On the system of word equations
\[ x_0 u_1^i x_1 u_2^i x_2 u_3^i x_3 = y_0 v_1^i y_1 v_2^i y_2 v_3^i y_3 \quad (i = 0, 1, 2, \ldots) \]
in a free monoid

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Abstract

We prove that if \( x_0 u_1^i x_1 u_2^i x_2 u_3^i x_3 = y_0 v_1^i y_1 v_2^i y_2 v_3^i y_3 \) holds for all \( i \in \{0, 1, 2, 3\} \), then it is true for each \( i \in \{0, 1, 2, 3, 4, 5\} \) holds for all \( i \in \{0, 1, 2, 3, 4, 5\} \), and that it is true for each \( i \in \{0, 1, 2, 3, 4, 5\} \). © 1999 Elsevier Science B.V. All rights reserved.

0. Introduction

In the theory of automata and formal languages many problems (for instance, the question of test sets and the equivalence problem of morphisms) induce systems of word equations in which pumpings in different places can exist.

The study started in [5] is carried on in this paper. We focus our attention to two special cases of the general system of word equations

\[ x_0 u_1^i x_1 \cdots u_n^i x_n = y_0 v_1^i y_1 \cdots v_m^i y_m \quad (i = 0, 1, 2, \ldots), \quad (*) \]

where \( n \) and \( m \) are positive integers, \( x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m \) are words over a finite alphabet \( \Sigma \) (the midwords of the system) and \( u_1, \ldots, u_n, v_1, \ldots, v_m \) are nonempty words over \( \Sigma \) (the loops of the system). The two special cases we concentrate on are

(i) \( \max\{n, m\} = 2 \) and

(ii) \( \max\{n, m\} = 3 \).

The famous Ehrenfeucht Conjecture solved a decade ago (see, for instance, [2, 8]) tells that any denumerable system of word equations over a finite alphabet is equivalent with some finite subsystem of it. Thus, as long as solutions are concerned, the system (*) is equivalent with a subsystem consisting of some \( d \) first equations of it, where \( d \) is a natural number depending probably on \( n \) and \( m \).

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The test set problem for context-free languages is solved in [1], results of which imply case (i) above.

Our article is organized as follows. In Section 1 some preliminaries, definitions and well-known results of combinatorics on words are given. In Section 2 the case \( \max\{n,m\} = 2 \) is studied. Section 3 contains the main results of this paper. We prove that if \( \max\{n,m\} = 3 \), then (*) is satisfied if the first six equations of it are. In Section 4 some open problems and further topics of investigation are stated.

1. Preliminaries

Let \( \mathbb{N} \) be the set of all natural numbers. Throughout the text \( \Sigma \) denotes a finite alphabet and \( \Sigma^* \) (\( \Sigma^+ \), resp.) is the free monoid (semigroup, resp.) generated by \( \Sigma \). The elements of \( \Sigma^* \) are called words. Let \( w \in \Sigma^* \). The length of \( w \), denoted by \( |w| \), is the number of occurrences of letters in \( w \). The word with length zero is the empty word, denoted by \( \varepsilon \). The \( n \)th power of \( w \) is defined inductively by \( w^0 = \varepsilon \), \( w^{n+1} = w^n w \), \( n \in \mathbb{N} \).

Denote by \( w^+ \) the set of all positive powers of the word \( w \). The word \( w \) is a subword of \( w \) if there exist words \( x \) and \( y \) such that \( w = xwy \). If above \( x = \varepsilon \) (\( y = \varepsilon \), resp.) then \( w \) is a prefix (suffix, resp.) of \( w \). Let \( \text{pref}(w) \) (\( \text{sufs}(w) \), resp.) be the set of all prefixes (suffixes, resp.) of \( w \).

A word \( v \in \Sigma^1 \) is primitive, if for each \( u \in \Sigma^1 \) and \( n \in \mathbb{N} \), the equality \( v = u^n \) implies \( n = 1 \) (and \( v = u \)). It is well known (see, for instance, [7]) that for each \( v \in \Sigma^+ \) there exists a unique primitive word \( t \in \Sigma^+ \), the primitive root of \( v \), such that \( v = tm \) for some positive \( m \in \mathbb{N} \).

