General Euler–Ostrowski formulae and applications to quadratures

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Abstract

The aim of this paper is to generalize inequality

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(s_1) \, ds_1 - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \leqslant \| f'' \|_{\infty} \cdot \frac{(b-a)^3}{6} \frac{A(x)}{b-a}
\]


\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(s_1) \, ds_1 - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \leqslant \| f'' \|_{\infty} \cdot \frac{A(x)}{(b-a)^3}
\]

obtained in [G.A. Anastassiou, Univariate Ostrowski inequalities, Revisited, Monatsh. Math. 135 (2002) 175–189]. To do this, first we derive general Euler–Ostrowski formulae which generalize extended Euler formulae, obtained in [Lj. Dedić, M. Matić, J. Pečarić, On generalizations of Ostrowski inequality via some Euler-type identities, Math. Inequal. Appl. 3(3) (2000) 337–353]. The main novelty is that a remainder is expressed in terms of \( B_n'(x - mt) \) which enables us to obtain a vide variety of quadrature formulae such as trapezoid, midpoint, bitrapezoid, twopoint formulae and their multipoint generalizations.

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1. Introduction

The following inequality was one of the main results in [2].

**Theorem 1.** Let \( f: [a, b] \rightarrow \mathbb{R} \) be 3-times differentiable on \([a, b]\). Assume that \( f'' \) is bounded on \([a, b]\). Let \( x \in [a, b] \). Then we obtain

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(s_1) \, ds_1 - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \\
- \frac{f'(b) - f'(a)}{2(b-a)} \cdot \left( x^2 - (a+b)x + \frac{a^2 + b^2 + 4ab}{6} \right) \leq \|f''\|_\infty \cdot \frac{A(x)}{(b-a)^3},
\]

where

\[
A(x) = abx^4 - \frac{1}{3} a^2b^3x + \frac{1}{3} a^3b^2x - ab^2x^3 - \frac{1}{6} a^3b^2x + \frac{1}{3} ab^3x^2 + a^3b^2x - 2ax^5 - \frac{1}{2} bx^5 + \frac{1}{6} x^6 \\
+ \frac{3}{4} x^5 + \frac{3}{4} x^5 + \frac{1}{3} a^2b^2x - \frac{2}{3} a^3x^2 - \frac{2}{3} b^3x^2 - \frac{1}{3} a^3b^3 + \frac{5}{12} a^4x^2 + \frac{5}{12} b^4x^2 + \frac{1}{3} a^2b^4 \\
- \frac{2}{15} a^3b - \frac{2}{15} ab^5 - \frac{1}{6} a^5x + \frac{1}{6} b^5x + \frac{a^6}{20} + \frac{b^6}{20}.
\]

Inequality (1) was improved in [1]. Namely,

**Theorem 2.** Let \( f: [a, b] \rightarrow \mathbb{R} \) be 3-times differentiable on \([a, b]\) and \( \|f''\|_\infty < \infty \). Then for every \( x \in [a, b] \) we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(s_1) \, ds_1 - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \\
- \frac{f'(b) - f'(a)}{2(b-a)} \cdot \left( x^2 - (a+b)x + \frac{a^2 + b^2 + 4ab}{6} \right) \leq \|f''\|_\infty \cdot \frac{(b-a)^3}{6} I \left( \frac{x-a}{b-a} \right),
\]

