FUZZY LOGIC OPERATORS IN
DECISION-MAKING

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The fuzzy logic operators are used to decide the following multicriteria decision-making problem. A finite set of alternatives is evaluated by a set of fuzzy criteria, i.e., fuzzy relations, which may be either fuzzy preference, or similarity, or likeness relations. The problem is to construct an evaluation procedure to compare the set of alternatives according to the whole set of criteria, i.e., to aggregate the private fuzzy relations in order to get the union relation as a fuzzy one, enhancing to solve the ranking, choice, or cluster problem. Some properties of these operators, required to decide the ranking, choice, or cluster problems and depending on the properties of the private relations, are proved. A numerical example is given as well.

The multicriteria decision-making problem considered here is based on:

— $A = \{A_1, \ldots, A_n\}$ a finite set of alternatives, among which a decision-maker has to choose (choice problem) or to rank (ranking problem), or to part (cluster problem);
— $K = \{K_1, \ldots, K_m\}$ a finite set of judges or criteria on the base of which the alternatives are evaluated;

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—\{R_1, \ldots, R_m\} matrices corresponding to the fuzzy relations by each criterion, which may be either fuzzy preference or similarity, or likeness relations.

The setting problem consists in aggregating the fuzzy relations given by the described above matrices in such a way, that the union relation would be the one providing a possibility to decide the choice, ranking, or cluster problem.

Most approaches deciding this problem consist of two stages:

1. The aggregation of fuzzy relations with respect to all criteria, i.e., premise aggregation;
2. The rank ordering or clustering of the decision alternatives according to the aggregated relations, i.e., result aggregation.

A purposeful approach for uniting fuzzy relations is to use the aggregation procedures that realize the idea of trade-offs between conflicting criteria, when compensation is allowed. These procedures may be realized with the help of fuzzy logic operators. The first ones introduced by L. Zadeh are for the logical operations AND, OR, and NOT as extensions of their Boolean origins. These operators are Min, Max, and 1 − μ (μ is the membership degree to a given relation). The degree of compensation through which humans aggregate criteria is not expressed by the Min, Max, and 1 − μ operators. There exist some fuzzy logic operators that more accurately represent human decision-making. The operators such as Weighted Mean and MaxMin ones (Von Altrock, 1997; Zimmermann, 1993) are examples of averaging operators. In fact, they are adequate models of human aggregation procedure and have empirically performed quite well. The resulting trade-off lies between the most optimistic lower bound and the most pessimistic upper bound, i.e., they map between the minimum and the maximum membership degree of the aggregated sets.

In order to decide a variety of phenomena in decision situations, several operators with different compensations are introduced. An operator, which is more general in the sense that the compensation between intersection and union of the fuzzy sets is expressed by a parameter γ, is suggested and empirically tested by Zimmermann and
Zysno (1980) under the name “compensatory and” or Gamma operator. Another operator, named MinAvg (Von Altrock, 1997; Zimmerman, 1993) is a combination of “fuzzy” and with the Weighted Mean. It allows compensation, based on a parameter $\lambda$, between the membership values of the aggregated sets and leads to very good results with respect to empirical data.

At the premise aggregation stage, the main contribution of this research lies in the possibility for choosing a combination of fuzzy relations’ properties, such that the obtained union relation by the help of fuzzy logic operators be a fuzzy relation with proved properties. In other words, out intention is not to suggest methods for fusing fuzzy relations, but to show that if there are source fuzzy relations with given properties, then by using proposed union operators one can get a fuzzy aggregated relation with proved properties. Four fuzzy logic operators are investigated and their properties (according to the private relations’ properties) are proved. These union relations may be transformed at the result aggregation stage in such a way enhancing to decide the ranking, choice, or cluster problem.

The rest of the paper is organized as follows. Definitions of some special fuzzy relations are presented in the first part. Premise aggregation with the help of four fuzzy logic operators are proposed in the second part. The propositions and theorems stated prove the properties of the aggregated relations according to the properties of the private relations. How can the result aggregation stage be decided is considered in the third part. An application solving a choice and ranking problems for a water supply system is presented in the last section.

**SPECIAL FUZZY RELATIONS**

Relations that are of interest in this research are fuzzy relations that pertain to the similarity of fuzzy sets and those that order fuzzy sets.

Let $A$ be a set of alternatives and $R$ be a binary fuzzy relation on $A$, i.e., for $\forall a, b \in A$ there exists a membership function $\mu_R: A \times A \rightarrow [0, 1]$.

**Definition 1**: A similarity relation (Dubois & Prade, 1980; Zadeh, 1971) is a fuzzy relation in $A$, if it is reflexive, symmetrical, and
max-min transitive, i.e.,
\[ \mu_R(a, a) = 1 \quad \text{for } \forall a \in A \] (1)
\[ \mu_R(a, b) = \mu_R(b, a) \quad \text{for } \forall a, b \in A \] (2)
\[ \mu_R(a, c) \geq \min(\mu_R(a, b), \mu_R(b, c)) \quad \text{for } \forall a, b, c \in A. \] (3)

According to Bezdek and Harris (1978) and Zadeh (1971) the max-min transitivity is a too strong property to impose on a fuzzy relation. Zadeh suggested several definitions that are compared in Venugopalan (1992). The weakest of all these definitions is the called max-\(\Delta\) transitivity, i.e.,
\[ \mu_R(a, c) \geq \max(0, \mu_R(a, b) + \mu_R(b, c) - 1) \quad \text{for } \forall a, b, c \in A. \] (4)

**Definition 2:** A likeness relation (Dubois & Prade, 1980; Ruspini, 1977) is a fuzzy relation in \(A\), if it is reflexive (1), symmetrical (2), and max-\(\Delta\) transitive (4).

