Level set method for the inverse elliptic problem in nonlinear electromagnetism

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Abstract

An inverse problem of inhomogeneity identification inside a nonlinear magnetic material from the local measurements of the magnetic induction is investigated. The representation of the shape of the inhomogeneity and its evolution during an iterative reconstruction process is achieved by the level set method. The reconstruction is realized by the minimization of a cost function using the steepest descent method. The gradient directions are evaluated using the sensitivity equation and the adjoint variable method. Simulations has been performed showing the robustness of the algorithm and its ability to reconstruct single inhomogeneities, convex and non-convex, as well as multiple inhomogeneities.

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1. Introduction

We model physical processes governed by quasi-linear equations where the properties of the material are described by piecewise smooth functions.

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Such description includes electromagnetic processes with nonlinear magnetic materials. More specifically, electromagnetic non-destructive evaluation aim at e.g. the localization of crack or inhomogeneities in the steel production process and these impurities can be described by piecewise smooth functions.

Non-destructive evaluation has gained a lot of interest during the last decade, since it is becoming an important tool in different research topics. Eddy current testing has been widely used for defects characterization in materials [1]. We aim at developing a numerical technique capable of proper characterization of defects in magnetically non-linear materials. Although the linear model used in the EIT is sometimes sufficient for the acceptable defect detection, the non-linear model introduced in the following section is more robust and able to reconstruct the shape of the crack or of the inhomogeneity even in the cases when the linear model is not. The superiority of the non-linear model is evident especially in the cases when measurements are not given over the whole workpiece but only at the part of it, mostly around the boundary. In that cases the linear model is not accurate enough to reconstruct the crack inside the material from the measurements near the boundary. The non-linear model reliably detects this air-gaps.

The paper is organized as follows. In Section 2 we describe the model and its mathematical description. In Section 3 we describe the level set method that is used in the representation of the unknown shapes. Then we analyze the direct problem in Section 4 and we define the inverse problem in Section 5. The inverse problem is solved by means of minimization of a cost function. Here we derive the sensitivity equation and the adjoint problem that together gives an explicit expression for the derivative of the cost function.

Section 6 is devoted to numerical simulations. We describe the algorithm and we test it for several scenarios including the determination of convex, non-convex and multiple cracks. Finally in Section 7 we draw several conclusions.

2. Model description

Suppose that the bounded domain $\Omega \in \mathbb{R}^2$ with $C^1$ boundary contains a subdomain $D \in \mathbb{R}^2$, possibly not connected. Assume that the function $\nu : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined piecewise by

$$
\nu(x, s) = \begin{cases} 
\nu_1(s) & \text{for } x \in D, \\
\nu_2(s) & \text{for } x \in \Omega \setminus D
\end{cases}
$$

(1)
where \( \nu_1, \nu_2 \) are nonlinear functions.

The direct problem is governed by the following quasi-linear partial differential equation

\[
\nabla \cdot (\nu(x, |\nabla A|^2)\nabla A) = J \quad \text{in } \Omega, \quad A = 0 \quad \text{on } \partial \Omega \tag{2}
\]

with \( J \) being a suitable function. Since \( \nu \) is discontinuous across the boundary \( \partial D \) we work with weak formulation of the problem.

The above mathematical model covers many physical and industrial applications. For a linear choice of \( \nu_1 \) and \( \nu_2 \), the problem (2) describes the EIT problem [2, 3, 4, 5, 6]. For a nonlinear \( \nu_2 \) the model describes the following electromagnetic problem.

The workpiece made of hard magnetic steel has a crack inside it. The crack is filled with air. By a function \( J \) one can model the current density that induces a magnetic field. In this setting, two materials occur, the air and the steel, with respective permeabilities \( \mu_1, \mu_2 \). The magnetic permeability \( \mu(x) \) as a space variable function defined in the whole \( \Omega \) has thus value \( \mu_1 \) inside \( D \), and value \( \mu_2 \) in \( \Omega \setminus D \). It has therefore a discontinuity at the boundary of \( D \). We are interested in the static distribution of the magnetic field under the induced current density \( J \).

The electromagnetic process in the above example is governed by the static magnetic vector potential formulation of the Maxwell equations with
the \(xy\) planar symmetry. In this setting, the only nonzero component of the vector potential \(A\) pointing out of the \(xy\) plane is denoted by \(A\). Due to the planar symmetry, the magnetic induction \(B\) has two nonzero components and can be expressed as
\[
B = \begin{pmatrix}
\frac{\partial A}{\partial y} & -\frac{\partial A}{\partial x} & 0
\end{pmatrix}^T.
\] (3)

For the strength of the magnetic induction it holds that \(|B| = |\nabla A|\).