where

\[
I(\lambda) = \begin{cases} 
-\frac{1}{2} t_1^4 + \frac{1}{2} t_1^4 - \frac{1}{2} t_1^4 - \frac{1}{2} \lambda^4 - \lambda^3 - \lambda^2 + \frac{1}{2} \lambda, & \lambda \in \left[ 0, \frac{3\sqrt{3}}{6} \right] \\
\frac{1}{3} t_1^4 - \frac{1}{2} t_1^4 - \frac{1}{2} \lambda^4 + 3\lambda^3 - 2\lambda^2 + \frac{1}{2} \lambda, & \lambda \in \left( \frac{3\sqrt{3}}{6}, \frac{1}{2} \right] \\
\frac{1}{4} t_1^4 - \frac{1}{2} t_1^4 + \frac{3}{2} \lambda^4 + 3\lambda^3 + \lambda^2 - \frac{1}{2} \lambda, & \lambda \in \left( \frac{1}{2}, \frac{3\sqrt{3}}{6} \right] \\
-\frac{1}{2} t_1^4 + \frac{1}{2} t_1^4 + \frac{3}{2} \lambda^4 - 3\lambda^3 + 2\lambda^2 - \frac{1}{2} \lambda, & \lambda \in \left( \frac{3\sqrt{3}}{6}, 1 \right]
\end{cases}
\]

and

\[
t_1 = \frac{3}{4} - \frac{1}{2} \lambda - 1 \sqrt{\frac{1}{4} + 3\lambda^2 - 3\lambda^2}, \\
t_2 = \frac{3}{4} - \frac{1}{2} \lambda + \frac{1}{2} \sqrt{\frac{1}{4} + 3\lambda^2 - 3\lambda^2}.
\]

Further, we can write \( A(x) \) from (1) as

\[
A(x) = \frac{(b-a)^6}{6} B \left( \frac{x-a}{b-a} \right),
\]

where

\[
B(\lambda) = \lambda^6 - 3\lambda^5 + \frac{9}{2} \lambda^4 - 4\lambda^3 + \frac{5}{2} \lambda^2 - \lambda + \frac{3}{10}.
\]
and for every \( \lambda \in [0, 1] \)
\[
\frac{1}{32} \leq I(\lambda) \leq \frac{\sqrt{3}}{36} < \frac{41}{320} \leq B(\lambda) \leq \frac{3}{10}.
\]

The aim of this paper is to generalize inequality (2), and therefore obtain a generalization and improvement of inequality (1). To do this, we will first derive general Euler–Ostrowski formulae which generalize extended Euler formulae, obtained in [3], in a sense that the value of the integral is approximated not only with the value of the function in a certain point, but with the values of the function in \( m \) equidistant points.

Extended Euler formulae extend the well known formula for the expansion of an arbitrary function in Bernoulli polynomials (cf. [4]). Namely, for \( f: [0, 1] \rightarrow \mathbb{R} \) such that \( f^{(n-1)} \) is continuous of bounded variation on \([0, 1]\) for some \( n \geq 1 \), for every \( x \in [0, 1] \) we have
\[
\int_0^1 f(t) \, dt = f(x) - \sum_{k=1}^{n} \frac{B_k(x)}{k!} [f^{(k-1)}(1) - f^{(k-1)}(0)] + \frac{1}{n!} \int_0^1 B_n(x - t) \, df^{(n-1)}(t), \tag{3}
\]
\[
\int_0^1 f(t) \, dt = f(x) - \sum_{k=1}^{n-1} \frac{B_k(x)}{k!} [f^{(k-1)}(1) - f^{(k-1)}(0)] + \frac{1}{n!} \int_0^1 [B_n(x - t) - B_n(x)] \, df^{(n-1)}(t). \tag{4}
\]

2. Main results

To derive our formulae, we will need an analogue of Multiplication Theorem, stated for periodic functions \( B_n \). Multiplication Theorem for Bernoulli polynomials \( B_n \) states (cf. [5]):
\[
B_n(mt) = m^{n} \sum_{k=0}^{m-1} B_n \left(t + \frac{k}{m}\right), \quad n \geq 0, \ m \geq 1. \tag{5}
\]

That (5) is true for \( B_n(t) \) and \( t \in [0, 1/m] \) is obvious. For \( t \in [j/m, (j+1)/m] \), \( 1 \leq j \leq m-1 \):
\[
B_n(mt) = B_n(m(t - j/m)) = m^{n-1} \sum_{k=0}^{m-1} B_n \left(t + \frac{k - j}{m}\right) = m^{n-1} \sum_{k=0}^{m-1} B_n \left(t + \frac{k}{m}\right),
\]

so the statement is true again. Thus, we have
\[
B_n(mt) = m^{n-1} \sum_{k=0}^{m-1} B_n \left(t + \frac{k}{m}\right), \quad n \geq 0, \ m \geq 1. \tag{6}
\]

Interval \([0, 1]\) is used for simplicity and involves no loss in generality.

