A Family of Multipoint Flux Mixed Finite Element Methods for Elliptic Problems on General Grids

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Abstract

In this paper, we discuss a family of multipoint flux mixed finite element (MFMFE) methods on simplicial, quadrilateral, hexahedral, and triangular-prismatic grids. The MFMFE methods are locally conservative with continuous normal fluxes, since they are developed within a variational framework as mixed finite element methods with special approximating spaces and quadrature rules. The latter allows for local flux elimination giving a cell-centered system for the scalar variable. We study two versions of the method: with a symmetric quadrature rule on smooth grids and a non-symmetric quadrature rule on rough grids. Theoretical and numerical results demonstrate first order convergence for problems with full-tensor coefficients. Second order superconvergence is observed on smooth grids.

Keywords: mixed finite element, multipoint flux approximation, cell-centered finite difference, full tensor, simplices, quadrilaterals, hexahedra, triangular prisms.

1. Introduction

We discuss the development of a family of numerical schemes for second order elliptic problems. These methods, referred to as multipoint flux mixed finite element (MFMFE) methods, allow for an accurate and efficient treatment of full tensor coefficients, irregular geometries and heterogeneities that require highly distorted grids and discontinuous coefficients. These schemes are shown to be cell-centered discretizations and to have convergent approximations for both the scalar variable and its flux. Following the terminology for Darcy flow, we refer to the scalar variable as pressure and the flux variable as velocity.

MFMFE methods can be viewed as variational counterpart of the multipoint flux approximation (MPFA) methods [1, 2, 3, 4]. In the MPFA finite volume framework, sub-edge (sub-face) fluxes are introduced, which allows for localization of velocity interactions around mesh vertices. Therefore fluxes can be easily eliminated, resulting in a
cell-centered pressure scheme. Similar elimination is achieved in the MFMFE variational framework, by employing appropriate finite element spaces and special quadrature rules. Our approach is based on the BDM1 [5] on triangles and quadrilaterals, the BDDF1 [6] on tetrahedra or hexahedra, or the CD1 [7] spaces on triangular prisms with a trapezoidal quadrature rule applied on the reference element. Related approaches have been developed in [8] on simplicial grids and in [9, 10] on quadrilateral grids using a broken Raviart-Thomas space. A key element of our methods is that the velocity space has \( n \) normal degrees of freedom on each edge (face), where \( n \) is the number of vertices of the edge (face). In the case of a reference element with some square faces, the original BDDF1 (cube) and CD1 (triangular prism) have only three degrees of freedom per face, which is insufficient for local flux elimination. We enhance the spaces by adding an appropriate number of curl basis functions, so that the resulting spaces have four degrees of freedom on square faces.

Due to their variational formulation, the MFMFE methods allow for multiscale and multiphysics extensions such as the mortar mixed finite element methods [11, 12, 13] and the enhanced velocity method [14]. These methods can handle non-matching grids and allow for coupling of different numerical algorithms and different physics in adjacent subdomains. The multiblock variational framework is useful in designing optimal parallel solvers that utilize efficient interface multiscale bases as interface preconditioners and subdomain solvers such as algebraic multigrid. These approaches have also been shown to be convergent and efficient when applying stochastic methods for uncertainty analyses [15, 16] and applying the MFMFE methods for multiscale modeling of nonlinear flow problems in porous media [17].

The remainder of the paper is organized as follows. The MFMFE methods on various grids are defined in Section 2. Theoretical convergence results are presented in Section 3. Numerical results confirming the theory are discussed in Section 4. Conclusions are given in Section 5.

2. Formulation of multipoint flux mixed finite element methods

Consider a boundary value problem on a domain \( \Omega \subset \mathbb{R}^d, d = 2, 3 \), with a Lipschitz continuous boundary \( \partial \Omega \),

\[
-\nabla \cdot \mathbb{A} \nabla p = f \quad \text{in} \ \Omega, \\
p = 0 \quad \text{on} \ \partial \Omega,
\]

where \( p \) is an unknown scalar function, \( \mathbb{A} \) is a symmetric, uniformly positive definite tensor with \( L^\infty(\Omega) \) components, and \( f \) is a source term. The choice of homogeneous Dirichlet boundary conditions is made for simplicity of the presentation; other boundary conditions can also be treated. Let \( H(\text{div}; \Omega) := \{ v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega) \} \) and let \((\cdot, \cdot)\) denote the inner product in \( L^2(\Omega) \). The weak mixed formulation of (2.1)-(2.2) reads: find \( u := -\mathbb{A} \nabla p \in H(\text{div}; \Omega) \) and \( p \in L^2(\Omega) \), such that

\[
(\mathbb{A}^{-1} u, v) - (p, \nabla \cdot v) = 0, \quad \forall v \in H(\text{div}; \Omega),
\]

\[
(\nabla \cdot u, w) = (f, w), \quad \forall w \in L^2(\Omega).
\] (2.4)