We take into account that the magnetic material of which the workpiece is made, is non-linear. Therefore, the magnetic permeability \(\mu_2\) is dependent on the strength of the magnetic induction and we have \(\mu_2 = \mu_2(|B|^2) = \mu_2(|\nabla A|^2)\). This makes a significant difference between our setting and the EIT setting, where \(\mu_2\) is a constant.

Let us now set \(\nu_1\) and \(\nu_2\) from (1) to be \(\nu_1(s) = \mu_0^{-1}\) and \(\nu_2(s) = \mu_2^{-1}(s)\). Assume that the magnetic potential \(A\) vanishes on the boundary.

After all these considerations, the above electromagnetic problem is governed by (2).

Typical behavior of nonlinear permeability \(\mu_2(s)\) is depicted in Figure 2. It first monotonically increases to some point \(s_{\text{max}}\) where it reaches the maximum value and then it monotonically decreases. The real experiments we have in mind guarantee that the values of \(|\nabla A|^2\) entering \(\mu_2\) as parameter \(s\) will always be greater than some positive \(s_0 > s_{\text{max}}\). Thus the non-monotone \(\mu_2(s)\) can be replaced by its monotone approximation \(\bar{\mu}_2(s)\). This approximation thus monotonically decreases on the whole \((0, \infty)\), its limit at zero is a positive number and its limit at infinity is \(\mu_0\), the permeability of the air. From now on we do not work with the notation \(\bar{\mu}_2(s)\) but instead we assume from the beginning that the permeability \(\mu_2(s)\) is monotone.

Let us define the properties for the inverse of \(\mu_2(s)\).

**A1** The function \(s \to \nu_2(s)\) is non-decreasing,

**A2** \(\lim_{s \to 0} \nu_2(s) = \nu_{\min} > 0\),

**A3** \(\lim_{s \to \infty} \nu_2(s) = \nu_{\max}\) and define \(\nu(s) = \nu_{\max}\) for \(s = \infty\).

The function \(\nu_1\) from the example (1)–(2) is equal to the constant \(\mu_0^{-1}\) and thus naturally satisfies the conditions **A1–A3**. For the considerations in the following sections it is however not crucial that \(\nu_1\) is a constant. We therefore let \(\nu_1\) be any function satisfying **A1–A3**.

We further put another condition on \(\nu_1, \nu_2\). We assume that
Figure 2: Typical behavior of nonlinear permeability.

\[ \nu_1, \nu_2 \text{ are differentiable with well-defined derivatives } \nu'_1, \nu'_2 \text{ satisfying} \]
\[ \nu'_\text{min} \leq \nu'_i \leq \nu'_\text{max}, \quad i = 1, 2. \tag{4} \]

3. Level set representation

We work in a two dimensional setting. A point in the space is either denoted by \( x \) or by \( (x, y) \).

For the description of the geometry we use a method of the level set type. For an overview we refer to [7, 8, 9] and the references therein. Recent results [10, 11, 12] show the efficiency of the level set method in the design, shape and topology optimization problems. The level set methods for ill-posed distributed parameter estimation problems have been elaborated in [13]. The pioneering work about the level set approach for inverse problems involving obstacles is [14]. This technique has been used in the determination of electromagnetic inclusion [15]; however, magnetic materials have been considered linear in [15]. The basic idea of our work follows an approach which was first proposed in [4].

We represent the boundary of \( D \) as a zero level set of a function \( \phi \), see Figure 3. We take \( \phi \) in such a manner that \( D = \{ x \in \Omega \mid \phi(x) > 0 \} \) and \( \Omega \setminus D = \{ x \in \Omega \mid \phi(x) < 0 \} \). We define the Heaviside function \( H \) in a classical
way by

\[ H(\phi) = \begin{cases} 
0, & \phi < 0 \\
1/2, & \phi = 0 \\
1, & \phi > 0.
\end{cases} \]

The derivative of the Heaviside function \( H'(\phi) \) is the Dirac delta function.