The subsets of \( \Sigma^* \) are called languages. For a language \( L \), let \( \text{pref}(L) = \bigcup_{x \in L} \text{pref}(x) \) and \( \text{sufs}(L) = \bigcup_{x \in L} \text{sufs}(x) \). If \( L \) is a singleton, i.e. \( L = \{x\} \) for some word \( x \), we identify \( L \) with \( x \). The product of the languages \( L_1 \) and \( L_2 \) is the set \( L_1 L_2 = \{xy \mid x \in L_1, y \in L_2 \} \).

The nonempty words \( u \) and \( v \) are conjugate (words of each other) if there exist words \( x \) and \( y \) such that \( u = xy \) and \( v = yx \). The word \( u \) and \( v \) commute if \( uv = vu \).

The following three results belong to the folklore of combinatorics on words. Proofs can be found in [7].

**Lemma 1.** Let \( x \) and \( y \) be nonempty words. The following three conditions are equivalent.

(i) The words \( x \) and \( y \) are conjugate;

(ii) The words \( x \) and \( y \) are of equal length and there exist unique words \( t_1 \) and \( t_2 \), with \( t_2 \) nonempty, such that \( t = t_1 t_2 \) is primitive and \( x \in (t_1 t_2)^+ \) and \( y \in (t_2 t_1)^+ \);

(iii) There exists a word \( z \) such that \( xz = yz \).

Furthermore, assume that any of the three conditions above holds and that \( t_1 \) and \( t_2 \) are as in condition (ii). Then, for each word \( w \), we have \( xw = yw \) if and only if \( w \in (t_1 t_2)^* t_1 \).
Lemma 2. Two nonempty words commute if and only if they are powers of the same (primitive) word, i.e. they have the same primitive root.

One of the strongest results in the elementary theory of combinatorics on words can be stated as follows (for the proof, see for instance [6, 7]).

Lemma 3. If two powers \( u^n \) and \( v^n \) of nonempty words \( u \) and \( v \) have a common subword of length at least \(|u| + |v| - d \) (\( d \) being the greatest common divisor of \(|u| \) and \(|v| \)), then the primitive roots of \( u \) and \( v \) are conjugate.

Note that if in the previous lemma \( u^n \) and \( v^n \) have a common prefix of length at least \(|u| + |v| - d \), then \( u \) and \( v \) have the same primitive root, so they are powers of the same (primitive) word.

Lemma 4. (a) Assume that \( x_1, x_2, \beta_1, \beta_2, w, y \) and \( z \) are words such that

\[
yx_i = w^i z \beta_i
\]

for \( i = 1, 2 \). If \(|y| > |w|\), then \( y \) and each nonempty word in the set \( \{z, w\} \) begins with the same symbol.

(b) Assume that \( x_i, \beta_i, i = 1, 2, 3 \), and \( w_1, w_2, y, z \) are words such that

\[
yx_i = w_i^i w_2^i z \beta_i
\]

for \( i = 1, 2, 3 \). If \(|y| > |w_1^2 w_2|\), then \( y \) and each nonempty word in the set \( \{z, w_1, w_2\} \) begins with the same symbol.

Proof. (a) Assume that \(|y| > |w|\). If \( w = \epsilon \) or \( z = \epsilon \), the claim is certainly true. Suppose that \( w \) and \( z \) are nonempty. Since \( yx_1 = wz \beta_1 \), there exists a nonempty word \( d \in \text{pref}(z \beta_1) \) such that \( y = wd \). The equality \( yx_2 = w^2 z \beta_2 \) then implies that \( dx_2 = wz \beta_2 \). Thus all the words \( y, z, w \) begin with the same symbol.