MFMFE methods have been developed and analyzed in [18] on simplicial and quadrilateral grids, in [19, 20, 13] on hexahedral grids and in [21] on triangular-prismatic grids. The method is defined as: find \( u_h \in V_h \) and \( p_h \in W_h \) such that

\[
(\mathbb{A}^{-1} u_h, v)_Q - (p_h, \nabla \cdot v) = 0, \quad \forall v \in V_h,
\]

\[
(\nabla \cdot u_h, w) = (f, w), \quad \forall w \in W_h
\] (2.5)

There are two key ingredients in the method. The first one is an appropriate choice of mixed finite element spaces \( V_h \) and \( W_h \) and degrees of freedom. The second one is a specific choice of the numerical integration rules for \((\cdot, \cdot)_Q\) in (2.5). These two choices allow for flux variables associated with a vertex to be expressed by cell-centered pressures surrounding the vertex. This results in a 9 point or 27 point pressure stencil on logically rectangular 2D or 3D grids.

The quadrature rule (2.22) can be symmetric or non-symmetric. We call the method symmetric or non-symmetric MFMFE method depending on the choice of quadrature rule. On affine or smooth grids, both the symmetric and non-symmetric MFMFE methods give first-order accurate velocities and pressures, as well as second order accurate
face fluxes and pressures at the cell centers [18, 19, 20, 21]. On highly distorted quadrilateral and hexahedral grids with non-planar faces, the convergence of the symmetric MFMFE can deteriorate while the non-symmetric MFMFE still gives a first order accuracy [20]. This is due to the fact that the non-symmetric quadrature rule satisfies some critical properties on physical elements. On affine grids, the two quadrature rules in (2.22) are the same if the tensor \( \hat{h} \) is constant in each element, since the Jacobian is a constant matrix. The non-symmetric quadrature rule was originally proposed in [10] for quadrilateral grids.

We now discuss the two ingredients in detail.

2.1. Finite element spaces

Let \( \Omega \) be a polyhedral domain partitioned into a union of finite elements of characteristic size \( h \). The elements can be triangles or quadrilaterals in 2D, tetrahedra, hexahedra, or triangular prisms in 3D. Let us denote the partition by \( \mathcal{T}_h \) and assume that it is shape-regular and quasi-uniform [22]. The velocity and pressure finite element spaces on any physical element \( E \) are defined, respectively, via the Piola transformation

\[
v \leftrightarrow \hat{v} : \hat{v} = \frac{1}{J_E} D \hat{F}_E \hat{v} \circ F_E^{-1},
\]

and the scalar transformation

\[
w \leftrightarrow \hat{w} : w = \hat{w} \circ F_E^{-1},
\]

where \( F_E \) denotes a mapping from the reference element \( \hat{E} \) to the physical element \( E \), \( D \hat{F}_E \) is the Jacobian of \( F_E \), and \( J_E \) is its determinant. The Piola transformation preserves the normal components of the vectors. The finite element spaces \( V_h \) and \( W_h \) on \( \mathcal{T}_h \) are given by

\[
V_h = \left\{ v \in H(\text{div} ; \Omega) : v|_E \leftrightarrow \hat{v}, \hat{v} \in \hat{V}(\hat{E}), \forall E \in \mathcal{T}_h \right\},
\]

\[
W_h = \left\{ w \in L^2(\Omega) : w|_E \leftrightarrow \hat{w}, \hat{w} \in \hat{W}(\hat{E}), \forall E \in \mathcal{T}_h \right\},
\]

where \( \hat{V}(\hat{E}) \) and \( \hat{W}(\hat{E}) \) are finite element spaces on the reference element \( \hat{E} \).

**Triangular elements.** In the case of triangles, \( \hat{E} \) is the reference triangle with vertices \( \hat{r}_1 = (0, 0)^T \), \( \hat{r}_2 = (1, 0)^T \), and \( \hat{r}_3 = (0, 1)^T \). Let \( \hat{r}_i (i = 1, 2, 3) \) be the corresponding vertices on the physical element. The linear mapping \( F_E \) has the form

\[
F_E(\hat{r}) = \hat{r}_1 (1 - \hat{x} - \hat{y}) + \hat{r}_2 \hat{x} + \hat{r}_3 \hat{y},
\]

and the spaces are chosen as the lowest order BDM\(_1\) [5] spaces:

\[
\hat{V}(\hat{E}) = P_1(\hat{E})^2, \quad \hat{W}(\hat{E}) = P_0(\hat{E}),
\]

where \( P_k \) denotes the space of polynomials of degree at most \( k \).