In computations, to achieve numerical robustness, the use of a smeared out Heaviside function is recommended e.g. by Osher and Fedkiw [7]. We use the following smooth approximation of the Heaviside function

\[ H_k(\phi) = \frac{1}{2} + \frac{1}{\pi} \arctan(k\phi), \tag{5} \]

with \( k \) being a parameter influencing how steep is the approximation around zero. For \( k \to \infty \), \( H_k(\phi) \) converges pointwise to \( H(\phi) \). The derivative of \( H_k(\phi) \) is

\[ H'_k(\phi) = \frac{k}{\pi(1 + k^2 \phi^2)}. \]

Finally we rewrite (1) and we use the level set representation for \( \nu(x, s) \)

\[ \nu(x, s) = H_k(\phi(x))\nu_1(s) + [1 - H_k(\phi(x))]\nu_2(s). \tag{6} \]

4. Direct problem

Using the level set formulation we can define the solution of the direct problem. We will further work with weak solutions. Given the level set function \( \phi \), uniquely determining domain \( D \), the solution of the direct problem is such \( A \in W^{1,2}_0(\Omega) \) for which the identity

\[ \int_{\Omega} \left( H_k(\phi)\nu_1(|\nabla A|^2) + [1 - H_k(\phi)]\nu_2(|\nabla A|^2) \right) \nabla A \cdot \nabla \varphi \, dx = \int_{\Omega} J \varphi \, dx \]  \tag{7} \]

is fulfilled for all \( \varphi \in W^{1,2}_0(\Omega) \). The previous PDE defining the direct problem is a quasi-linear elliptic equation of the second order. The existence and uniqueness of the solutions to such equation has been thoroughly studied in [16, 17]. The coefficients of (7), however, have to satisfy several smoothness properties. We work with smooth \( H_k \) and assuming A1–A4, we meet those smoothness properties. Therefore, using the results from [16, 17] we can state that there exists a unique solution to (7).
Figure 3: Level set function $\phi$. The intersection of the surface $z = \phi(x, y)$ with the plane $z = 0$ forms the circle representing the boundary of $D$.

5. Inverse problem

We are interested in the reconstruction of the domain $D$ for given $\nu_1, \nu_2$ and given measurements $\mathbf{B}$ of the magnetic induction in a specific subset of $\Omega$ denoted by $\Gamma$. For example, if we have measurements $\mathbf{B}$ on the outer boundary of the workpiece, then $\Gamma$ is a one dimensional domain, if we possess the measurements over the whole $\Omega$ or over a subdomain of $\Omega$, then $\Gamma$ is a two dimensional domain. Similar problem in 1D has been studied in [18].

For a given $D$, characterized by $\phi$, one can compute the direct problem (7) to obtain the solution $A(\phi)$. The reconstructed $D$ should be such that $A(\phi)$ corresponds to the measurements as close as possible. To measure the misfit between the measured data $\mathbf{B}$ and the computed induction $\nabla A$ we introduce the following cost functional

$$F(\phi) = \frac{1}{2} \int_{\Gamma} |\nabla A(\phi) - \mathbf{M}|^2 dx.$$ 

We introduced $\mathbf{M} = (-\bar{B}_2, \bar{B}_1)^T$ where $\bar{B}_1$ and $\bar{B}_2$ are the components of $\mathbf{B}$ so that (3) is satisfied.

We emphasize the dependence of $A$ on $\phi$ because $A$ is the solution to (7) for that specific $\phi$. Since we use the level set representation of the geometry,
The inverse problem is defined as follows: Find the level set function $\phi$ for which the cost function $F(\phi)$ is minimal, i.e. find $\phi_{\text{min}} \in W^{1,2}(\Omega)$ such that

$$\phi_{\text{min}} = \arg \min_{\phi \in W^{1,2}(\Omega)} F(\phi). \quad (8)$$

To this end we employ a gradient-type minimization method to minimize the cost function. For this we need to compute the gradient $DF$ of $F$ with respect to $\phi$.

We introduce the notation for the Gâteaux derivative of an arbitrary function $f(\phi)$ in the direction $h$

$$\delta_h(f) = \delta_h(f(\phi)) = \frac{df(\phi)}{d\phi} h. \quad (10)$$

In order to compute the gradient of $F$ with respect to $\phi$ we first compute the Gâteaux derivative in the direction $h$

$$\delta_h(F) = \lim_{\varepsilon \to \infty} \frac{F(\phi + \varepsilon h) - F(\phi)}{\varepsilon} = \int_{\Gamma} \nabla \delta_h(A) \cdot (\nabla A(\phi) - \bar{M}) dx. \quad (9)$$

We will use a finite element method for the finite dimensional approximation of $\phi$. We choose Lagrange finite elements of the first order. This however means that if the mesh has $N$ vertices, the parameter space will have $N$ degrees of freedom. Thus, for the full gradient of $F$ one needs to compute $\delta_h A$ for $h = \varphi_1, \ldots, \varphi_N$, where $\varphi_i$ are the basis functions of our finite element space. Once we know $\delta_{\varphi_i} F$, the gradient will be expressed as

$$DF = \sum_{i=1}^{N} \delta_{\varphi_i} F \varphi_i. \quad (10)$$

For the computation of the gradient we use the adjoint variable method. A similar approach of an adjoint variable has been used in many applications [19, 20, 21, 22, 23, 24]. We choose this method because of its computational cost reduction in comparison with the conventional method of perturbations or with the method of sensitivity equation. This speed up is caused by the fact that the direct problem is nonlinear and therefore it must be solved iteratively.