Fuzzy-order relations play a very important role in the models for decision-making in fuzzy environment. They are defined in different ways by different authors, but we use the ones given more systematically in Nakamura (1986).

**Definition 3:** Relation \(R\) is a weak F-weak order iff it is weakly F-comparable and weakly F-transitive, i.e.,
\[ \max(\mu_R(a, b), \mu_R(b, a)) \geq 1/2 \quad \text{for } \forall a, b \in A \] (5)
if \(\mu_R(a, b) \geq 1/2, \mu_R(b, c) \geq 1/2 \rightarrow \mu_R(a, c) \geq 1/2 \)
\[ \text{for } \forall a, b, c \in A. \] (6)

**Definition 4:** Relation \(R\) is a moderate F-weak order iff \(R\) is moderately F-comparable and moderately F-transitive, i.e.,
\[ \mu_R(a, b) + \mu_R(b, a) \geq 1, \quad \text{for } \forall a, b \in A \] (7)
if \(\mu_R(a, b) \geq 1/2, \)
\[ \mu_R(b, c) \geq 1/2 \rightarrow \mu_R(a, c) \geq \min(\mu_R(a, b), \mu_R(b, c)) \]
\[ \text{for } \forall a, b, c \in A. \] (8)
**Definition 5:** $R$ is a fuzzy total ordering iff it is F-reciprocal and weakly F-transitive, i.e.,

$$\mu_R(a, b) + \mu_R(b, a) = 1$$  \hspace{1cm} (9)

if $\mu_R(a, b) > 1/2$, $\mu_R(b, c) > 1/2 \rightarrow \mu_R(a, c) > 1/2$

for $\forall a, b, c \in A$.

There exists an opinion (Yuan, 1991) that the definition of a fuzzy total ordering is a strong one and, consequently, is suitable for representing a string preference relation only. Then, if fuzzy relations are used to compare alternatives, the properties should be satisfied to a certain degree to derive a desired fuzzy ordering.

**Definition 6:** A fuzzy relation that is reflexive (1), perfect antisymmetric, i.e.,

for $\forall a, b \in A$ if $\mu_R(a, b) > 0$ then $\mu_R(b, a) = 0$

and max-$\Delta$ transitive (4) is called a perfect fuzzy-order relation or a fuzzy partial-order relation (Venugopalan, 1992; Zadeh, 1971).

**Definition 7:** A fuzzy relation that is reflexive (1) and max-$\Delta$ transitive (4) is called a fuzzy preorder (Dubois & Prade, 1980).

**Definition 8:** A fuzzy linear ordering $R$ is a fuzzy partial ordering such that for $\forall a, \forall b$ if $a \neq b$, either $\mu_R(a, b) > 0$ or $\mu_R(b, a) > 0$ (Dubois & Prade, 1980).

Any $\alpha$-cut of a fuzzy linear ordering is a nonfuzzy linear ordering.

**PREMISE AGGREGATION WITH FUZZY LOGIC OPERATORS**

**Weighted Mean Operator**

The aggregation of $m$ fuzzy relations $R_i$ is investigated with the help of the operator $R = \sum_{i=1}^{m} c_i R_i$ where $\sum_{i=1}^{m} c_i = 1$ (Von Altrock, 1997). It is a convex composition of $m$ fuzzy relations, where $c_i$ expresses the “percentage” of $R_i$ required to build the fuzzy relation $R$. The membership function of this operator is $\mu_R(a, b) = \sum_{i=1}^{m} c_i \mu_{R_i}(a, b)$. The following properties of the aggregated relation $R$, according to the
private relations’ properties, are proved in Peneva and Popchev (1996a, 1998):

—If \( R_i, i = 1, \ldots, m \) are likeness relations, then \( R \) is a likeness one (Peneva & Popchev, 1996a).

—If \( R_i, i = 1, \ldots, m \) are moderately F-comparable (7) and max-\( \Delta \) transitive (4), then \( R \) a moderate F-weak order (Peneva & Popchev, 1998).

—If \( R_i, i = 1, \ldots, m \) are F-reciprocal (9) and max-\( \Delta \) transitive (4), then \( R \) is a fuzzy total ordering (Peneva & Popchev, 1998).

**MaxMin Operator**

The mathematical model of the MaxMin operator is \( R = \alpha \bigcup_{i=1}^{m} R_i + (1-\alpha) \bigcap_{i=1}^{m} R_i \), where \( \alpha \in [0,1] \). This fuzzy operator reflects a combination of the optimistic and pessimistic views using an index of optimism \( \alpha \). Its membership function is (Von Altrock, 1997):

\[
\mu_R(a, b) = \alpha \max_i \{ \mu_{R_i}(a, b) \} + (1-\alpha) \min_i \{ \mu_{R_i}(a, b) \}.
\]

The following properties of the aggregated relation \( R \), according to the private relations’ properties, are proved in Peneva and Popchev (1996a, 1998):

—If \( R_i, i = 1, \ldots, m \) are likeness relations, then \( R \) is a likeness one (Peneva & Popchev, 1996a).