**Convex quadrilaterals.** In the case of convex quadrilaterals, \( \hat{E} \) is the unit square with vertices \( \hat{r}_1 = (0, 0)^T \), \( \hat{r}_2 = (1, 0)^T \), \( \hat{r}_3 = (1, 1)^T \), and \( \hat{r}_4 = (0, 1)^T \). Denote by \( \hat{r}_i, i = 1, \ldots, 4 \), the corresponding vertices of \( E \). In this case \( F_E \) is the bilinear mapping given as

\[
F_E(\hat{r}) = \hat{r}_1 (1 - \hat{x})(1 - \hat{y}) + \hat{r}_2 \hat{x}(1 - \hat{y}) + \hat{r}_3 \hat{x}\hat{y} + \hat{r}_4 (1 - \hat{x})\hat{y},
\]

and the spaces are the lowest order BDM\(_1\) [5] spaces

\[
\hat{V}(\hat{E}) = P_1(\hat{E})^2 + r \text{curl}(\hat{x}^2 \hat{y}) + s \text{curl}(\hat{x}\hat{y}^2), \quad \hat{W}(\hat{E}) = P_0(\hat{E}),
\]

where \( r \) and \( s \) are real constants.
**Tetrahedra.** In the case of tetrahedra, \( \hat{E} \) is the reference tetrahedron with vertices \( \hat{r}_1 = (0, 0, 0)^T, \hat{r}_2 = (1, 0, 0)^T, \hat{r}_3 = (0, 1, 0)^T, \) and \( \hat{r}_4 = (0, 0, 1)^T \). Let \( \mathbf{r}_i (i = 1, \ldots, 4) \) be the corresponding vertices of \( E \). The linear mapping for tetrahedra has the form

\[
F_E(\hat{r}) = \mathbf{r}_1(1 - \hat{x} - \hat{y} - \hat{z}) + \mathbf{r}_2 \hat{x} + \mathbf{r}_3 \hat{y} + \mathbf{r}_4 \hat{z},
\]

and the spaces are the BDM1 spaces [5]:

\[
\hat{\mathbf{v}}(\hat{E}) = P_1(\hat{E})^3, \quad \hat{\mathbf{w}}(\hat{E}) = P_0(\hat{E}).
\]

**Hexahedra.** In the case of hexahedra, \( \hat{E} \) is the unit cube with vertices \( \hat{r}_1 = (0, 0, 0)^T, \hat{r}_2 = (1, 0, 0)^T, \hat{r}_3 = (1, 1, 0)^T, \hat{r}_4 = (0, 1, 0)^T, \hat{r}_5 = (0, 0, 1)^T, \hat{r}_6 = (1, 0, 1)^T, \hat{r}_7 = (1, 1, 1)^T, \) and \( \hat{r}_8 = (0, 1, 1)^T \). Denote by \( \mathbf{r}_i = (x_i, y_i, z_i)^T, i = 1, \ldots, 8 \), the eight corresponding vertices of \( E \). We note that the element can have non-planar faces. In this case \( F_E \) is a trilinear mapping given by

\[
F_E(\hat{r}) = \mathbf{r}_1(1 - \hat{x} - \hat{y} - \hat{z}) + \mathbf{r}_2 \hat{x}(1 - \hat{y} - \hat{z}) + \mathbf{r}_3 \hat{y}(1 - \hat{z}) + \mathbf{r}_4(1 - \hat{x})\hat{y}(1 - \hat{z}) + \mathbf{r}_5(1 - \hat{x})(1 - \hat{y})\hat{z} + \mathbf{r}_6 \hat{x}(1 - \hat{y})\hat{z} + \mathbf{r}_7 \hat{y}\hat{z} + \mathbf{r}_8(1 - \hat{x})\hat{y}\hat{z},
\]

and the spaces are defined by enhancing the BDDF1 spaces [19]:

\[
\hat{\mathbf{v}}(\hat{E}) = \text{BDDF}_1(\hat{E}) + s_2 \text{curl}(0, 0, \hat{x}^2 \hat{z})^T + s_3 \text{curl}(0, 0, \hat{x}^2 \hat{y} \hat{z})^T + t_2 \text{curl}(\hat{y} \hat{z}, 0, 0)^T + t_3 \text{curl}(\hat{y} \hat{z}^2, 0, 0)^T + w_2 \text{curl}(0, \hat{y} \hat{z}^2, 0)^T + w_3 \text{curl}(0, \hat{y} \hat{z}^2, 0)^T,
\]

\[
\hat{\mathbf{w}}(\hat{E}) = P_0(\hat{E}),
\]