To obtain $\delta_h(F)$, we use (9) and thus we formally differentiate the direct problem (7). We get the following sensitivity equation

$$\int_{\Omega} \delta_h[\nu(x, |\nabla A|^2)] \nabla A \cdot \nabla \varphi \ dx + \int_{\Omega} \nu(x, |\nabla A|^2) \nabla \delta_h A \cdot \nabla \varphi \ dx = 0$$

8
We start with the first term. We must still keep in mind that $A$ is dependent on $\phi$.

$$
\delta_h [\nu(x, |\nabla A|^2)] = \delta_h [H_k(\phi)](\nu_1(|\nabla A|^2) - \nu_2(|\nabla A|^2)) + 2H_k(\phi)\nu'_1(|\nabla A|^2) \nabla \delta_h A \cdot \nabla A + 2\left[1 - H_k(\phi)\right]\nu'_2(|\nabla A|^2) \nabla \delta_h A \cdot \nabla A(\phi).
$$

Further we compute

$$
\delta_h [H_k(\phi)] = H'_k(\phi) h.
$$

Finally, putting all particular results together we write down the sensitivity equation

$$
\int_{\Omega} 2\left(H_k(\phi)\nu'_1(|\nabla A|^2) + [1 - H_k(\phi)]\nu'_2(|\nabla A|^2)\right) \nabla \delta_h A \cdot \nabla A \cdot \nabla \varphi \, dx + \int_{\Omega} \left(H_k(\phi)\nu_1(|\nabla A|^2) + [1 - H_k(\phi)]\nu_2(|\nabla A|^2)\right) \nabla \delta_h A \cdot \nabla \varphi \, dx = - \int_{\Omega} H'_k(\phi) h \left(\nu_1(|\nabla A|^2) - \nu_2(|\nabla A|^2)\right) \nabla A \cdot \nabla \varphi \, dx. \tag{12}
$$

To avoid the computation of $\delta_h A$ from (12) for $h = \varphi_i, i = 1, \ldots, N$ we introduce the adjoint variable $b$ being the solution of the following problem

$$
\int_{\Omega} 2\left(H_k(\phi)\nu'_1(|\nabla A|^2) + [1 - H_k(\phi)]\nu'_2(|\nabla A|^2)\right) \nabla b \cdot \nabla A \nabla A \cdot \nabla \psi \, dx + \int_{\Omega} \left(H_k(\phi)\nu_1(|\nabla A|^2) + [1 - H_k(\phi)]\nu_2(|\nabla A|^2)\right) \nabla b \cdot \nabla \psi \, dx = \int_{\Gamma} (\nabla A(\phi) - \mathbf{M}) \cdot \nabla \psi \, dx. \tag{13}
$$

Now, we take special test functions in the weak formulation of the sensitivity equation and of the equation for the adjoint variable. Take $\varphi = b$ in (12) and $\psi = \delta_h A$ in (13). We see that the left-hand sides of (12) and (13) are equal. Hence, we get the equality of the right-hand sides

$$
- \int_{\Omega} H'_k(\phi) h \nu_1(|\nabla A|^2) - \nu_2(|\nabla A|^2) \nabla A \cdot \nabla b \, dx = \int_{\Gamma} (\nabla A(\phi) - \mathbf{M}) \cdot \nabla \delta_h A \, dx.
$$

The adjoint problem was constructed in such a way that the right-hand side of the above expression is exactly the Gâteaux derivative of $F$ with respect to...
to $\phi$ in the direction $h$. Eventually we obtained

$$
\delta_h F = - \int_\Omega H'_k(\phi) \left( \nu_1(|\nabla A|^2) - \nu_2(|\nabla A|^2) \right) \nabla A \cdot \nabla b \, dx,
$$

(14)

where $A$ is the solution to the direct problem (7) and $b$ is the solution of the adjoint problem (13).

Here we see why we need to use a smeared-out Heaviside function. The previous integral in its non-smeared-out form represents an integral over lower-dimensional interface and it is unlikely that any standard numerical approximation based on sampling will give a good approximation to this integral.