—\( R \) is moderately F-comparable (7) iff \( R_i, i = 1, \ldots, m \) are moderately F-comparable (Peneva & Popchev, 1998).

—If \( R_i, i = 1, \ldots, m \) are F-reciprocal (9), then \( R \) is moderately F-comparable (7) (Peneva & Popchev, 1998).

—If \( R_i, i = 1, \ldots, m \) are max-min transitive (3), then \( R \) is max-\( \Delta \) transitive (4) (Peneva & Popchev, 1998).

**MinAvg Operator**

Another operator that has not only been suggested in research papers, but has also been applied in some practical applications, is the MinAvg operator (Von Altrock, 1997; Zimmermann, 1993). It consists of a linear
combination between the minimum operator and the average one with a parameter $\lambda$. The membership function of this operator is

$$\mu_R(a, b) = \frac{\lambda}{m} \sum_{i=1}^{m} \mu_{R_i}(a, b) + (1 - \lambda) \min_{i=1}^{m} \{ \mu_{R_i}(a, b) \}. \quad (11)$$

**Theorem 1:** The aggregated relation $R$ by MinAvg operator is a likeness relation iff the private relations $R_i$, $i = 1, \ldots, m$ are similarity relations.

**Proof:** It is easy to prove that if $R_i$ are reflexive (1) and symmetrical (2) relations for $\forall i = 1, \ldots, m$ then $R$ is reflexive and symmetrical one as well.

Let $R_i$, $i = 1, \ldots, m$ are max-min transitive (3) relations. Then

$$\mu_R(a, c) = \frac{\lambda}{m} \sum_{i=1}^{m} \mu_{R_i}(a, c) + (1 - \lambda) \min_{i=1}^{m} \{ \mu_{R_i}(a, c) \}$$

$$\geq \frac{\lambda}{m} \sum_{i=1}^{m} \min \left( \mu_{R_i}(a, b), \mu_{R_i}(b, c) \right)$$

$$+ (1 - \lambda) \min_{i=1}^{m} \{ \min \left( \mu_{R_i}(a, b), \mu_{R_i}(b, c) \right) \}$$

$$\geq \lambda \sum_{i=1}^{m} \min \left( \frac{1}{m} \mu_{R_i}(a, b), \frac{1}{m} \mu_{R_i}(b, c) \right)$$

$$+ (1 - \lambda) \min_{i=1}^{m} \{ \mu_{R_i}(a, b), \mu_{R_i}(b, c) \}$$

$$\geq \lambda \sum_{i=1}^{m} \max \left( 0, \frac{1}{m} \mu_{R_i}(a, b) + \frac{1}{m} \mu_{R_i}(b, c) - 1 \right)$$

$$+ (1 - \lambda) \min_{i=1}^{m} \{ \mu_{R_i}(a, b) \} \cdot \min_{i=1}^{m} \{ \mu_{R_i}(b, c) \}.$$
But \( \max(0, x) + \max(0, y) \geq \max(0, x + y) \). Hence,

\[
\mu_R(a, c) \geq \max \left( 0, \frac{\lambda}{m} \sum_{i=1}^{m} \mu_{R_i}(a, b) + \frac{\lambda}{m} \sum_{i=1}^{m} \mu_{R_i}(b, c) - \lambda \right) \\
+ (1 - \lambda) \max \left( 0, \min_{i=1}^{m} \{ \mu_{R_i}(a, b) \} + \min_{i=1}^{m} \{ \mu_{R_i}(b, c) \} - 1 \right)
\]

\[
= \max \left( 0, \frac{\lambda}{m} \sum_{i=1}^{m} \mu_{R_i}(a, b) + \frac{\lambda}{m} \sum_{i=1}^{m} \mu_{R_i}(b, c) - \lambda \right) \\
+ \max \left( 0, (1 - \lambda) \min_{i=1}^{m} \{ \mu_{R_i}(a, b) \} \right) \\
+ (1 - \lambda) \min_{i=1}^{m} \{ \mu_{R_i}(b, c) \} - (1 - \lambda)
\]

\[
\geq \max \left( 0, \frac{\lambda}{m} \sum_{i=1}^{m} \mu_{R_i}(a, b) + (1 - \lambda) \min_{i=1}^{m} \{ \mu_{R_i}(a, b) \} \right) \\
+ \frac{\lambda}{m} \sum_{i=1}^{m} \mu_{R_i}(b, c) + (1 - \lambda) \min_{i=1}^{m} \{ \mu_R(b, c) \} - 1
\]

\[
= \max(0, \mu_R(a, b) + \mu_R(b, c) - 1).
\]

Therefore, \( R \) is \( \text{max-}\Delta \) transitive and according to Definition 2, \( R \) is a likeness relation.

**Theorem 2:** The aggregated relation \( R \) by MinAvg operator is a fuzzy preorder relation iff the private relations \( R_i \) are reflexive and max-min transitive for \( \forall i = 1, \ldots, m \).

