v \in B$ and $u \in V(G - B)$. If $(u, v) \in D(G)$, then we have $\pi|_{N_G(u)} : N_G(u) \to N_G(u)$ is bijective for any $\tilde{u} \in \pi^{-1}(u)$. Let $u_1$ be the vertex of $(G - B)^2$ corresponding to $\tilde{u}$. Then we have $(\tilde{u}, v) \in D(\tilde{G})$ and $(u_1, v) \in D(G_B^2)$.

If $(v, u) \in D(G)$, then we have $(v, u) \in D(\tilde{G})$ and $(u_1, v) \in D(G_B^2)$ similar to the above case. Therefore, it follows that $G_B^2 \cong \tilde{G}$. □

Next, let $G_B^2$ be a branched $n$-covering of $G$ with branch set $B$ of index 1, where $\pi : D(G - B) \to S_n$ is a permutation voltage assignment. We define a graph $K = K_1 \sqcup K_2$ by $K_1 = \langle B \rangle_G$ and $K_2 = G - B$. Furthermore, let $N_1 = \{1\}$, $N_2 = \{1, \ldots, n\}$ and $I : D(\langle B \rangle_G) \to N_1$ a permutation voltage assignment such that $I(e) = 1, e \in D(\langle B \rangle_G)$. Then we have $G_B^2 = K_1^1 \ast K_2^2$. Thus, we obtain the following result.

For an induced subgraph $K$ of a connected graph $G$, let the matrix $D_{G,K}$ be a $|V(K)| \times |V(K)|$ diagonal matrix with diagonal elements $\deg_G w_1, \ldots, \deg_G w_m$, where $V(K) = \{w_1, \ldots, w_m\}$.

**Theorem 6.** Let $G$ be a connected graph with $v$ vertices and $e$ edges, $S_n$ the symmetric group on $N = \{1, 2, \ldots, n\}$ and $G_B^2$ a branched $n$-covering of $G$ with branch set $B$ of index 1, where $\pi : D(G - B) \to S_n$ is a permutation voltage assignment.

Set $e_1 = |D(\langle B \rangle_G)|$, $e_2 = |D(G - B)|$, $v_1 = |B|$, $v_2 = |V(D - B)|$ and $\Gamma = \langle \pi(e) \mid e \in D(G - B) \rangle$. Furthermore, let $\rho_1, \rho_2, \ldots, \rho_k$ be the inequivalent irreducible representations of $\Gamma$, and $f_i$ the degree of $\rho_i$ for each $i$, where $f_1 = 1$. Let $P : \Gamma \to GL(n, \mathbb{C})$ be a permutation representation of $\Gamma$ such that $P(g) = P_g$. Assume that $(G - B)^2$ is connected and $P = 1 + m_2 \rho_2 + \cdots + m_k \rho_k$.

Then the reciprocal of the Bartholdi zeta function of $G_B^2$ is

$$
\zeta(G_B^2, u, t)^{-1} = (1 - (1 - u)^2 t^2)^{e_1 + n(e - e_1 - e_2) - v_1} \det(I_v - tA' + (1 - u)t^2(D_G - (1 - u)I_v))
\times \prod_{i=2}^{k} \left\{ (1 - (1 - u)^2 t^2)^{(e_2 - v_2) f_i} \det(I_{v_2 f_i} - t \sum_{h \in \Gamma} \rho_i(h) \otimes A_h^n)
+ (1 - u)t^2(I_{f_i} \otimes (D_{G,G-B} - (1 - u)I_v)) \right\} .
$$

Here

$$
A' = \begin{bmatrix} A(\langle B \rangle_G) & nF \\ tF & A(G - B) \end{bmatrix}, \quad A = \begin{bmatrix} A(\langle B \rangle_G) & F \\ tF & A(G - B) \end{bmatrix}
$$

and $D_G = D_{G,\langle B \rangle} \oplus D_{G,G-B}$.

In the case that $\pi$ is an ordinary voltage assignment from $D(G - B)$ to a finite group $\Gamma$, the following result follows.