The following theorem is crucial for our further investigations but is also of independent interest. Namely, the remainder is expressed in terms of \( B_n(x - mt) \).

**Theorem 3.** Let \( f: [0, 1] \rightarrow \mathbb{R} \) be such that \( f^{(n-1)} \) is continuous of bounded variation on \([0, 1]\) for some \( n \geq 1 \). Then, for \( x \in [0, 1] \) and \( m \in \mathbb{N} \), we have
\[
\int_0^1 f(t) \, dt = \frac{1}{m} \sum_{k=0}^{m-1} f \left(\frac{x+k}{m}\right) - T_n(x) + \frac{1}{n! \cdot m^n} \int_0^1 B_n(x - mt) \, df^{(n-1)}(t), \tag{7}
\]
where
\[
T_n(x) = \sum_{j=1}^{n} \frac{B_j(x)}{j! \cdot m^j} [f^{(j-1)}(1) - f^{(j-1)}(0)].
\]

**Proof.** From (6) we get
\[
B_n(x - mt) = m^{n-1} \sum_{k=0}^{m-1} B_n \left(\frac{x+k}{m} - t\right).
\]

Multiplying this with \( df^{(n-1)}(t) \) and integrating over \([0, 1]\) produces formula (7) after applying (3). \( \square \)
Formula (7) can easily be rewritten as:

$$\int_0^1 f(t)\,dt = \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) - T_{n-1}(x) + \frac{1}{n! \cdot m^n} \int_0^1 |B_n^s(x - mt) - B_n(x)| |f^{(n-1)}(t)| \,dt$$  \hfill (8)

with $T_0(x) = 0$.

We will call formulae (7) and (8) general Euler–Ostrowski formulae.

**Theorem 4.** Assume $(p, q)$ is a pair of conjugate exponents, that is $1 < p, q < \infty$, $1/p + 1/q = 1$. Let $f: [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(n)} \in L_p[0, 1]$ for some $n \geq 1$. Then, for $x \in [0, 1]$ and $m \in \mathbb{N}$, we have

$$\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \leq K(n, q) \cdot \|f^{(n)}\|_p,$$  \hfill (9)

$$\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \leq K^*(n, q) \cdot \|f^{(n)}\|_p,$$  \hfill (10)

where

$$K(n, q) = \frac{1}{n! \cdot m^n} \left[ \int_0^1 |B_n^s(t)|^p \,dt \right]^{\frac{1}{p}}, \quad K^*(n, q) = \frac{1}{n! \cdot m^n} \left[ \int_0^1 |B_n^s(t) - B_n(x)|^p \,dt \right]^{\frac{1}{p}}.$$  

These inequalities are sharp for $1 < p \leq \infty$ and best possible for $p = 1$.

**Proof.** Inequalities (9) and (10) follow immediately after applying Hölder’s inequality to the remainders in formulae (7) and (8) and using the fact that functions $B_n^s(t)$ are periodic. To prove inequalities are sharp, put

$$f^{(n)}(t) = \text{sgn} B_n^s(x - mt) \cdot |B_n^s(x - mt)|^{1/(p-1)} \quad \text{for } 1 < p < \infty \text{ and}$$

$$f^{(n)}(t) = \text{sgn} B_n^s(x - mt) \quad \text{for } p = \infty \text{ in (9),}$$

$$f^{(n)}(t) = \text{sgn}(B_n^s(x - mt) - B_n(x)) \cdot |B_n^s(x - mt) - B_n(x)|^{1/(p-1)} \quad \text{for } 1 < p < \infty$$

$$f^{(n)}(t) = \text{sgn}(B_n^s(x - mt) - B_n(x)) \quad \text{for } p = \infty \text{ in (10).}$$

The proof that these inequalities are best possible for $p = 1$ is the same as the proof of Theorem 2 in [6].