Next, we have two possibilities: either we compute $N$ integrals to obtain $\delta \phi_i F$ for $i = 1, \ldots, N$ and then from (10) we obtain $DF$, or, we simply project $H'_k(\phi)[\nu_1(|\nabla A|^2) - \nu_2(|\nabla A|^2)]\nabla A \cdot \nabla b$ onto the finite element space by solving the following linear equation

$$
- \int_\Omega H'_k(\phi)[\nu_1(|\nabla A|^2) - \nu_2(|\nabla A|^2)]\nabla A \cdot \nabla b \phi \, dx = \int_\Omega DF \phi \, dx
$$

(15)

with the unknown $DF$. Simulations confirm that the second choice is faster.

6. Numerical implementation

Throughout this section we consider $\Omega \in \mathbb{R}^2$ to be a square $(-0.5, 0.5) \times (-0.5, 0.5)$. We assume that we possess the measurements over the whole $\Omega$. The direct problem (7) can be seen as an operator equation $G(A) = J$ where $G$ is a mapping $G : A \in W^{1,2}_0(\Omega) \rightarrow G(A) \in W^{1,2}_0(\Omega)$ such that

$$
\int_\Omega \left( H_k(\phi)\nu_1(|\nabla A|^2) + [1 - H_k(\phi)]\nu_2(|\nabla A|^2) \right) \nabla A \cdot \nabla \varphi \, dx = \int_\Omega G(A) \varphi \, dx.
$$

This operator equation is nonlinear and therefore it will be solved for all the numerical examples by the same iterative algorithm. Starting from the initial guess $A_0$, we use the Newton-Raphson algorithm based on the following update

$$
A_{i+1} = A_i - [DG(A_i)]^{-1}(G(A_i) - J).
$$

Notice that for each iteration one linear PDE has to be solved.

The material parameter functions $\nu_1, \nu_2$ are chosen in such a way that they mimic the real setting described in the Introduction. $\nu_1$ is a constant
since it represents the inverse of the magnetic permeability of the air and \( \nu_2 \) is chosen to be

\[
\nu_2(s) = d_1 + \frac{c_1 s^{b_1}}{a_1^{b_1} + s^{b_1}}.
\]

From the graph of \( \nu_2(s) \) in Figure 4 one can see that the Assumptions A1–A4 are satisfied. The concrete values are set to be

\[
a_1 = 0.5, \ b_1 = 4, \ c_1 = 3, \ d_1 = 0.2.
\]

For solving the optimization problem (8) we use an iterative gradient-type method. We choose the steepest descent algorithm. Starting from an initial guess \( \phi_0 \), we update the current approximation by

\[
\phi_{i+1} = \phi_i - \lambda_i DF(\phi_i),
\]

where the direction \( DF(\phi_i) \) is computed from (15) and the step size \( \lambda_i \) is chosen by the following rules:

i) If the cost \( F \) drops, then double the step size.

ii) If the cost \( F \) does not drop, divide the step size by 2 until it does drop.

Since in every step we have the descent direction \( DF \), theoretically, the part ii) should never make a dead loop. However in practice, it sometimes occurs
that the computed gradient is not accurate enough and in fact it is not a descent direction any more. Therefore, we sometimes proceed against the descent direction in order to get to a more stable area.

Initially, all simulations featured oscillations of the zero level set. To stabilize the optimization process we introduce the regularization and we add a Tikhonov stabilizing term. We choose the squared norm of the gradient of the level set function. We use the coefficient $\alpha$ to control the trade-off between the fidelity term and the regularizing term. The resulting cost function reads as

$$F(\phi) = \frac{1}{2} \int_{\Gamma} |\nabla A(\phi) - \bar{M}|^2 dx + \alpha \int_{\Omega} |\nabla \phi|^2 dx.$$  

The expression for the Gâteaux derivative of $F$ with respect to $\phi$ in the direction $h$ will be derived from (14) by adding the corresponding derivative of the regularization term leading to

$$\delta_h F = - \int_{\Omega} H^t_k(\phi) h \left[ \nu_1(|\nabla A|^2) - \nu_2(|\nabla A|^2) \right] \nabla A \cdot \nabla b dx + 2\alpha \int_{\Omega} \nabla h \cdot \nabla \phi dx. \quad (16)$$

The linear problems are solved on the regular triangular mesh with $2 \text{dim}^2$ triangles constructed by splitting of the square into $\text{dim}^2$ small squares and next splitting each of them into two triangles.