**Corollary 1.** Let $G$ be a connected graph, $\Gamma$ a finite group and $G_B^2$ a branched $|\Gamma|$-covering of $G$ with branch set $B$ of index 1, where $\pi : D(G - B) \to \Gamma$ is an ordinary voltage assignment. Let the parameters $v, e, e_1, e_2, v_1$ and $v_2$ be as in Theorem 6. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_k$ be the inequivalent irreducible representations of $\Gamma$, and $f_i$ the degree of $\rho_i$ for each $i$, where $f_1 = 1$. Then the reciprocal of the Bartholdi zeta function of $G_B^2$ is

$$
\zeta(G_B^2, u, t)^{-1} = (1 - (1 - u)^2 t^2)^{e_1 + n(e - e_1 - e_2) - v_1} \det(I_v - tA' + (1 - u)t^2(D_G - (1 - u)I_v))
\times \prod_{i=2}^{k} \left\{ (1 - (1 - u)^2 t^2)^{(e_2 - v_2) f_i} \det(I_{v_2 f_i} - t \sum_{h \in \Gamma} \rho_i(h) \otimes A_h^n)
+ (1 - u)t^2(I_{f_i} \otimes (D_{G,G-B} - (1 - u)I_v)) \right\} .
$$
6. Example

We give an example. Let $L_1$ and $L_2$ be the complete graph with three vertices $v_1, v_2, v_3$ and $w_1, w_2, w_3$, respectively, and let $K = L_1 \cup L_2$ be the graph vertex set $V(L_1) \cup V(L_2)$ and edge set $E(L_1) \cup E(L_2) \cup \{v_2w_1, v_3w_3\}$. Then we have

\[
\mathbf{A}(K) = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]

Let $S_2$ and $S_3$ be the symmetric group on the set $\{1, 2\}$ and $\{1, 2, 3\}$, respectively. Furthermore, let $\alpha : D(L_1) \rightarrow S_2$ and $\beta : D(L_2) \rightarrow S_3$ be the permutation voltage assignments such that $\alpha(v_1, v_2) = (12), \alpha(v_1, v_3) = \alpha(v_2, v_3) = 1$, and $\beta(w_1, w_2) = (12), \beta(w_1, w_3) = (23), \beta(w_2, w_3) = 1$. Then we consider the Bartholdi zeta function of the graph $G = L_1^2 \ast L_2^2$.

At first, the matrix $\mathbf{A}'$ is given as follows:

\[
\mathbf{A}' = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 3 & 0 & 0 \\
1 & 1 & 0 & 0 & 3 & 0 \\
0 & 2 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 2 & 1 & 1 & 0
\end{bmatrix}, \quad \mathbf{D}_K' = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 4
\end{bmatrix}.
\]

Then we have

\[
\det(\mathbf{I}_6 - t\mathbf{A'} + (1-u)t^2(\mathbf{D}_K' - (1-u)\mathbf{I}_6)) = \det \begin{bmatrix}
a & -t & -t & 0 & 0 & 0 \\
-t & b & -t & -3t & 0 & 0 \\
-t & -t & b & 0 & 0 & -3t \\
0 & -2t & 0 & c & -t & -t \\
0 & 0 & 0 & -t & a & -t \\
0 & 0 & -2t & -t & -t & c
\end{bmatrix} = (bc + (b+c)t - 5t^2)(a^2bc - a(ab + ac + 2c)t - (5a^2 + 2ab - 2a)r^2 + 2(a + 2r^2)),
\]

where $a = 1 + (1 - u^2)r^2, b = 1 + (4 - 3u - u^2)r^2$ and $c = 1 + (3 - 2u - u^2)r^2$.