where the BDDF1(\( \hat{E} \)) space is defined as [6]:

\[
\text{BDDF}_1(\hat{E}) = P_1(\hat{E})^3 + s_1 \text{curl}(0, 0, \hat{x} \hat{y} \hat{z})^T + s_1 \text{curl}(0, 0, \hat{x} \hat{z}^2)^T + t_0 \text{curl}(\hat{x} \hat{y} \hat{z}, 0, 0)^T + t_1 \text{curl}(\hat{y} \hat{z}^2, 0, 0)^T + w_0 \text{curl}(0, \hat{y} \hat{z}^2, 0)^T + w_1 \text{curl}(0, \hat{y} \hat{z}^2, 0)^T.
\]

In above equations, \( s_i, t_i, w_i (i = 0, \ldots, 3) \) are real constants.

**Triangular prisms.** Consider a prism with two triangular bases and three parallelogram faces. Let \( \hat{E} \) be a reference prism with vertices \( \hat{r}_1 = (0, 0, 0)^T, \hat{r}_2 = (1, 0, 0)^T, \hat{r}_3 = (0, 1, 0)^T, \hat{r}_4 = (0, 0, 1)^T, \hat{r}_5 = (1, 0, 1)^T, \) and \( \hat{r}_6 = (0, 1, 1)^T \). Denote by \( \mathbf{r}_i = (x_i, y_i, z_i)^T, i = 1, \ldots, 6 \), the six corresponding vertices of element \( E \). Note that the three side faces are parallelograms. In this case, \( F_E \) is a linear mapping given by

\[
F_E(\hat{r}) = \mathbf{r}_1(1 - \hat{x} - \hat{y} - \hat{z}) + \mathbf{r}_2 \hat{x} + \mathbf{r}_3 \hat{y} + \mathbf{r}_4 \hat{z}.
\]

The spaces are given by [21]

\[
\hat{\mathbf{v}}(\hat{E}) = P_1(\hat{E})^3 + s_0 \text{curl}(0, 0, \hat{y}^2 \hat{z}) + s_1 \text{curl}(\hat{x} \hat{z}^2, 0, 0) + t_0 \text{curl}(\hat{x} \hat{y} \hat{z}, 0, 0) + t_1 \text{curl}(0, -\hat{x} \hat{z}^2, 0) + w_0 \text{curl}(0, -3/2 \hat{x}^2 \hat{z} - 1/2 \hat{x} \hat{z}^2 \hat{y}) + w_1 \text{curl}(3/2 \hat{y} \hat{z}^2 \hat{x}, 0, 1/2 \hat{y} \hat{z} \hat{x}),
\]

\[
\hat{\mathbf{w}}(\hat{E}) = P_0(\hat{E}).
\]

In all cases the velocity degrees of freedom (DOF) are chosen to be the normal components at \( n \) points on each face where \( n \) is the number of vertices of that face. We choose these points to be the vertices. The dimension of the space is \( d n_v \), where \( d \) is the dimension and \( n_v \) is the number of vertices in \( E \). Note that the original BDDF1 and CD1 spaces have only three DOF on square faces. These spaces have been enhanced to have four DOF on square faces. This special choice is needed in the reduction to a cell-centered pressure stencil as described later in this section. In addition, the normal components of the velocity vector across element faces are continuous, which is needed for an \( H(\text{div}; \Omega) \)-conforming velocity space as required by (2.7).
2.2. Numerical quadratures

The computation of the velocity integral on a physical element is performed by mapping to the reference element and choosing a quadrature rule on \( \hat{E} \). Using the Piola transformation, we write \((\hat{h}^{-1}, \cdot)\) in (2.3) as

\[
(\hat{h}^{-1} \mathbf{q}, \mathbf{v})_E = \left( \frac{1}{J_E} \mathbf{D}F_E^T \hat{h}^{-1}(F_E(\hat{x})) \mathbf{D}F_E \hat{\mathbf{q}}, \hat{\mathbf{v}} \right)_E \equiv (M_{qE} \hat{\mathbf{q}}, \hat{\mathbf{v}})_E,
\]

where

\[
M_{qE} = \frac{1}{J_E} \mathbf{D}F_E^T \hat{h}^{-1}(F_E(\hat{x})) \mathbf{D}F_E.
\]

Define a perturbed \( \hat{M}_{qE} \) as

\[
\hat{M}_{qE} = \frac{1}{J_E} \mathbf{D}F_E^T (\hat{\mathbf{r}}_c) \hat{h}^{-1} \mathbf{D}F_E,
\]

where \( \hat{\mathbf{r}}_c \) is the centroid of \( \hat{E} \) and \( \hat{h} \) denotes the mean of \( h \) on \( E \). In addition, denote the trapezoidal rule on \( \hat{E} \) by \( \text{Trap}(\cdot, \cdot)_{\hat{E}} \):

\[
\text{Trap}(\hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{E}} \equiv \frac{\hat{E}}{n_v} \sum_{i=1}^{n_v} \hat{\mathbf{q}}(\hat{\mathbf{r}}_i) \cdot \hat{\mathbf{v}}(\hat{\mathbf{r}}_i),
\]

where \( \{\hat{\mathbf{r}}_i\}_{i=1}^{n_v} \) are the vertices of \( \hat{E} \).