**Corollary 1.** Let $f: [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(n)} \in L_\infty[0, 1]$. Let $x \in [0, 1]$. If $n$ is odd, then we have

$$\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \leq \frac{(4 - 2^{1-n}) |B_{n+1}|}{m^n \cdot (n+1)!} \cdot \|f^{(n)}\|_\infty$$  \hfill (11)

and for $n = 1$

$$\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) \right| \leq \frac{1}{m} \left[ \frac{1}{4} + \left( x - \frac{1}{2} \right)^2 \right] \cdot \|f'\|_\infty,$$  \hfill (12)

while for $n \geq 3$

$$\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \leq \frac{\|f^{(n)}\|_\infty}{m^n \cdot n!} \left( (1 - 2|x - x_1|) \cdot |B_n(x)| + \frac{2}{n + 1} |B_{n+1}(x) - B_{n+1}(x_1)| \right),$$  \hfill (13)
where \( x_1 \in [0,1] \) is such that \( B_n(x_1) = B_n(x) \) and \( x_1 \neq x \), except when \( B_{n-1}(x) = 0 \). If \( x = 0 \) or \( x = 1 \), take \( x_1 = 1/2 \).

If \( n \) is even, then we have

\[
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x+k}{m} \right) + T_n(x) \right| \leq \frac{4\|f^{(n)}\|_\infty}{m^n \cdot (n+1)!} \cdot |B_{n+1}(x_1)| = \frac{4\|f^{(n)}\|_\infty}{m^n \cdot (n+1)!} \max_{t \in [0,1]} |B_{n+1}(t)|, \tag{14}
\]

where \( B_n(x_1) = 0 \), and

\[
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x+k}{m} \right) + T_{n-1}(x) \right| \leq \|f^{(n)}\|_\infty \left( (-1)^{n/2} (1 - 4|x - 1/2|)B_n(x) + \frac{4}{n+1} |B_{n+1}(x)| \right). \tag{15}
\]

**Proof.** Put \( p = \infty \) in Theorem 4. Inequality (11) follows straightforward since it is known that, for an odd \( n \), Bernoulli polynomials have constant sign on \((0,1/2)\) and on \((1/2,1)\). Inequality (12) also follows by direct calculation.

To prove (13), assume first that \( 0 \leq x \leq 1/2 \). For an odd \( n \) we have \( B_n(1-t) = -B_n(t) \), so we can rewrite \( K^*(n,1) \) as

\[
\int_0^{1/2} |B_n(t) - B_n(x)| \, dt + \int_0^{1/2} |B_n(t) + B_n(x)| \, dt.
\]

The second integral has no zeros on \((0,1/2)\), so we can calculate it easily. The first integral, however, has two zeros. One is obviously \( x \) and the other is \( x_1 \), where \( x_1 \in [0,1/2] \) and \( B_n(x_1) = B_n(x) \). When \( 1/2 \leq x \leq 1 \), the statement follows similarly.

Next, assume \( 0 \leq x \leq 1/2 \). Since \( B_n(t) \) are symmetric about \( t = 1/2 \) for an even \( n \), we can rewrite \( K^*(n,1) \) as

\[
2 \int_0^{1/2} |B_n(t) - B_n(x)| \, dt.
\]

As Bernoulli polynomials are monotonous on \((0,1/2)\) for an even \( n \), inequality (15) follows. For \( 1/2 \leq x \leq 1 \) the statement follows analogously. Using similar arguments we get (14). \( \square \)

**Corollary 2.** Let \( f: [0,1] \to \mathbb{R} \) be such that \( f^{(n)} \in L_1[0,1] \) and \( x \in [0,1] \). For \( n = 1 \), we have

\[
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x+k}{m} \right) + B_1(x) \frac{|f(1) - f(0)|}{2m} \right| \leq \frac{\|f'\|_1}{2m}, \tag{16}
\]

\[
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x+k}{m} \right) \right| \leq \frac{\|f'\|_1}{m} \left( \frac{1}{2} + |x - \frac{1}{2}| \right). \tag{17}
\]