It is often the case that in the iteration process the level set function become too steep or too flat around its zero level set. This causes difficulties and errors in determination of the zero level set. To overcome this drawback, a re-initialization is frequently used. This means that if the norm of $\nabla \phi_i$ becomes too high or too small around the zero level set of $\phi_i$, then $\phi_i$ is reset to the distance function with respect to its zero level set [7]. Several strategies without using re-initialization have been described in [25].

We however do not need to use re-initialization. During the simulations we regularly check the value $|\nabla \phi_i|$ around the interface and it always remains between 0.7 and 3 which is acceptable.

We design three numerical examples with different exact shapes in order to demonstrate the convergence of the adjoint variable method implemented to solve the optimization problem (8). The first example shows how the algorithm finds the exact circular shape from the initial ellipsoidal shape. The second example tests the algorithm on a more complicated non-convex shrimp-like exact shape. The third numerical example covers the case of an exact shape consisting of several large disjoint parts.
To quantify the convergence of the method we introduce a distance between two shapes. Since any shape is represented by a zero level set of a level set function, we define the distance \( \text{dist}(\phi_1, \phi_2) \) between two shapes using its level set representations \( \phi_1, \phi_2 \) in the following way

\[
\text{dist}(\phi_1, \phi_2) = \|H_k(\phi_1) - H_k(\phi_2)\|_{L^2(D)}
\]

and we say that the sequence of shapes represented by \( \phi_n \) converges to a shape represented by \( \phi \) iff \( \text{dist}(\phi_n, \phi) \to 0 \). Notice, that we can manipulate with the distance by changing the value of \( k \). Indeed, regardless what value of \( k \) has been used in the determination of the shape, for the measuring of the obtained shape we can use a different \( k \). The higher \( k \) means that the interface is sharper and therefore two different shapes are distinguished better.

![Figure 5: Evolution of the cost function for various exact shapes.](image)

6.1. Example #1. (The exact shape is a circle.)

We set the current density function \( J \) to be a positive constant on the upper strip of \( \Omega \), and a negative constant on the bottom strip of \( \Omega \)

\[
J(x, y) = \begin{cases} 
  J_1, & y > 0.4 \\
  -J_1, & y < -0.4 \\
  0, & \text{otherwise}. 
\end{cases}
\]
Figure 6: Convergence for all examples. The two lowest curves show the convergence in Example #1 with different distance functions, one with $k = 80$ as has been used in computation self, and one with $k = 500$ which corresponds to much sharper interface.

This setting models the case when the current density is induced by the wires wrapped around the workpiece.

The concrete values are

$$J_1 = 500, \ k = 80, \ dim = 60,$$

where $k$ is from (5) and $dim$ is the discretization parameter. The ratio $k/dim$ is chosen such that the width of the interface is somewhere between 3 and 5 grid cells.

For the measurements we use a synthetic data generated for the level set function $\phi_{ex}$. We chose $\phi_{ex}$ such that its zero level set corresponds to the circle with the center $(0.2, 0.15)$ and radius 0.1. This level set function is defined as a signed distance function from that circle. By solving the direct problem (7) with $\phi = \phi_{ex}$, we obtain $A_{ex}$ and set $M = \nabla A_{ex}$. The initial guess $\phi_0$ is set to be an ellipse with the center $(0.15, 0.125)$, $x$-radius 0.15, and $y$-radius 0.125, see Figure 7(a).

We present the results in Figure 7. We see that the initial shape has directly been modified in the direction of the exact shape. The optimization without regularization lead to an oscillating zero level set. To remove the
oscillations we use the Tikhonov regularization with coefficient $\alpha = 0.01$. Eventually, the optimization process ends up finding the exact shape.

6.2. Example #2. (The exact shape has a shrimp-like form.)

In this numerical example and also in the examples #3 and # 4 we take the values

$$J_1 = 500, \; k = 60, \; \text{dim} = 40,$$

and the current density function $J$ to be a positive constant $J_1$ over the whole $\Omega$.

In the example #2 we choose for the exact domain a more exotic shrimp-like shape. The initial guess $\phi_0$ is set to be a circle inside the shrimp with the center $(0.0, 0.1)$, and radius 0.05, see Figure 8(a).

We use the Tikhonov regularization with coefficient $\alpha = 0.02$.

The evolution of the zero level set is depicted in Figure 8. We see that during the iterations 0–16 the initial small circle gets larger and maintains the circular shape until it hits the borders of the exact shape. Then, until the iteration 19 it adapts to the shrimp-like shape and the rest of the evolution is simply an adjustment to fit the exact shape.

In Figure 5 we see a significant drop in the value of the cost. The drop appears in between the iterations 15 and 20. That precisely corresponds to the phase when the approximating shape hits the borders of the exact shape and tries to adjust to the shrimp-like form.