**Proof:** Let \( R \) be a fuzzy relation with a membership function (11) and \( R_i, i = 1, \ldots, m \) are reflexive (1) and transitive (3) ones. Then as it is proved in Theorem 1, \( R \) is reflexive and max-\( \Delta \) transitive. Therefore, according to Definition 7, \( R \) is a fuzzy preorder relation.
Gamma Operator

The operator is defined by the membership function

\[ \mu_R(a, b) = \left[ \prod_{i=1}^m \mu_{R_i}(a, b) \right]^{1-\gamma} \left[ 1 - \prod_{i=1}^m \left( 1 - \mu_{R_i}(a, b) \right) \right]^{\gamma}, \]

\[ 0 \leq \gamma < 1. \quad (12) \]

This operator represents more accurately human decision-making with the help of a parameter \( \gamma \), which denotes the degree of compensation and can be set from 0 to 1, representing no compensation at all, or complete compensation. Thus, this operator covers the entire range of human aggregation. In most practical implementations (Von Altrock, 1997) \( \gamma \) takes values between 0.1 and 0.4. The Gamma operator is pointwise injective (except at zero and one), continuous, monotonous, and commutative (Zimmermann & Zysno, 1980, 1983; Zimmermann, 1993). It also satisfies the DeMorgan laws and is in accordance with the truth tables of dual logic. The parameter \( \gamma \) indicates where the actual operator is located between the logical “and” and “or.”

The following theorems point out of the operator’s properties. Let us introduce the denotations (for simplicity):

\[ \prod_{i=1}^m \mu_{R_i}(a, b) = A, \quad \prod_{i=1}^m \left( 1 - \mu_{R_i}(a, b) \right) = B \rightarrow \mu_R(a, b) = A^{1-\gamma} (1 - B)^\gamma \]

\[ \prod_{i=1}^m \mu_{R_i}(b, c) = C, \quad \prod_{i=1}^m \left( 1 - \mu_{R_i}(b, c) \right) = D \rightarrow \mu_R(b, c) = C^{1-\gamma} (1 - D)^\gamma \]

\[ \prod_{i=1}^m \mu_{R_i}(a, c) = N, \quad \prod_{i=1}^m \left( 1 - \mu_{R_i}(a, c) \right) = M \rightarrow \mu_R(a, c) = N^{1-\gamma} (1 - M)^\gamma. \]
It is obvious, that \( A + B < 1, C + D < 1, N + M < 1 \) and
\[
0 < A < A^{1-\gamma}(1 - B) < (1 - B) < 1;
\]
\[
0 < C < C^{1-\gamma}(1 - D) < (1 - D) < 1;
\]
\[
0 < N < N^{1-\gamma}(1 - M) < (1 - M) < 1.
\]

**Theorem 3:** The aggregated relation \( R \) by Gamma operator is a likeness relation for \( 0 < \gamma < 0.5 \) iff the private relations \( R_i \), \( i = 1, \ldots, m \) are similarity relations.

**Proof:** 1) If \( R_i \), \( i = 1, \ldots, m \) are reflexive, then \( R \) is reflexive as well, i.e., if \( \mu_{R_i}(a, a) = 1 \) for \( \forall a \in A \), then
\[
\mu_R(a, a) = \left[ \prod_{i=1}^{m} \mu_{R_i}(a, a) \right]^{1-\gamma} \left[ 1 - \prod_{i=1}^{m} (1 - \mu_{R_i}(a, a)) \right]^{\gamma} = 1. 
\]

2) If \( R_i \), \( i = 1, \ldots, m \) are symmetrical, then \( R \) is symmetrical as well, i.e., if \( \mu_{R_i}(a, b) = \mu_{R_i}(b, a) \) for \( \forall i = 1, \ldots, m \), then \( \mu_R(a, b) = \mu_R(b, a) \), because
\[
\mu_R(a, b) = \left[ \prod_{i=1}^{m} \mu_{R_i}(a, b) \right]^{1-\gamma} \left[ 1 - \prod_{i=1}^{m} (1 - \mu_{R_i}(a, b)) \right]^{\gamma} = \mu_R(b, a).
\]

3) If \( R_i \) are max-min transitive (3) for \( i = 1, \ldots, m \), one can prove that \( R \) is max-\( \Delta \) transitive.

4) For \( \gamma \in [0, 0.5] \), i.e., the following inequality (19) is succeeded using the notations (13), (14), (15):
\[
N^{1-\gamma}(1 - M) > \max(0, A^{1-\gamma}(1 - B) + C^{1-\gamma}(1 - D) - 1).
\]
Let $\mu_R(a, c) \triangleright \min(\mu_R(a, b), \mu_R(b, c))$ for $\forall i = 1, \ldots, m$. Then,
\[
N = \prod_{i=1}^{m} \mu_R(a, c) \triangleright \prod_{i=1}^{m} \min(\mu_R(a, b), \mu_R(b, c))
= \frac{AC}{\prod_{i=1}^{m} \max(\mu_R(a, b), \mu_R(b, c))}
> \prod_{i=1}^{m} \mu_R(a, b) \mu_R(b, c) = AC.
\]

Therefore, $N \triangleright AC \triangleright \max(0, A + C - 1)$, i.e., $R$ is max-$\Delta$ transitive for $\gamma = 0$.

a) Let $N \triangleright A \to M \triangleleft B \to 1 - M \triangleright 1 - B$. Then,
\[
\mu_R(a, c) = N^{1-\gamma}(1 - M)^{\gamma} > A^{1-\gamma}(1 - B)^{\gamma}
> A^{1-\gamma}(1 - B)^{\gamma} C^{1-\gamma}(1 - D)^{\gamma}
> \max(0, A^{1-\gamma}(1 - B)^{\gamma} + C^{1-\gamma}(1 - D)^{\gamma} - 1)
= \max(0, \mu_R(a, b) + \mu_R(b, c) - 1),
\]
i.e., $R$ is max-$\Delta$ transitive for $\gamma \in [0, 1]$ in this case.