(b) Let \(|y| > |w_1^2 w_2|\). If either \( w_1 = \epsilon \) or \( w_2 = \epsilon \) or \( z = \epsilon \), we are through by part (a) of the lemma. Assume that all the words \( w_1, w_2, z \) are nonempty. Since \( yx_1 = w_1 w_2 z \beta_1 \) and \( yx_2 = w_1^2 w_2^2 z \beta_2 \), there exist nonempty words \( d_1 \in \text{pref}(z \beta_1) \) and \( d_2 \in \text{pref}(w_2 z \beta_2) \) such that \( y = w_1 w_2 d_1 = w_1^2 w_2^2 d_2 \). Certainly \( w_2 d_1 = w_1 w_2 d_2 \) implying that there exist a word \( w \) such that \( w_1 w_2 = w_2 w \). Then \( d_1 = wd_2 \). Substituting \( y = w_1^2 w_2^2 d_2 \) into the equality \( yx_3 = w_1^3 w_2^3 z \beta_3 \) implies that \( w_2 d_2 x_3 = w_1 w_2^3 z \beta_3 \), from which, remembering that \( w_1 w_2 = w_2 w \), we deduce that \( d_2 x_3 = w w_2^2 z \beta_3 \). Clearly all the words \( y, w_1, w_2 \) begin with a common symbol, as well as all the words \( d_1, d_2, w \). Thus all the words \( y, z, w_1, w_2 \) begin with the same symbol.

2. The system with at most two loops on both sides of the word equations

To prove the main result of this section, we need two auxiliary lemmas.
Lemma 5. Let $u_1, u_2, v_1, v_2$ and $x$ be words such that $u_1'xu_2' = v_1'xv_2'$ for $i = 1, 2$. Then $u_1'xu_2' = v_1'xv_2'$ for each $i = 1, 2, 3, \ldots$.

Proof. We proceed by induction on $n = |x|$.

If $n = 0$ we are clearly through (for instance by Theorem 2 of [5]).

Assume thus that $n > 0$. If $u_1 = v_1$, then $u_2 = v_2$, so the claim holds. Suppose that $u_1 \neq v_1$. By Lemma 4(a), each nonempty word in the set $\{u_1, v_1, x\}$ begins with the same symbol, $b$, say. Thus $u_1b = bu$, $v_1b = bv$ and $x = bz$ for some words $u$, $v$ and $z$. Clearly, by assumption, $(ub)zv_2' = (vb)zv_2'$ for $i = 1, 2$. By the induction hypothesis, $u'zv_2' = v'zv_2'$ for each $i = 1, 2, 3, \ldots$, and therefore $u_1'xu_2' = v_1'xv_2'$ for each $i = 1, 2, 3, \ldots$. $\Box$

Lemma 6. Let $u_1, u_2, v_1, v_2$ and $x$ be words such that $xu_1'u_2' = v_1'v_2'x$ for $i = 1, 2, 3$. Then $xu_1'u_2' = v_1'v_2'x$ for each $i = 1, 2, 3, \ldots$.

Proof. We proceed again by induction on $n = |x|$.

If $n = 0$ we are through as above. Assume that $n > 0$.

Consider first the case $|xu_1| > |v_1|$. By Lemma 4(a), each nonempty word in the set $\{v_1, v_2, x\}$ begins with the same symbol, $c$, say. Certainly there exist words $w_1$ and $w_2$ such that $v_1c = cw_1$, $v_2c = cw_2$ and $x = cz$. Then, by assumption, $zu_1'u_2' = w_1'w_2'z$ for $i = 1, 2, 3$. By the induction hypothesis, $zu_1'u_2' = w_1'w_2'z$ for each $i = 1, 2, 3, \ldots$, so therefore $xu_1'u_2' = v_1'v_2'x$ for each $i = 1, 2, 3, \ldots$.

Let now $|xu_1| \leq |v_1|$. Then, by assumption, there exists a word $d$ such that

$$
\begin{align*}
v_1 &= xu_1d, \\
u_2 &= dv_2x, \\
u_1u_2d &= dv_1v_2, \\
u_1^2u_2^2d &= dv_1^2v_2^2.
\end{align*}
$$

(1)

The second and third equalities of (1) clearly imply that there exist words $p_1$ and $p_2 = v_2xd$ for which $u_1d = dp_1$ and $u_2d = dp_2$. Thus from the third and fourth equality of (1) we deduce that $p_1'p_2' = v_1'v_2'$ for $i = 1, 2$. Then $p_1'p_2' = v_1'v_2'$ for each $i = 1, 2, 3, \ldots$. Obviously, $xu_1'u_2' = xu_1'(dv_2x)' = xu_1'dp_1'^{-1}p_2'^{-1}v_2x = v_1'v_2'x$ for each $i = 1, 2, 3, \ldots$. $\Box$

Note. Lemma 6 does not remain true if we assume $xu_1'u_2' = v_1'v_2'x$ for $i = 1, 2$ only. For if we choose $x = ab$, $u_1 = a$, $u_2 = bab$, $v_1 = aba$ and $v_2 = b$, where $a$ and $b$ are distinct symbols, then $xu_1'u_2' = v_1'v_2'x$ for $i = 1, 2$, but $xu_1'u_2' \neq v_1'v_2'x$.