6.3. Example #3. (The exact shape consists of three large ellipses.)

To test the ability of the algorithm to detect disconnected regions we choose three ellipses as the exact shape. The constants $J_1, k$ and dim are set as in Example #2. We use the Tikhonov regularization with coefficient $\alpha = 0.1$.

The evolution of the zero level set can be seen in Figure 9. Three small initial circles uniformly grow until they hit the borders of the exact ellipses, see the iterations 0–17. The upper right circle covers approximately only one half of the exact ellipse. Then the approximate circles almost fill the exact ellipses, see iterations 17–40 and afterwards, the algorithm just fine-tunes the exact shape.

Again, we can see this behavior in Figure 5. The cost lowers its value slightly during the iterations 0–17, then it suddenly drops and after the iteration 40 it goes down again slightly.
6.4. Convergence

The convergence of the methods for all three examples is depicted in Figure 6. We plotted $\text{dist}(\phi_n, \phi_{ex})$ against the number of iterations $n$. As was already mentioned at the end of Section 5, the use of smeared-out Heaviside function in the simulations is needed since the integral in (14) is taken over a lower-dimensional interface. However, the distance function $\text{dist}(\phi_n, \phi_{ex})$ is defined as an integral over the whole domain. Therefore to better distinguish between two shapes, a sharper interface is more suitable. We thus use a higher $k$ for the evaluation of the distance, namely we take $k = 500$.

We see that the convergence curves in Figure 6 and the curves for the evolution of the cost functional in Figure 5 are very similar.

The behavior of two convergence curves for Example #1 (for $k = 500$ and for $k = 80$) shows an interesting phenomenon. Around the 60th iteration, the value of the distance function $\text{dist}(\phi_n, \phi_{ex})$ is the lowest and later on, this distance even slightly raises. Graphically, the approximating shape coincides with the exact shape just around the 60th iteration and thereafter both shapes remain identical. The reason for the raise of the distance $\text{dist}(\phi_n, \phi_{ex})$ for $n > 60$, is that the distance function is an integral over the whole domain and with further evolution of $\phi$, $\phi_n$ can differ from $\phi_{ex}$ on the locations outside the interface and thus the distance can raise (in Example #1) even if the shapes remain identical. In other two examples, the distance function decreases during the whole evolution.

The evolution of $\phi_n$ for $n > 60$ is driven by the gradient $DF$ which minimizes $F$, since the interface is not sharp. The zero level set however does not change.

This phenomenon is diminished by using a higher $k$ in the distance function $\text{dist}(\phi_n, \phi_{ex})$ however, the higher $k$, the more oscillations appear.

6.5. Sensitivity to the noise

To demonstrate the capability of the algorithm to cope with the noise in the data we run the following simulation. We use the settings from Example #2. We show four simulations, with data corrupted by the 5%, 10%, 15% and 20%, respectively. The results are depicted in Figure 10. With noise levels 5% and 10% we used Tikhonov regularization with coefficient $\alpha = 0.1$. With higher noise level we had to increase the regularization weight to 0.15 to obtain non-oscillatory reconstructions. We see that even for quite high noise levels of 5% and 10%, the algorithm is still able to reconstruct the air-gap with a reasonable accuracy. Of course, the chosen regularization was
successful only because of the smooth exact domain. If the exact shape \( D_{ex} \) has sharp corners we will need to use other kinds of regularization such as bounded (or sometimes called total) variation regularization. With 15\% and 20\% noise the reconstruction becomes unacceptable.

6.6. Gradient-for-the-initiation (GFI) method

In the numerical example \#1 we set the initial guess to be an ellipse being positioned not ”far” from the exact circular shape. In the examples \#2 and \#3 we set the initial guess to be small circles inside the exact shapes. All this can however be done only in the case that we know the number of disconnected regions that form the exact shape. Moreover, we need to know their approximate position too. This is a quite strong assumption and it is almost never the case that we know this information in advance.

We suggest a heuristic approach how to find the number of the air gaps and their approximate position. We call this approach a Gradient-for-initiation method (the GFI method).

Assume that the initial guess for \( D_0 \) is an empty set. That means that the domain \( \Omega \) is completely occupied by the ferromagnetic material without any air gaps. We compute the cost function for this particular \( D_0 \). Further we try to locally remove a tiny amount of the material. Our hypothesis is that if the material is removed from the places located inside \( D_{ex} \), then the cost should drop and, if the material is removed from the places located outside \( D_{ex} \), the cost will raise.