b) Let $N \triangleleft A, N \triangleleft C \to M \triangleright B, M \triangleright D \to 1 - M \triangleleft 1 - B, 1 - M \triangleleft 1 - D$

Consider two cases:

b1) $1 - M \triangleright (1 - B)(1 - D)$. Then,
\[
(1 - M)^{\gamma} > (1 - B)^{\gamma} (1 - D)^{\gamma}
\text{and from (20) } N^{1-\gamma} > A^{1-\gamma} C^{1-\gamma}.
\]
It follows that
\[
\mu_R(a, c) = N^{1-\gamma}(1 - M)^{\gamma} > A^{1-\gamma}(1 - B)^{\gamma} C^{1-\gamma}(1 - D)^{\gamma}
> \max(0, A^{1-\gamma}(1 - B)^{\gamma} + C^{1-\gamma}(1 - D)^{\gamma} - 1)
= \max(0, \mu_R(a, b) + \mu_R(b, c) - 1),
\]
i.e., $R$ is max-$\Delta$ transitive for $\gamma \in [0, 1]$ in this case.
b2) \(1 - M \leq (1 - B)(1 - D)\). Then the inequality
\[
N^{1-\gamma} (1 - M)^{\gamma} > A^{1-\gamma} (1 - B)^{\gamma} C^{1-\gamma} (1 - D)^{\gamma}
\]  
(21)

and (19) are not performed for \(\gamma = 1\). To be succeeded (21) for \(\gamma \neq 1\) one has
\[
\left[\frac{N}{AC}\right]^{1-\gamma} > \left[\frac{(1 - B)(1 - D)}{(1 - M)}\right]^\gamma.
\]  
(22)

But \(N/AC > 1\) and \((1 - B)(1 - D)/(1 - M) > 1\). Therefore, (21) cannot be performed for arbitrary \(0 < A, B, C, D, N, M < 1\) and \(0.5 > \gamma > 1\).

Let \(\gamma = 0.5\) and (19) can be rewritten as
\[
\sqrt{N(1 - M)} > \max(0, \sqrt{A(1 - B)} + \sqrt{C(1 - D)} - 1).
\]

Then, if \(\sqrt{A(1 - B)} + \sqrt{C(1 - D)} < 1\), it follows that (19) is performed.

Let \(\sqrt{A(1 - B)} + \sqrt{C(1 - D)} > 1\). From (20) one has
\[
N \geq \frac{AC}{\prod_{i=1}^{m} \max_i (\mu_{R_i}(a, b), \mu_{R_i}(b, c))}.
\]  
(23)

It is obvious that if
\[
S = \frac{\sqrt{A(1 - B)} + \sqrt{C(1 - D)} - 1}{\sqrt{A(1 - B)} \sqrt{C(1 - D)}} < 1,
\]
then
\[
(1 - M) > (1 - B)(1 - D)S^2 \prod_{i=1}^{m} \max_i (\mu_{R_i}(a, b), \mu_{R_i}(b, c)).
\]  
(24)

Therefore, (21) follows from (23), (24), and (19) is succeeded for \(\gamma = 0.5\). The inequality (19) is performed for \(0 < \gamma < 0.5\) as well, because the function \(f(\gamma) = N^{1-\gamma} (1 - M)^{\gamma} - A^{1-\gamma} (1 - B)^{\gamma} - C^{1-\gamma} (1 - D)^{\gamma} - 1\) decrease, when \(\gamma\) increase from 0 to 1. According to a), b) (19) is performed for arbitrary \(0 < A, B, C, D, N, M < 1\), if \(0 < \gamma < 0.5\).
Therefore $R$ is a max-$\Delta$ transitive relation for $\gamma \in [0, 0.5]$.

From Definition 2 and 1), 2), 3) it follows that $R$ is a likeness relation for $\gamma \in [0, 0.5]$. □

**Theorem 4:** The aggregated relation $R$ by Gamma operator is a fuzzy preorder for $\gamma \in [0, 0.5]$ iff the private relations $R_i$ are reflexive and max-min transitive ones.

**Proof:** Let $R$ be the fuzzy relation with a membership function (12) and $R_i$ be reflexive and max-min transitive relations for $i = 1, \ldots, m$. Then,

1) According to (17) $\mu_R(a, a) = 1$ if $\mu_{R_i}(a, a) = 1$ for $\forall a \in A$ and $i = 1, \ldots, m$.

2) $\mu_R(a, c) > \max(0, \mu_R(a, b) + \mu_R(b, c) - 1)$ for $\forall a, b, c \in A$ and $0 < \gamma < 0.5$ as it is proved in Theorem 3.

Therefore from Definition 7 and 1), 2) it follows that $R$ is a fuzzy preorder relation. □

**CLUSTERING AND ORDERING ACCORDING TO THE AGGREGATED RELATION**

The final judgments are represented by fuzzy sets after the premise aggregation stage. Then the second stage can be decided as follows (Roubens, 1995; Zimmermann, 1993).