For an odd \( n \), \( n \geq 3 \), we have

\[
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x+k}{m} \right) + T_n(x) \right| < \frac{2\|f^{(n)}\|_1}{(1-2^{-n-1})(2\pi m)^n}, \tag{18}
\]

\[
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x+k}{m} \right) + T_{n-1}(x) \right| < \frac{2\|f^{(n)}\|_1}{m^n \cdot n!} \frac{2n!}{(1-2^{-n-1})(2\pi n)^n} + |B_n(x)|. \tag{19}
\]
If $n$ is even, then we have

$$
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x + k}{m} \right) + T_n(x) \right| \leq \frac{|B_n|}{m^n \cdot n!} \cdot \|f^{(n)}\|_1, \tag{20}
$$

$$
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x + k}{m} \right) + T_{n-1}(x) \right| \leq \frac{\|f^{(n)}\|_1}{m^n \cdot n!} \left( (1 - 2^{-n})|B_n| + |2^{-n}B_n - B_n(x)| \right). \tag{21}
$$

**Proof.** Put $p = 1$ in Theorem 4. Inequalities (16) and (17) follow by direct calculation. Using estimations of the maximal value of Bernoulli polynomials (cf. [5]), we get (18)–(20). Finally, since $B_n(t)$ are symmetric about $t = 1/2$ for an even $n$, it is enough to consider them on $(0,1/2)$ and there they are monotonous. So the maximal value of $|B_n(t) - B_n(x)|$ is obtained either for $t = 0$ or for $t = 1/2$. Using formula

$$
\max\{|A|, |B|\} = \frac{1}{2}(|A + B| + |A - B|),
$$

Inequality (21) follows. □

**Corollary 3.** Let $f:[0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L^2[0,1]$ and $x \in [0,1]$. Then we have

$$
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x + k}{m} \right) + T_n(x) \right| \leq \frac{\|f^{(n)}\|_2}{m^n \cdot (2n)!} \left( \frac{|B_{2n}|}{(2n)!} \right)^{1/2}, \tag{22}
$$

$$
\left| \int_0^1 f(t) \, dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left( \frac{x + k}{m} \right) + T_{n-1}(x) \right| \leq \frac{\|f^{(n)}\|_2}{m^n \cdot n!} \left( \frac{(n)!^2}{(2n)!} \right)^{1/2} \left( |B_{2n}| + B_n^2(x) \right)^{1/2}. \tag{23}
$$

**Proof.** Both inequalities follow by direct calculation after taking $p = 2$ in Theorem 4. □

### 3. Concluding remarks

In this section we will give some remarks connecting results of this paper with known results.

For $m = 1$, Inequality (12) becomes the classical Ostrowski inequality. For $m = 1$ and $n = 2$, Inequality (15) produces an improvement of a result obtained in [7], which was explained in detail in [3].

Taking $m = 1$ and $n = 1$ in (13) produces (2), so (13) is a generalization of that result. It is clear from Theorem 2 that inequality (2) gives a better estimation than inequality (1) from Theorem 1 because the extremal values of the right-hand sides of both inequalities are calculated. That this is so is also obvious from the proofs of these inequalities. Namely, in [2], the left-hand side of (1) is expressed in terms of iterated integration of Peano kernels which, when integrated, give Bernoulli polynomials (which is implicitly clear from that proof). The estimation of the absolute value of the left-hand side is then obtained by inserting the absolute value on each of the Peano kernels in the iterated integrals in the remainder. In [1], however, a single absolute value is put on the integrand of the remainder. So, it is quite clear why (2) is better.