Theorem 7. Assume that $u_i, v_i$, $i = 1, 2$, and $x_j, y_j$, $j = 0, 1, 2$, are words such that $x_0u_1'u_2'x_2 = y_0v_1'y_1v_2'y_2$ for $i = 0, 1, 2, 3$. Then $x_0u_1'u_2'x_2 = y_0v_1'y_1v_2'y_2$ for each $i = 0, 1, 2, 3, \ldots$. 

Proof. We may clearly assume that \( y_0 = e \) and that either \( x_2 = e \) or \( y_2 = e \). Consider first

\[
x_0 u_1^i x_1^i u_2^i x_2 = v_1^i y_1 v_2^i
\]

for \( i = 0, 1, 2, 3 \). The cases \( i = 0 \) and \( i = 1 \) in (2) imply that \( y_1 = x_0 x_1 x_2 \) and \( x_0 u_1^i x_1^i u_2^i x_2 = v_1 y_1 v_2^i \). There thus exist words \( w_1, w_2 \) such that \( v_1 x_0 x_1 x_2 = v_1 w_1 \) and \( x_2 v_2 = w_2 x_2 \). Then (2) implies

\[
u_1^i x_1^i u_2^i = w_1^i x_1^i w_2^i
\]

for \( i = 0, 1, 2, 3 \). Lemma 5 now implies that \( u_1^i x_1^i u_2^i = w_1^i x_1^i w_2^i \) for \( i = 0, 1, 2, 3, \ldots \) from which \( x_0 u_1^i x_1^i x_2 = v_1^i y_1 v_2^i \) for \( i = 0, 1, 2, 3, \ldots \). Consider then

\[
x_0 u_1^i x_1^i u_2^i = v_1^i y_1 v_2^i v_2
\]

for \( i = 0, 1, 2, 3 \). If \( x_0 = e \) or \( y_2 = e \), the techniques used above can be applied. Assume thus that \( x_0 \neq e \) and \( y_2 \neq e \).

Proceed by induction on \( m = \left| x_1 \right| + \left| y_1 \right| \). If \( m = 0 \), we are through by Lemma 6. Assume thus that \( m > 0 \), and, without loss of generality, that \( y_1 \neq e \). Since \( x_0 x_1 = y_1 y_2 \), there exists a symbol \( b \) and words \( x \) and \( y \) such that \( x_0 = bx \) and \( y_1 = by \). Since \( x_0 u_1 x_1 u_2 = v_1 y_1 v_2 y_2 \), we have \( v_1 b = bw \) for some word \( w \). Now (4) implies

\[
x u_1^i x_1^i u_2^i = w^i y v_2^i y_2
\]

for \( i = 0, 1, 2, 3 \). By the induction hypothesis, (5) is true for each \( i = 0, 1, 2, 3, \ldots \), and we deduce that (4) is true for all \( i = 0, 1, 2, 3, \ldots \). \( \square \)

3. The system with at most three loops on both sides of the word equations

Let \( u_i, v_i, \ i = 1, 2, 3, \) and \( x_1, y_1, k = 0, 1, 2, 3, \) be words such that

\[
x_0 u_1^i x_1^i x_2 u_2^i x_3 = y_0 v_1^i y_1 v_2^i y_2 v_3^i y_3 \quad (i = 0, 1, 2, 3, \ldots).
\]

\([**]\)

To show that (**) is satisfied if \( x_0 u_1^i x_1^i x_2 u_2^i x_3 = y_0 v_1^i y_1 v_2^i y_2 v_3^i y_3 \) for \( i = 0, 1, 2, 3, 4, 5 \), we need several smaller results.