The symmetric quadrature rule is based on the original \( M_{qE} \) while the non-symmetric one is based on the perturbed \( \hat{M}_{qE} \). The quadrature rule on an element \( E \) is defined as

\[
(\hat{h}^{-1} \mathbf{q}, \mathbf{v})_{Q,E} \equiv \begin{cases} 
\text{Trap}(M_{qE} \hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{E}} & \text{symmetric}, \\
\text{Trap}(\hat{M}_{qE} \hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{E}} & \text{non-symmetric}.
\end{cases}
\]

The non-symmetric quadrature rule has certain critical properties on the physical elements that lead to a convergent method on rough quadrilaterals and hexahedra [20].

The global quadrature rule on \( \Omega \) is then given as

\[
(\hat{h}^{-1} \mathbf{q}, \mathbf{v})_{Q} \equiv \sum_{E \in T_h} (\hat{h}^{-1} \mathbf{q}, \mathbf{v})_{Q,E}.
\]

2.3. Cell-centered finite differences

The trapezoidal quadrature rule localizes the interactions of the velocity DOF. In particular, the DOF associated with a vertex become decoupled from the rest of the DOF, resulting in a block-diagonal flux mass matrix in (2.5) with one block per grid vertex. The dimension of each block equals the number of faces that share the vertex. For example, this dimension is 12 for logically rectangular hexahedral grids, see Figure 1. Inverting each local block in the mass matrix in (2.5) allows for expressing the velocity DOF associated with a vertex in terms of the scalars at the centers of the elements that share the vertex (there are eight such elements in Figure 1). Substituting these expressions into the mass conservation equation (2.6) leads to a cell-centered system for the scalar. The stencil is 27 points on logically rectangular hexahedral grids. The local linear systems and the resulting global system are positive definite and therefore invertible for the symmetric MFMFE method and, under a mild restriction on the shape regularity of the grids and/or the anisotropy of the permeability, for the non-symmetric MFMFE method; see (3.6) below. The reader is referred to [18, 19, 20, 21] for further details on the reduction to a cell-centered finite difference system.

3. Theoretical convergence results

Let \( W^{k,\infty}_{T_h} \) consist of functions \( \phi \) such that \( \phi|_E \in W^{k,\infty}(E) \) for all \( E \in T_h \). Here \( k \) is a multi-index with integer components and \( W^{k,\infty}(E) \) denotes the Sobolev space of functions whose derivatives of order \( k \) belong to \( L^\infty(E) \). In addition, let \( \| \cdot \|_k \) be the norm in the Hilbert space \( H^k(\Omega) \) with functions whose derivatives of order \( k \) belong to \( L^2(\Omega) \). The norm in \( L^2(\Omega) \) is denoted by \( \| \cdot \| \). Let \( X \lesssim (\gtrsim) Y \) denote that there exists a constant \( C \), independent of the mesh size \( h \), such that \( X \leq (\geq) CY \). The notation \( X \sim Y \) means that both \( X \lesssim Y \) and \( X \gtrsim Y \) hold.

We introduce the following definitions [18, 19, 13].
Definition 3.1. The (possibly non-planar) faces of a hexahedral element $E$ defined via a trilinear mapping in three dimensions are called generalized quadrilaterals.

Definition 3.2. A generalized quadrilateral with vertices $r_1, \ldots, r_4$ is called an $h^2$-parallelogram if
\[ |r_{34} - r_{21}|_{R^d} \lesssim h^2, \] (3.1)
where $| \cdot |_{R^d}$ is the Euclidean norm in $R^d$.

Definition 3.3. A hexahedral element is called a $h^2$-parallelepiped if all of its faces are $h^2$-parallelograms.