Assuming our hypothesis is correct, we need to determine those areas leading to the drop of the cost. We do it in the following way. We set the level set function \( \phi_0 \) equal to a small negative number. By this we ensure that \( D_0 = \emptyset \). Further we set \( k \) to be a high positive number so that the smooth approximation of the Heaviside function is steep around zero. In this way we ensure that even a small positive perturbation \( \phi_\delta \) of \( \phi_0 \) will lead to a nonempty \( D_{0+\delta} = \{ x \in \Omega \mid \phi_0(x) + \phi_\delta(x) > 0 \} \). Of course, \( D_{0+\delta} \) will be located around the support of \( \phi_\delta \).

Now we compute the derivative \( DF \) of the cost with respect to \( \phi \). The derivative in general has the property that it locally informs how fast the differentiated function will raise. That means that in the locations where the derivative \( DF \) is positive, we can expect the raise of the cost, and in the locations where it is negative, we can expect the drop of the cost. Therefore, the domain \( \{ x \in \Omega \mid DF(x) < 0 \} \) should define a good approximation of \( D_{ex} \).
We present the results of the GFI method on three numerical examples in Figures 11 and 12. In Figure 11 the dashed line represents the exact shape while the solid line represents the zero level set of the $DF$. The domain \( \{ x \in \Omega \mid DF(x) < 0 \} \) is close to be inside the solid line. We see that the number of the air gaps and their approximate position was detected very well.

6.7. Example #4

This example shows the result of the optimization when the initial shape has been determined using the GFI method. The other settings are the same as in Example #3.

In Figure 12 we see that starting even from a "wild" initial shape, the algorithm eventually finds the exact shape.

7. Conclusions

We have implemented a level set method to compute the quasi-linear elliptic equation describing the magnetic processes inside a nonlinear magnetic material that contains air gaps or cracks. In this way we obtained an efficient solver for the non-linear direct problem.

Furthermore, we solved the inverse problem for the determination of these air gaps by minimization of the cost function that was defined using the measurements of the magnetic induction. During the solution process we had to overcome several difficulties. For the iterative procedure based on the gradient-type minimization algorithms we implemented the adjoint variable method. This method brings a tremendous reduction of the computational costs in the computation of the gradient direction compared to the classical perturbation methods.

We tested the algorithm in several numerical examples. We successfully computed the inverse problems for three different exact shapes. In each case the algorithm found a very close approximation of the exact shape. The three exact shapes with different topological properties were devised to test the algorithm. We tested non-convex domains in the example #2 and disconnected large regions in the example #3.

Finally, we designed a heuristic algorithm called the Gradient-for-initiation method to find a good initial shape for the optimization algorithm. The GFI method successively found the number and the approximate positions of the air gaps in all the examples.
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Figure 7: Example #1. (a) The circular exact shape $D_{ex}$ plotted by dashed line, the ellipsoidal initial approximation $D_0$ plotted as solid line. (b) The final reconstruction after 344 iterations without any regularization. (c)-(h) Evolution of the zero level set after 2, 4, 13, 20, 30, 58 iterations.
Figure 8: Example #2. (a) The shrimp-like exact shape $D_{ex}$ plotted by dashed line, the small circular initial guess plotted as solid line. (b) The final reconstruction after 300 iterations without any regularization. (c)-(h) Evolution of the zero level set after 13, 16, 19, 30, 70, 200 iterations.
Figure 9: Example #3. (a) The exact $D_{ex}$ consists of three ellipses, plotted with dashed line. Three small circles inside those ellipses, plotted with the solid line are the initial guess. (b) The final reconstruction after 1000 iterations without any regularization. (c)-(h) Evolution of the zero level set after 12, 17, 22, 31, 40, 377 iterations.
Figure 10: Sensitivity to noise. The exact $D_{ex}$ consists of one shrimp-like shape, plotted with dashed line. Reconstructions are plotted with solid line. (a) 5% noise and $\alpha = 0.1$, (b) 10% noise and $\alpha = 0.1$, (c) 15% noise and $\alpha = 0.15$, (d) 20% noise and $\alpha = 0.15$. 
Figure 11: GFI method. Various exact shapes (dashed line) are being considered to test the GFI method. The full line depicts the outer border of the set \( \{ x \in \Omega \mid DF(x) < 0 \} \).
Figure 12: Example #4. (a) The exact $D_{ex}$ consists of three ellipses, plotted with dashed line. The initial guess is determined from the GFI method, plotted with the solid line. (b) The final reconstruction after 319 iterations without any regularization. (c)-(h) Evolution of the zero level set after 8, 99, 205, 330, 416, 574 iterations.