A similarity relation of a finite number of alternatives can be represented by a similarity tree, similar to a dendogram. In this tree each level represents an $\alpha$-cut ($\alpha$-level set) of the similarity relation. The set of elements on a specific $\alpha$-level can be considered as similarity classes (fuzzy clusters) of $\alpha$-level.

Let $R_\alpha$ be the $\alpha$-cut of a likeness relation. Let $\Pi_\alpha$ be the partition induced on $A$ by $R_\alpha$. If $\alpha' > \alpha$, then $\Pi_{\alpha'}$ is a partition of $\Pi_\alpha$. A nested sequence of partitions $\Pi_{a_1}, \Pi_{a_2}, \ldots, \Pi_{a_k}$ for $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ may be represented diagrammatically in the form of a partition tree.

Then the likeness relation can be interpreted in terms of fuzzy likeness clusters $R(a_j)$ in the following way: $\mu_{R(a_j)}(a_i) = \mu_R(a_i, a_j)$ is the membership degree of $a_i$ to the fuzzy cluster $R(a_j)$.

To decide a ranking or choice problem after the premise aggregation with considered fuzzy logic operators can be made the following.
The aggregated degree to which “a is not worse then b” obtained at the end of the aggregation process does not always present any ordering property, having in mind the max-min transitivity. Then the aggregated graph (Zimmermann, 1993) cannot be interpreted in terms of ranking and choice. But the relation $R$ may be transformed in such a way to obtain a modified max-min transitive relation. For example, every $\alpha$-cut of $R$ is transitive in the crisp sense and corresponds to a quasiorder, which can be represented by Hasse diagram. Other transitive relations close to a given preference relation are given in Roubens (1995).

The main contribution (Dubois & Prade, 1980) of the notion of fuzzy preorder is to propose membership degrees of a preference relation without violating the choice problem itself. Any antisymmetric fuzzy preorder relation $R$ is a fuzzy partial ordering. It is possible to represent $R$ as a triangular matrice or a Hasse diagram. Owing to perfect antisymmetry and transitivity, this graph has no cycle. An $\alpha$-cut of $R$ is a nonfuzzy partial ordering.

**NUMERICAL EXAMPLE**

This example is based on the groundwater management system described in Czyżak and Slovínški (1997). There are some groundwater resources on a circular island. The water is accumulated in a single layer and has to be used for water supply of a distribution center located in the middle of the island. The management problem consists in finding the location of the wells and corresponding pumping rates, satisfying the hydrodynamic model. Moreover, the solution should reflect the decision-maker’s preferences and should yield the best compromise among several criteria.

Set $A$ of 13 potential locations of water supply wells is considered. They are denoted as $a_1, \ldots, a_{13}$. Three criteria are taken into account to evaluate the well’s quality:

- $K_1$ — the total pumping rates over the aquifer;
- $K_2$ — operating costs including water pumping and transport costs;
- $K_3$ — risk interpreted as the possible fluctuation in water quantity at the exploitation stage.

The first criterion is maximized while two others are minimized.
Due to the fact that the groundwater management problem requires long-time planning, Table 1 contains uncertain data. It assumes that the experts can estimate some ranges of variation of these data, together with possible distribution within them using past experience. So the alternatives’ evaluations from the set $A$ are triangular fuzzy numbers (t.f.n.), where $\tilde{a}_{ij}$, $i = 1, \ldots, 13$, $j = 1, 2, 3$ is the estimation of the alternative $a_i$ by the $j$th criterion. The t.f.n. $\tilde{a}_{ij}$ is the triple $(a_{ij}^1, a_{ij}^2, a_{ij}^3)$, where $a_{ij}^2$ is the “most possible” value and $a_{ij}^1, a_{ij}^3$ are nonnegative left and right “diversion” of $a_{ij}$.

The fuzzy performances of alternatives are defined as follows:

---the values given in Table 1 are equal to $a_{ij}^2$;
---the left and right spreads depend on criteria in the following way:

- for $K_1$, are constants for each alternative and equal to 2,
- for $K_2$, are equal to 5% of $a_{ij}^2$ for each alternative,
- for $K_3$, are equal to 10% of $a_{ij}^2$ for each alternative plus 0.2.

The weights (importances) of each criterion, assigned by the decision-maker are

\[ c_1 = 0.21 \quad c_2 = 0.42 \quad c_3 = 0.37. \]
The geometrical index $F(\tilde{a}_{ij})$ is computed for each fuzzy number $\tilde{a}_{ij}$ by the expressions given in Peneva and Popchev (1996b, 1998a). Let $\max_i F(\tilde{a}_{ij}) = F_{\text{max}}^j$, $\min_i F(\tilde{a}_{ij}) = F_{\text{min}}^j$, and $F_{\text{max}}^j - F_{\text{min}}^j = F^j$. Then the following preference relation $R_j$ for criterion $K_j$ is defined based on the dependence (Peneva & Popchev, 1996b, 1998a), which gives its membership function:

$$
\mu_{R_j}(\tilde{a}_{ij}, \tilde{a}_{kj}) = \begin{cases} 
1 & \text{if } i = k \\
0.5 + \frac{F(\tilde{a}_{ij}) - F(\tilde{a}_{kj})}{2F^j} & \text{if } i \neq k.
\end{cases}
$$