3.1. Convergence of the symmetric MFMFE

Theorem 3.1 ([18, 19, 21]). Let $T_h$ consist of simplices, $h^2$-parallelograms, $h^2$-parallelepipeds or triangular prisms. If $A^{-1} \in W_{1,\infty}^{1,\infty}$, then, the flux $u_h$ and scalar $p_h$ of the symmetric MFMFE method (2.5)–(2.6) satisfy
\[ \|u - u_h\| \lesssim h\|u\|_1, \] (3.2)
\[ \|\nabla \cdot (u - u_h)\| \lesssim h\|\nabla \cdot u\|_1, \] (3.3)
\[ \|p - p_h\| \lesssim h(\|u\|_1 + \|p\|_1). \] (3.4)

3.2. Convergence of the non-symmetric MFMFE

On simplicial grids, $h^2$-parallelograms, $h^2$-parallelepipeds, and triangular prisms, the non-symmetric MFMFE method has same order of accuracy as the symmetric method. In addition, the non-symmetric method has first order convergence for the flux and scalar on general quadrilaterals and for the face flux and scalar on general hexahedra with non-planar faces. These convergence properties are not shared by the symmetric method.

The stability and accuracy of the non-symmetric MFMFE method rely on some properties of the bilinear form $(A^{-1}, \cdot)_Q$ defined on the space $V_h$. We have
\[ (A^{-1} q, v)_Q = \sum_{E \in T_h} (A^{-1} q, v)_E = \sum_{c \in C_h} v^T c M_c q, \] (3.5)
where $C_h$ denotes the set of corner or vertex points in $T_h$, $v_c := (v \cdot n_c)(x_c)_{i=1}^{n_c}$, $x_c$ is the coordinate vector of point $c$, and $n_c$ is the number of faces (or edges in 2D) that share the vertex point $c$.

Lemma 3.1 ([20]). If $M_c$ is uniformly positive definite for all $c \in C_h$:
\[ h^d \xi^T c \xi \lesssim \xi^T M_c \xi, \quad \forall \xi \in R^n, \] (3.6)
then the bilinear form $(A^{-1}, \cdot)_Q$ is coercive in $V_h$ and induces a norm in $V_h$ equivalent to the $L^2$-norm:
\[ (A^{-1} v, v)_Q \approx \|v\|^2, \quad \forall v \in V_h. \] (3.7)

If in addition
\[ \xi^T M_c^T M_c \xi \lesssim h^{2d} \xi^T c \xi, \quad \forall \xi \in R^n, \] (3.8)
then the following Cauchy-Schwarz type inequality holds:
\[ (A^{-1} q, v)_Q \lesssim \|q\|\|v\| \quad \forall q, v \in V_h, \] (3.9)
Conditions (3.6) and (3.8) impose mild restrictions on the element geometry and the anisotropy of the permeability tensor $K$, see [10, 23].

**Theorem 3.2** ([20]). Let $K_h \in W_{T_h}^{1,\infty}(\Omega)$ and $K_h^{-1} \in W_{T_h}^{0,\infty}(\Omega)$. If (3.6) and (3.8) hold, then the flux $u_h$ and the scalar $p_h$ of the non-symmetric MFMFE method (2.5)—(2.6) satisfy

$$
\|\Pi u - u_h\| + \|Q_h p - p_h\| \lesssim h(\|u\|_1 + \|p\|_2),
$$

where $\Pi$ is the canonical interpolation operator onto $V_h$ [24] and $Q_h$ is the $L^2$-orthogonal projection onto $W_h$.

This result further implies convergence of the computed normal velocity to the true normal velocity on the element faces in the norm

$$
\|v\|_{T_\Omega}^2 := \sum_{E \in T_h} \sum_{e \in E} \frac{|E|}{|e|} \|v \cdot n\|_e^2,
$$

where $|E|$ is the volume of $E$ and $|e|$ is the area of $e$. This norm gives an appropriate scaling of $|\Omega|^{1/2}$ for a unit vector.

**Theorem 3.3** ([20]). Let $K_h \in W_{T_h}^{1,\infty}(\Omega)$ and $K_h^{-1} \in W_{T_h}^{0,\infty}(\Omega)$. If (3.6) and (3.8) hold, then the flux $u_h$ of the non-symmetric MFMFE method (2.5)—(2.6) satisfies

$$
\|u - u_h\|_{T_\Omega} \lesssim h(\|u\|_1 + \|p\|_2).
$$

4. **Numerical results**

In this section, we test the convergence for both symmetric and non-symmetric MFMFE methods on quadrilaterals and hexahedra. The flux error $\|u - u_h\|_{T_\Omega}$ is approximated by using 9-point or 3-point Gaussian quadrature rules on faces or edges respectively. The scalar error $\|p - p_h\|$ is approximated by 27-point Gauss quadrature rule.

The resulting linear algebraic system is solved using the software HYPRE (high performance preconditioners) developed by researchers at Lawrence Livermore National Laboratory\(^4\). More precisely, we employ the generalized minimum residual (GMRES) method with one V-cycle of algebraic multigrid as a preconditioner. The stopping criteria of GMRES is that the relative residual error is less than $10^{-9}$.

