NOTE

ENUMERATION OF PACKED GRAPHS

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Received 26 August 1987
Revised 21 March 1988

Let \( |G| \) be the number of vertices of a graph \( G \), let \( \omega(G) \) be the density of \( G \), and \( K(G) \) be the clique graph of \( G \). A graph \( G \) is called a \((p, n)\)-packed graph if
\[
|G| = p, \quad \omega(G) = p - n \quad \text{and} \quad |K(G)| = 2^n.
\]
We obtain the number of non-isomorphic \((p, n)\)-packed graphs.

All graphs considered here are finite, undirected and simple. We denote the number of vertices of a graph \( G \) by \(|G|\). A clique of a graph \( G \) is a maximal complete subgraph of \( G \). The clique graph \( K(G) \) of a graph \( G \) is the intersection graph of the vertex sets of cliques of \( G \). The density \( \omega(G) \) of a graph \( G \) is the number of the vertices in the largest clique of \( G \). Hedman [4] showed that, for every graph \( G \),
\[
|K(G)| \leq 2^{\lfloor \omega(G) \rfloor}.
\]
We call a graph \( G \) packed if
\[
|K(G)| = 2^{\lfloor \omega(G) \rfloor}.
\]
Moreover, Hedman [4] showed that, for any packed graph \( G \),
\[
\omega(G) \geq \frac{1}{2} |G|.
\]
A packed graph \( G \) is called a \((p, n)\)-packed graph if
\[
|G| = p \quad \text{and} \quad |G| - \omega(G) = n,
\]
where \( n \leq \frac{1}{2} p \). Hedman [5] proved the following result.

**Theorem A.** Among graphs \( G \) on \( p \) vertices with \( \omega(G) < \frac{1}{2} p \), the number of non-isomorphic graphs maximizing the number of cliques is one.

We shall examine the number of non-isomorphic \((p, n)\)-packed graphs \((n \leq \frac{1}{2} p)\). If \( U \) is a nonempty subset of the vertex set \( V(G) \) of a graph \( G \), then the subgraph \( \langle U \rangle_G \) of \( G \) induced by \( U \) is the graph having vertex set \( U \) and whose edge set consists of those edges of \( G \) incident with two elements of \( U \). A subgraph \( H \) of \( G \) is called induced, denoted \( H < G \), if \( H \cong \langle U \rangle_G \) for some subset \( U \) of \( V(G) \). A \( 2n \)-Neumann graph \( H_{2n} \) is the complement of a matching between \( 2n \) vertices. A graph \( G \) is called an \( H(p, n)\)-graph \((n \leq \frac{1}{2} p)\) if there is a partition
\[
V_1 \cup V_2 \cup V_3 \text{ of } V(G) \text{ such that}
\]
(i) \(|G| - p, |V_1| = p - 2n \text{ and } |V_2| - |V_3| = n, \)
(ii) \( \langle V_1 \cup V_2 \rangle_G \) is complete,
(iii) \( \langle V_2 \cup V_3 \rangle_G \cong H_{2n}. \)

We require the following result.

**Lemma** (Sato [6]). If \( G' < G \), then \( K(G') \) is a subgraph of \( K(G) \).
The structure of \((p, n)\)-packed graphs is given as follows:

**Theorem 1.** For two positive integers \(p\) and \(n\), with \(n \leq \frac{1}{2}p\), a graph \(G\) is a \((p, n)\)-packed graph if and only if \(G\) is an \(H(p, n)\)-graph.

**Proof.** Let \(G\) be a \((p, n)\)-packed graph. Then we have \(\omega(G) = p - n\). Let \(K_1\) be a clique of \(G\) such that \(|K_1| = \omega(G)\). Denote by \(G_1\) the graph \(G - K_1\) obtained from \(G\) by removing all vertices of \(K_1\). Similarly to the proof of [4, Theorem 3.1], there are \(n\) vertices \(v_1, v_2, \ldots, v_n\) of \(K_1\) such that \(\langle \{v_1, v_2, \ldots, v_n\} \cup V(G_1) \rangle \cong H_{2n}\).

Now we may set \(V_1 = V(K_1) - \{v_1, v_2, \ldots, v_n\}\), \(V_2 = \{v_1, v_2, \ldots, v_n\}\) and \(V_3 = V(G_1)\).

Conversely, let \(G\) be an \(H(p, n)\)-graph. It is clear that \(\omega(G) = p - n\). Moreover we have \(G \supset H_{2n}\). By the lemma, \(K(H_{2n})\) is a subgraph of \(K(G)\). Thus, it follows that \(|K(G)| \geq |K(H_{2n})| = 2^n\). By [4, Theorem 2.1], we have \(|K(G)| \leq 2^n\). Hence \(G\) is a \((p, n)\)-packed graph. \(\Box\)

Let \(S_k\) be the symmetric group of degree \(k\) and \(Z(G)\) be the cycle indicator of a group \(G\) (see [1]).

The following theorem determines the number of non-isomorphic \((p, n)\)-packed graphs.

**Theorem 2.** For two positive integers \(p\) and \(n\), with \(n \leq \frac{1}{2}p\), the number of non-isomorphic \((p, n)\)-packed graphs is equal to the sum of all coefficients in \(Z(S_{p-2n} \times S_n; 1 + x)\).

**Proof.** By Theorem 1, the number of non-isomorphic \((p, n)\)-packed graphs is equal to that of non-isomorphic \(H(p, n)\)-graphs. Note that, for two graphs \(H\) and \(K\) with the same vertex set, \(H \cong K\) if and only if \(\overline{H} \cong \overline{K}\), where \(\overline{H}\) and \(\overline{K}\) are the complements of \(H\) and \(K\), respectively. Thus, the number of non-isomorphic \((p, n)\)-packed graphs is equal to that of non-isomorphic complements of \(H(p, n)\)-graphs.

Let \(G\) be an \(H(p, n)\)-graph and \(V_1 \cup V_2 \cup V_3\) be a partition of \(V(G)\) satisfying the conditions (i), (ii) and (iii). Then it follows that \(\langle V_1 \cup V_2 \rangle_{\overline{G}}\) is empty and \(\langle V_2 \cup V_3 \rangle_{\overline{G}} \cong nK_{2n}\). \(\overline{G}\) can be regarded as a bipartite graph with partite sets \(V_i\) and \(E(\langle V_2 \cup V_3 \rangle_{\overline{G}})\). Hence the number of non-isomorphic complements of \(H(p, n)\)-graphs is equal to that of non-isomorphic spanning subgraphs of the complete bipartite graph \(L\) with partite sets \(V_i\) and \(E(\langle V_2 \cup V_3 \rangle_{\overline{G}})\).

Let \(\mathcal{E}(L)\) be the edge-group of \(L\) (see [1]). Then we have \(\mathcal{E}(L) \cong S_{p-2n} \times S_n\). Let \(g_k\) be the number of non-isomorphic spanning subgraphs of \(L\) with \(k\) edges \((0 \leq k \leq (p - 2n)n)\). Then, by Pólya's Theorem, the generating function of \(g_k\) is given by \(\sum_{k>0} g_k x^k = Z(S_{p-2n} \times S_n; 1 + x)\). \(\Box\)
Corollary. For two positive integers $p$ and $n$, with $n \leq \frac{1}{2}p$ and $p \neq 3n$, the number of non-isomorphic $(p, n)$-packed graphs is equal to that of non-isomorphic spanning subgraphs of the complete bipartite graph $K_{p - 2n, n}$.

Proof. By Theorem 2 and [3].

Acknowledgment

I would like to thank the referees for their helpful comments and suggestions.

References