Next, let $\Gamma_1 = \langle \{(1, 12)\} \rangle = S_2 \cong Z_2$, the cyclic group of order 2. The characters of $\mathbb{Z}_2$ are given as follows: $\chi_1 = 1, \chi_2((-1)^i) = (-1)^i(i = 0, 1)$. Let $P_1 : \Gamma_1 \rightarrow GL(2, \mathbb{C})$ be a permutation representation of $\Gamma_1$ such that $P_1(\gamma) = \mathbf{P}_7$. Then we have $P_1 = 1 + \chi_2$.

Furthermore, we have $\Gamma_2 = \langle \{(12), (23)\} \rangle = S_3$. Then $S_3$ has three irreducible representations $\rho_1 = 1, \rho_2$ (the sign representation) and $\rho_3$ with degrees $f_1 = f_2 = 1$ and $f_3 = 2$, respectively. The representation $\rho_3$ is given by

\[
\rho_3(1) = \mathbf{I}_2, \quad \rho_3((123)) = \begin{bmatrix}
\eta & 0 \\
0 & \eta^2
\end{bmatrix}, \quad \rho_3((132)) = \begin{bmatrix}
\eta^2 & 0 \\
0 & \eta
\end{bmatrix},
\]

\[
\rho_3((12)) = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad \rho_3((23)) = \begin{bmatrix}
0 & \eta \\
\eta^2 & 0
\end{bmatrix}, \quad \rho_3((13)) = \begin{bmatrix}
0 & \eta^2 \\
\eta & 0
\end{bmatrix},
\]

where $\eta = \exp(2\pi \sqrt{-1}/3) = (-1 + \sqrt{-3})/2$. Let $P_2 : \Gamma_2 \rightarrow GL(3, \mathbb{C})$ be a permutation representation of $\Gamma_2$ such that $P_2(\gamma) = \mathbf{P}_7$. Then we have $P_2 = 1 + \rho_3$.

But, the matrices $\mathbf{A}_{1,g}$ and $\mathbf{A}_{2,h}(g \in S_2, h \in S_3)$ are given as follows:

\[
\mathbf{A}_{1,1} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}, \quad \mathbf{A}_{1,(12)} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
and

\[ A_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_{2,(12)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{2,(23)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \]

\[ A_{2,(13)} = A_{2,(123)} = A_{2,(132)} = 0. \]

Let \( \rho = \rho_3 \). By Theorem 3, we have

\[
\zeta(L_2^\ast \ast L_2^\beta, u, t)^{-1} = (1 - (1 - u)^2 t^2)^{12} \det(I_6 - tA' + (1 - u^2)t^2(D'_K - (1 - u)I_6)) \\
\times \det \left( I_3 - t \sum_{g \in S_2} \zeta_2(g)A_{1,g} + (1 - u)t^2(D'_{L_1} - (1 - u)I_3) \right) \\
\times \det \left( I_6 - t \sum_{h \in S_3} \rho(h) \otimes A_{2,h} + (1 - u)t^2(I_2 \otimes (D'_{L_2} - (1 - u)I_3)) \right) \\
= (1 - (1 - u)^2 t^2)^{12} \det(I_6 - tA' + (1 - u^2)t^2(D'_K - (1 - u)I_6)) \\
\times \det \left( \begin{bmatrix} a & t & -t \\ t & b & -t \\ -t & -t & b \end{bmatrix} \right) \det \left( \begin{bmatrix} c & 0 & 0 & 0 & -t & -t \eta \\ 0 & a & -t & -t & 0 & 0 \\ 0 & -t & c & -t \eta & 0 & 0 \\ 0 & -t & -t \eta^2 & c & 0 & 0 \\ -t & 0 & 0 & 0 & a & -t \\ -t \eta^2 & 0 & 0 & 0 & -t & c \end{bmatrix} \right) \\
= (1 - (1 - u)^2 t^2)^{12}(bc + (b + c)t - 5t^2)(a^2 bc - a(ab + ac + 2c)t - (5a^2 + 2ab - 2a)t^2 + 2(a + 2)t^3) \\
\times (ab^2 - (a + 2b)t^2 + 2t^3)(ac^2 - (a + 2c)t^2 + t^3)^2.
\]

References