Lemma 8. Let \( u_i, v_i, \ i = 1, 2, 3, \) and \( x \) be words such that

\[
u_1^i x u_2^i u_3^i = v_1^i x v_2^i v_3^i
\]

for \( i = 1, 2, 3 \). Then (1) is true for each \( i = 1, 2, 3, \ldots \) .

Proof. We proceed induction on the length \( n \) of the word \( x \).

If \( n = 0 \), we are through by Theorem 2 in [5].

If \( u_1 = v_1 \), then \( u_2^i u_3^i = v_2^i v_3^i \) for \( i = 1, 2, 3 \). By Lemma 5, \( u_2^i u_3^i = v_2^i v_3^i \) for each \( i = 1, 2, 3, \ldots \), implying that \( u_1^i x u_2^i u_3^i = v_1^i x v_2^i v_3^i \) for each \( i = 1, 2, 3, \ldots \).

Assume that $n \neq 0$ and, without loss of generality, that $|u_1| > |v_1|$. By Lemma 4(a), all nonempty words in the set $\{u_1, v_1, x\}$ begin with the same symbol, $b$, say. Then there exist words $u, v$ and $z$ such that $bu = u_1 b$, $bv = v_1 b$ and $x = by$. Now (6) for $i = 1, 2, 3$ implies that

\[ u^i yu_2 u_3^i = v^i yv_2 v_3^i \]  \hspace{1cm} (7)

for $i = 1, 2, 3$. By the induction hypothesis, equality (7) is true for all $i = 1, 2, 3, \ldots$. We deduce that $bu^i yu_2 u_3^i = u_1 x u_2 u_3 = bv^i yv_2 v_3^i = v_1 x v_2 v_3^i$ for each $i = 1, 2, 3, \ldots$. \hfill \Box

A symmetric counterpart of the preceding lemma is, of course,

**Corollary 9.** Let $u_i, v_i$, $i = 1, 2, 3$, and $x$ be words such that

\[ u_1^i u_2 x u_3^i = v_1^i v_2 x v_3^i \]

for $i = 1, 2, 3$. Then $u_1^i u_2 x u_3^i = v_1^i v_2 x v_3^i$ is true for each $i = 1, 2, 3, \ldots$.

**Lemma 10.** Let $u_j, v_j$, $j = 1, 2, 3$, and $x$ be words such that

\[ u_1^i x u_2 u_3^i = v_1^i v_2 x v_3^i \]  \hspace{1cm} (8)

for $i = 1, 2, 3, 4$. Then (8) is true for each $i = 1, 2, 3, \ldots$.

**Proof.** We proceed by induction on the length $n$ of the word $u_1 x u_2 u_3^i (= v_1 v_2 x v_3^i)$.

If $|x| = 0$, we are through by Theorem 2 in [5].

If $u_i = e$ and $v_j = e$ for some $i, j \in \{1, 2, 3\}$, the lemma is true by Theorem 7.

Assume that $x \neq e$ and, without loss of generality, that $u_i \neq e$ for $i = 1, 2, 3$.

If $v_1 = e$, then Lemma 8 implies the result. Let $v_1 \neq e$.

For a while, suppose that the words $u_1$ and $v_1$ have the same primitive root. Let $r$ and $s$ be positive integers and $t$ a word such that $u_1 = t^r$ and $v_1 = t^s$. Assume, without loss of generality, that $r > s$. Then (8) implies

\[ i^{(r-s)} u_1^i u_2 u_3^i = v_1^i v_2 x v_3^i \]  \hspace{1cm} (9)

for $i = 1, 2, 3, 4$. By the induction hypothesis, equality (9) is true for all $i = 1, 2, 3, \ldots$.

If $|u_1| \leq 3 |v_1|$ and $|v_1| \leq 3 |u_1|$, we deduce by Lemma 3, that $u_1$ and $v_1$ are powers of the same (primitive) word. Assume thus that either (A) $|u_1| > 3 |v_1|$ or (B) $|v_1| > 3 |u_1|$.