It is easy to verify that $\mu_{R_j}(\tilde{a}_{ij}, \tilde{a}_{kj}) + \mu_{R_j}(\tilde{a}_{kj}, \tilde{a}_{ij}) = 1$, i.e., $R_j$ is a reciprocal relation. Besides $R_j$ is max-min transitive relation, because if $a, b, c$ are the geometrical indices of three alternatives by one criterion, then taking into account that

$$
\min(a, b) = \frac{1}{2}(a + b - |a - b|)
$$

one has

$$
\min(\mu_{R_j}(a, b), \mu_{R_j}(b, c))
$$

$$
= \frac{1}{2} \left(0.5 + \frac{a - b}{2F^j} + 0.5 + \frac{b - c}{2F^j} \right)
$$

$$
- \left|0.5 + \frac{a - b}{2F^j} - 0.5 - \frac{b - c}{2F^j} \right|
$$

$$
= \frac{1}{2} \left(1 + \frac{a - c}{2F^j} - \left|\frac{a - 2b + c}{2F^j}\right|\right) < \frac{1}{2} \left(1 + \frac{a - c}{F^j}\right) = \mu_{R_j}(a, c).
$$

Therefore $R$ is fuzzy total ordering (Yuan, 1991).

The so-defined relations $R_j$, $j = 1, 2, 3$ provide the possibility to get three matrices with the membership degrees of the couples of alternatives to these relations. The union relation $R$ computed by means of Weighted Mean and MaxMin operators are presented in Peneva and Popchev (1998b). The quasiorders represented by Hasse diagrams for 0.5-cuts of aggregated relations are built as well. The aggregated relation $R$ with the membership function (11) of the MinAvg operator for
Table 2. Aggregated relation by MinAvg operator

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
<th>$a_7$</th>
<th>$a_8$</th>
<th>$a_9$</th>
<th>$a_{10}$</th>
<th>$a_{11}$</th>
<th>$a_{12}$</th>
<th>$a_{13}$</th>
</tr>
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<td>0.136</td>
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<td>0.146</td>
<td>0.150</td>
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<td>0.132</td>
<td>0.134</td>
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<td>0.146</td>
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<td>0.118</td>
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<tr>
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<td>0.126</td>
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<td>0.074</td>
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<td>0.067</td>
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<td>0.126</td>
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<td>0.052</td>
<td>0.045</td>
<td>0.050</td>
<td>0.043</td>
<td>0.044</td>
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<td>$a_6$</td>
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<td>0.126</td>
<td>0.126</td>
<td>0.128</td>
<td>0.138</td>
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<td>0.141</td>
<td>0.147</td>
<td>0.148</td>
<td>0.124</td>
<td>0.116</td>
<td>0.116</td>
<td>0.118</td>
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<td>0.121</td>
<td>0.121</td>
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<td>0.131</td>
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<td>0.142</td>
<td>0.148</td>
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<td>0.122</td>
<td>0.122</td>
<td>0.125</td>
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<tr>
<td>$a_8$</td>
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<td>0.115</td>
<td>0.115</td>
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<td>0.136</td>
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<td>0.133</td>
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<td>0.129</td>
<td>0.129</td>
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<td>0.141</td>
<td>0.139</td>
<td>0.144</td>
<td>0.144</td>
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<td>0.144</td>
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<td>0.120</td>
<td>0.120</td>
<td>0.124</td>
<td>0.134</td>
<td>0.130</td>
<td>0.135</td>
<td>0.141</td>
<td>0.147</td>
<td>0.123</td>
<td>0.127</td>
<td>1</td>
<td>0.147</td>
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<tr>
<td>$a_{13}$</td>
<td>0.109</td>
<td>0.109</td>
<td>0.109</td>
<td>0.111</td>
<td>0.121</td>
<td>0.118</td>
<td>0.124</td>
<td>0.130</td>
<td>0.135</td>
<td>0.111</td>
<td>0.115</td>
<td>0.124</td>
<td>1</td>
</tr>
</tbody>
</table>

$\lambda = 0.5$ is written in Table 2. The weight coefficients $c_i$, $i = 1, 2, 3$ (25) are taken into account for the computations, i.e., $\mu_{R_i}(a, b)$ is multiplied with $c_i$ for $\forall a, b$ and $i = 1, 2, 3$. As $R_i$, $i = 1, 2, 3$ are reflexive and max-min transitive, then $R$ is a fuzzy preorder according to Theorem 2. Let $R$ be the antisymmetrized $R$, i.e., if $\mu_R(a, b) > \mu_R(b, a)$, then $\mu_R(a, b) = \mu_R(b, a)$ and $\mu_R(b, a) = 0$. Hence, $R$ is a fuzzy partial order from Definition 6 with membership degrees presented in Table 3. It is obvious that $R'$ is a fuzzy linear ordering taking into account Definition 8.

The matrice of the aggregated relation $R$ computed by means of the Gamma operator with membership function (12) and $\gamma = 0.5$ is presented in Table 4. The weight coefficients $c_i$, $i = 1, 2, 3$ (25) are used in the computations as well. As $R_i$, $i = 1, 2, 3$ are reflexive and max-min transitive, then $R$ is a fuzzy partial preorder taking into account Theorem 4.

Let $R$ be the antisymmetrized $R$ obtained as above. Therefore, $R$ is a fuzzy partial order relation (Table 5) according to Definition 6. It is obvious from Table 5 that the set of undominated alternatives $\{a_{11}, a_{10}, a_{12}, a_1\}$ is a fuzzy linear ordering. The set of undominating alternatives consists of $a_9, a_5$.