For $m = 1$, formulae (7) and (8) reduce to (3) and (4), and thus produce all other results obtained in [3], i.e. the generalizations of Ostrowski’s inequality. Furthermore, for $m = 1$ and $x = 0$ those formulae become the Euler trapezoid formulae derived in [8]. Similarly, for $m = 1$ and $x = 1/2$, we get Euler midpoint formulae derived in [9] and of course all other results from that paper follow directly. In the last sections of those papers repeated Euler trapezoid and repeated Euler midpoint formulae are considered (the interval is divided into $v$ subintervals of equal length and then Euler trapezoid i.e. midpoint formulae are applied to each of them). Those formulae are in fact (7) and (8) for $x = 0$, i.e. $x = 1/2$.

Adding Euler trapezoid formula and Euler midpoint formula and dividing by 2 produces Euler bitrapezoid formula. Therefore, all results from [10] can be obtained. In the last section of that paper repeated Euler
bitrapezoid formulae are considered. Those formulae can be obtained directly from formulae (7) and (8) analogously as Euler bitrapezoid formulae.

Consider formulae (7) and (8) for $m = 1$ and $x \in [0, 1/2]$ and then for $1 - x \in [1/2, 1]$, add them and divide by 2. As a result two-point formulae are obtained, and those were studied in [6].

It is interesting to consider which $x \in [0, 1]$ gives the optimal estimation in inequalities (13) and (15). In (12) it is obvious that $x = 1/2$ is such point. Differentiating the function on the right-hand side of (15)—this is the case when $n$ is even—it is easy to see that it obtains its minimum for $x = 1/4$ and $x = 3/4$ (for $n \geq 2$) while its maximal value is in $x = 0$ and $x = 1$ (for $n \geq 4$). Of course, the minimal value is of greater interest. In that case, the quadrature formulae take the following form
\[
\int_0^1 f(t) \, dt \approx \frac{1}{4} \left( f(1) + 4f\left(\frac{1}{4}\right) - f(0) \right),
\]
\[
\int_0^1 f(t) \, dt \approx \frac{1}{4} \left( f(0) + 4f\left(\frac{3}{4}\right) - f(1) \right).
\]
Also, if we take these parameters and put them in (8), then add and divide by 2, we get a two-point formula where the integral is approximated with values of the function in $x = 1/4$ and $x = 3/4$. The error estimation for this formula can be deduced from the following, more general, estimation. Using triangle inequality, we get
\[
\left| \int_0^1 f(t) \, dt - \frac{1}{2m} \sum_{k=0}^{m-1} \left( f\left(\frac{x+k}{m}\right) + f\left(\frac{1-x+k}{m}\right) \right) + \sum_{j=1}^{(n-2)/2} \frac{B_{2j}(x)}{(2j)! \cdot m^{2j}} f^{(2j-1)}(1) - f^{(2j-1)}(0) \right|
\leq \frac{\|f^{(n)}\|_{\infty}}{m^n \cdot n!} \left( -1 \right)^{n/2} (1 - 4|x - 1/2|) B_n(x) + \frac{4}{n+1} |B_{n+1}(x)|.
\]
Therefore, this formula gives the best error estimate for $x = 1/4$.

On the other hand, inequality (13) behaves quite oppositely (this is the case when $n$ is odd and $n \geq 3$). Observe that $x_1$ is a decreasing function of $x$ and it is differentiable on $(0, 1/2)$. This is sufficient since the function on the right-hand side of that inequality (denote it by $F(x)$) obtains the same value for $x$ and $1 - x$. For $0 \leq x \leq 1/2$, we get
\[
F'(x) = (-1)^{(n+1)/2} \cdot n(1 - 2|x - x_1|) B_{n-1}(x).
\]
Since $F'(x)$ changes sign from positive to negative when passing through point $x \in (0, 1/2)$ such that $B_{n-1}(x) = 0$, we conclude $F(x)$ obtains maximal value at $x$. Note that $x$ is close to $1/4$, but a bit smaller. Minimum is then obtained at the end points of the interval i.e. for $x = 0$ and $x = 1/2$ (the same value is obtained at both of these points).

References