4.1. **Quadrilaterals**

Consider the problem (2.1)—(2.2) with a given analytical solution and a full tensor as

$$
p(x, y) = \sin(\pi x)^3 \sin(2\pi y), \quad K_h = \begin{pmatrix} 5 & 3 \\ 3 & 7 \end{pmatrix}.
$$

We consider both smooth and non-smooth meshes. The smooth mesh is defined as a $C^\infty$ map of a uniform mesh on the unit square and the map is given as

$$
x = \hat{x} - 0.07 \sin(3\pi \hat{x}) \cos(2\pi \hat{y}), \quad y = \hat{y} + 0.06 \cos(2\pi \hat{x}) \sin(3\pi \hat{y}).
$$

The mesh is shown on the left in Figure 2. It is easy to check that the smooth mapping makes the elements to be $h^2$-parallelograms. The second mesh consists of highly distorted quadrilaterals generated by randomly perturbing each uniform mesh point within a square with edge length $0.6h$ centered at the grid point, see the right Figure 2. The grids on the different levels of refinement are obtained by mapping or perturbing refinements of the uniform grid.

Table 1 shows the numerical results on $h^2$-parallelogram meshes. As the theory predicts for both the symmetric and non-symmetric MFMFE methods, we observe first order convergence for the scalar and the flux.

Table 2 demonstrates the convergence behavior for the randomly $h$-perturbed grids. Clearly the convergence of the flux and the scalar of the symmetric MFMFE method deteriorates. As Theorem 3.3 predicts the non-symmetric MFMFE method has first order convergence for both the flux and the scalar on these highly distorted grids.

\(^4\)https://computation.llnl.gov/casc/hypre/software.html
Table 1: Convergence of symmetric (left) and non-symmetric (right) MFMFE on $h^2$-parallelogram grids

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$|p - p_h|$</th>
<th>$|u - u_h|_{\mathcal{F}_h}$</th>
<th>$|p - p_h|$</th>
<th>$|u - u_h|_{\mathcal{F}_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.36E-01</td>
<td>6.73E+00</td>
<td>1.41E-01</td>
<td>5.92E+00</td>
</tr>
<tr>
<td>16</td>
<td>6.54E-02</td>
<td>3.31E+00</td>
<td>6.61E-02</td>
<td>2.91E+00</td>
</tr>
<tr>
<td>32</td>
<td>3.23E-02</td>
<td>1.62E+00</td>
<td>3.24E-02</td>
<td>1.43E+00</td>
</tr>
<tr>
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<td>1.61E-02</td>
<td>8.06E-01</td>
<td>1.61E-02</td>
<td>7.13E-01</td>
</tr>
<tr>
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<td>8.06E-03</td>
<td>4.02E-01</td>
<td>8.06E-03</td>
<td>3.56E-01</td>
</tr>
<tr>
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<td>2.01E-01</td>
<td>4.03E-03</td>
<td>1.78E-01</td>
</tr>
<tr>
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<td>2.01E-03</td>
<td>1.00E-01</td>
<td>2.01E-03</td>
<td>8.89E-02</td>
</tr>
</tbody>
</table>

Table 2: Convergence of symmetric (left) and non-symmetric (right) MFMFE on $h$-perturbed quadrilateral grids

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$|p - p_h|$</th>
<th>$|u - u_h|_{\mathcal{F}_h}$</th>
<th>$|p - p_h|$</th>
<th>$|u - u_h|_{\mathcal{F}_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
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<td>4.94E+00</td>
<td>1.28E-01</td>
<td>4.14E+00</td>
</tr>
<tr>
<td>16</td>
<td>6.15E-02</td>
<td>2.54E+00</td>
<td>6.19E-02</td>
<td>1.77E+00</td>
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<tr>
<td>32</td>
<td>3.05E-02</td>
<td>1.21E+00</td>
<td>3.03E-02</td>
<td>9.80E-01</td>
</tr>
<tr>
<td>64</td>
<td>1.59E-02</td>
<td>1.83E+00</td>
<td>1.52E-02</td>
<td>4.61E-01</td>
</tr>
<tr>
<td>128</td>
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<td>1.86E+00</td>
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<td>2.27E-01</td>
</tr>
<tr>
<td>256</td>
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<td>1.86E+00</td>
<td>3.80E-03</td>
<td>1.18E-01</td>
</tr>
<tr>
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<td>5.78E-03</td>
<td>1.88E+00</td>
<td>1.90E-03</td>
<td>5.96E-02</td>
</tr>
</tbody>
</table>

Table 3: Convergence of symmetric (left) and non-symmetric (right) MFMFE on $h^2$-parallelogram grids