Resulting orders obtained by the 4th fuzzy logic operators and the methods ELECTRE III, multiattribute utility function (MUF), UTA, and Concordance-discordance approach (CDA), presented in Czyżak and Sloviński (1997) are given in the comparative Table 6.
Table 3. Matrice of the antisymmetrical relation $R$

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_{10}$</th>
<th>$a_{11}$</th>
<th>$a_{12}$</th>
<th>$a_2$</th>
<th>$a_6$</th>
<th>$a_7$</th>
<th>$a_8$</th>
<th>$a_9$</th>
<th>$a_{13}$</th>
<th>$a_3$</th>
<th>$a_4$</th>
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<tr>
<td>$a_1$</td>
<td>0.138</td>
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<td>0.132</td>
<td>0.136</td>
<td>0.146</td>
<td>0.150</td>
<td>0.148</td>
<td>0.142</td>
<td>0.134</td>
<td>0.136</td>
<td>0.138</td>
<td>0.148</td>
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<tr>
<td>$a_{10}$</td>
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<td>0.133</td>
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<td>0.147</td>
<td>0.140</td>
<td>0.160</td>
<td>0.133</td>
<td>0.136</td>
<td>0.146</td>
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</tr>
<tr>
<td>$a_{11}$</td>
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<td>0</td>
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<td>0.144</td>
<td>0.129</td>
<td>0.139</td>
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<td>0.156</td>
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<td>0.141</td>
</tr>
<tr>
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<td>0.130</td>
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<td>0.147</td>
<td>0.147</td>
<td>0.120</td>
<td>0.124</td>
<td>0.134</td>
</tr>
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<td>0.134</td>
<td>0.130</td>
<td>0.118</td>
<td>0.113</td>
<td>0.136</td>
<td>0.138</td>
<td>0.148</td>
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<tr>
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<td>1</td>
<td>0.141</td>
<td>0.147</td>
<td>0.148</td>
<td>0.118</td>
<td>0.126</td>
<td>0.128</td>
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<tr>
<td>$a_7$</td>
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<td>0</td>
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<td>0</td>
<td>1</td>
<td>0.142</td>
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<td>0</td>
<td>1</td>
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<td>0.133</td>
<td>0.115</td>
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<tr>
<td>$a_9$</td>
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<td>0</td>
<td>0</td>
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<td>0.110</td>
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<td>0.122</td>
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</tr>
<tr>
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<td>1</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Table 6 is separated into three parts horizontally: the top part contains “the good” alternatives and the down part “the poor” ones in the respective orders.

Obviously, the alternative $a_{11}$ is selected as “a good” one and $a_5$ is selected as “a poor” one by each method. The aggregated relation obtained by Weighted Mean of the private relations gives the same order as the first four methods in Table 6, which shows that this method

Table 4. Aggregated relation by Gamma operator

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
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<th>$a_3$</th>
<th>$a_4$</th>
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<td>0.045</td>
<td>0.042</td>
<td>0.041</td>
<td>0.039</td>
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<td></td>
</tr>
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Table 5. Matrix of the antisymmetrical relation $R'$

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does not lead to surprising results. The aggregated relation with MaxMin operator gives an order, which is quite different in its middle part from the others. This is explained by the fact that all other methods average the membership degrees of the private relations, using weight coefficients, while this method can prefer the alternative with maximum (minimum) membership degrees, according to the selected value of the optimism index. So this method differs from others conceptually, and at

Table 6. Comparison of the final rankings obtained by different methods

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<tr>
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<th>MUF</th>
<th>UTA</th>
<th>CDA</th>
<th>W.Mean</th>
<th>MaxMin</th>
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<th>Gamma</th>
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</table>
the same time it gives possibility to vary the index of optimism to get an order satisfying the decision-maker.

The MinAvg operator gives the possibility to vary with the weights of the average membership degrees and the minimum degrees of the private relations by means of parameter $\lambda$. In this case, $\lambda = 0.5$ shows that the two parts obtained equal weights and, therefore, this order is between the ones computed by the Min operator and by the Weighted Mean one.

The choice of the parameter $\gamma = 0.5$ for the gamma operator aims to obtain an average between no compensation at all for $\gamma = 0$ and complete compensation for $\gamma = 1$. As complete compensation is practically impossible and $\gamma$ takes value between 0.1 and 0.4 in most practical implementations (Von Altrock, 1997) then this choice is justified. The result obtains by the Gamma operator is an average between the Max and Min operators in this case.

CONCLUDING REMARKS

The multicriteria decision-making problem considered consists in constructing an evaluation procedure to compare the set of alternatives according to whole set of fuzzy criteria given as fuzzy relations. The aggregation of the private fuzzy relations is done with the help of fuzzy logic operators in such a way that the union relation be the fuzzy one giving a possibility to decide a ranking, choice, or cluster problem. Some special fuzzy relations as similarity (likeness) and order relation are chosen for this purpose.

In order to describe a variety of phenomena in decision situations, the fuzzy operators with a compensatory parameters are chosen. Four operator families in parameter form are included in this research: Weighted Mean, MinMax, MinAvg, and Gamma operators. They are relevant for practical applications (Von Altrock, 1997). The proved properties of these operators, depending on the properties of the private relations, are required to decide the ranking, choice, or cluster problems.

REFERENCES