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$|p - p_h|$</th>
<th>$|u - u_h|_{\mathcal{F}_h}$</th>
<th>$|p - p_h|$</th>
<th>$|u - u_h|_{\mathcal{F}_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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<td>1.49E-01</td>
<td>2.59E+00</td>
</tr>
<tr>
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</tr>
<tr>
<td>16</td>
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<td>6.04E-01</td>
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<td>5.91E-01</td>
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<td>1.76E-02</td>
<td>2.92E-01</td>
</tr>
<tr>
<td>64</td>
<td>8.79E-03</td>
<td>1.49E-01</td>
<td>8.79E-03</td>
<td>1.46E-01</td>
</tr>
</tbody>
</table>

Table 4: Convergence of symmetric (left) and non-symmetric (right) MFMFE on $h$-perturbed hexahedral grids

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$|p - p_h|$</th>
<th>$|u - u_h|_{\mathcal{F}_h}$</th>
<th>$|p - p_h|$</th>
<th>$|u - u_h|_{\mathcal{F}_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.51E-01</td>
<td>2.89E+00</td>
<td>1.53E-01</td>
<td>2.83E+00</td>
</tr>
<tr>
<td>8</td>
<td>7.20E-02</td>
<td>1.44E+00</td>
<td>7.27E-02</td>
<td>1.28E+00</td>
</tr>
<tr>
<td>16</td>
<td>3.63E-02</td>
<td>8.68E-01</td>
<td>3.61E-02</td>
<td>5.46E-01</td>
</tr>
<tr>
<td>32</td>
<td>1.93E-02</td>
<td>7.97E-01</td>
<td>1.80E-02</td>
<td>2.86E-01</td>
</tr>
<tr>
<td>64</td>
<td>1.17E-02</td>
<td>7.73E-01</td>
<td>9.01E-03</td>
<td>1.44E-01</td>
</tr>
</tbody>
</table>
We consider the problem (2.1)–(2.2) with a given analytical solution and a full tensor as

\[ p(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z), \quad A = \begin{pmatrix} 2 & 1.25 & 1.5 \\ 1.25 & 3 & 2 \\ 1.5 & 2 & 4 \end{pmatrix}. \]

We consider both smooth and distorted hexahedral meshes. The smooth ones are generated using the following smooth map of uniform grids

\[ \begin{align*}
  x &= \hat{x} + 0.04 \sin(\pi \hat{x}) \cos(2\pi \hat{y}) \cos(3\pi \hat{z}), \\
  y &= \hat{y} - 0.05 \cos(3\pi \hat{x}) \sin(\pi \hat{y}) \cos(2\pi \hat{z}), \\
  z &= \hat{z} + 0.06 \cos(2\pi \hat{x}) \cos(3\pi \hat{y}) \sin(\pi \hat{z}).
\end{align*} \]

This mapping yields an \( h^2 \)-parallelepiped mesh. The second mesh is generated by randomly perturbing each uniform mesh point within a cube with edge length 0.5\( h \) centered at the grid point. Element faces in both meshes are non-planar. The meshes are shown in Figure 3.

Tables 3–4 show the convergence behavior of both the symmetric and non-symmetric MFMFE methods on the two hexahedral meshes. The symmetric MFMFE method has first order convergence for the flux and scalar on the smooth mesh. However on randomly perturbed hexahedral meshes, the convergence of symmetric MFMFE method deteriorates. The non-symmetric MFMFE method has first order convergence for both flux and scalar on both smooth and distorted hexahedra. This confirms the theory established in Theorems 3.1–3.3.
Remark 4.1. On smooth grids, we also observe a second order superconvergence in the discrete norms for the pressure and flux error:

\[ |p - p_h|^2 \equiv \sum_{E \in T_h} |E| (p - p_h)^2(m_e), \quad |u - u_h|^2_{E_f} \equiv \sum_{E \in T_h} \sum_{e \in \partial E} |E| \left( \frac{1}{|e|} \int_e u \cdot n_e - \frac{1}{|e|} \int_e u_h \cdot n_e \right)^2, \]

where \( m_e \) is the center of mass of the element \( E \). The reader is referred to [18, 19, 20] for details.

5. Conclusions

We presented a family of accurate and efficient cell-centered discretization methods, multipoint flux mixed finite element methods on simplicial, quadrilateral, hexahedral, and triangular-prismatic grids. Both the symmetric and non-symmetric methods have first order convergence on smooth and affine grids. In addition, the non-symmetric method has first order convergence on rough quadrilateral or hexahedral grids. We have also developed and analyzed a post-processing technique on general hexahedra that gives an accurate flux inside each element based on accurate MFME face fluxes [25].

References