Consider first the case (A). Suppose first, that $|u_1| > |v_1^i x v_2^i|$. By Lemma 4(b), the words $u_1$ and $x$ and all nonempty words in the set $\{v_1, v_2\}$ begin with the same symbol, $c$, say. Thus $x = c x_1$, $u_1 c = c u$, $v_1 c = c w_1$, and $v_2 c = c w_2$ for some words $x_1, u, w_1$ and $w_2$. Then (8) for $i = 1, 2, 3, 4$ implies

\[ u_1^i x u_2 u_3^i = w_1^i w_2 x_1 v_3^i \]  \hspace{1cm} (10)

for $i = 1, 2, 3, 4$. By the induction hypothesis, the equality (10) holds for each $i = 1, 2, 3, \ldots$, so we can write $u_1^i x u_2 u_3^i = c u_1^i x u_2 u_3^i = c w_1^i w_2 x_1 v_3^i = v_1^i v_2 x v_3^i$ for all $i = 1, 2, 3, \ldots$.\hfill \Box
Let $|u_1| \leq |v_1^2v_2|$. Since $|u_1| > 3|v_1|$, equality (8) for $i = 2$ implies that $u_1 = v_1^2d$ for some nonempty word $d \in \operatorname{pref}(v_2)$. Equality (8) for $i = 3$ allows us to deduce that $dv_1 = v_1d$. Thus $v_1$ and $d$, as well as $v_1$ and $u_1$ have the same primitive root.

Let us turn to the case (B).

Clearly $v_1 = u_3^2p$ for some word $p$. Equality (8) for $i = 4$ implies that there exists a word $q \in \operatorname{pref}(xu_2^2u_3^2)$ such that $u_1p = qu_1$. Equality (8) for $i = 3$ guarantees that $p \in \operatorname{pref}(xu_2^2u_3^2)$. If $|x| > |p|$, then $p - q$, the words $u_1$ and $p$, and thus the words $u_1$ and $v_1$ are powers of the same word, and we are through. Assume, that $|p| > |x|$. Since $x \in \operatorname{pref}(p)$ and $x \in \operatorname{pref}(q)$, the equality $u_1p = qu_1$ allows us to find a word $u$ such that $u_1u = xu$. We deduce that

$$

xu^i_1u_2^i = v_1^i v_2^i xu^i_3

$$

(11)

for $i = 1, 2, 3, 4$. Since $|xu| < |v_1|$, there exists a word $d$ such that $v_1 = xud$. Then $xu^i_1u_2^i = v_1^{i-1}(xud)v_2^i xu^i_3$ for $i = 1, 2, 3, 4$, so $v_1(xu) = (xu)v$ for some word $v$. Now (11) implies, for certain words $d_i$ that

$$

u^{i-1}_1u_2^i = v^{i-1}dd_i

$$

(12)

for $i = 1, 2, 3, 4$. Suppose that $|v| < |uv_2|$. Then $v = ut$ for some $t \in \operatorname{pref}(u_2)$ and (12) for $i = 3$ implies that $u^2t = utu$. Certainly $ut = tu$, so the words $u$ and $t$, and thus $u$ and $v$ have the same primitive root.

Assume that $v = uu_2z$ for some nonempty word $z$ (certainly shorter than $u_3$). Then we deduce from (12) that

$$

u^i_1u_2^i w_i = (uu_2z)^i

$$

(13)

for some word $w_i \in \operatorname{pref}(u_2u_3^2)$, when $i = 1, 2, 3$. Obviously $w_i = z^i$ for $i = 1, 2, 3$, so, by Theorem 1 in [5], the words $u_1, u_2, z$ and $v$ are powers of the same word.

In both of the cases above $u$ and $v$ have a common primitive root. Since $u_1x = xu$ and $v_1(xu) = (xu)v$ we find that also $u_1$ and $v_1$ have the same primitive root. □

**Corollary 11.** Let $u_j, v_j, j = 1, 2, 3, \text{and } x_k, y_k, k = 1, 2, \text{be words such that}

$$

u_1^i x_1 u_2^i x_2 u_3^i = v_1^i y_1 v_2^i y_2 v_3^i

$$

(14)

for $i = 0, 1, 2, 3, 4$. Then (14) is true for each $i = 0, 1, 2, 3, \ldots$.

**Proof.** By induction on length $n$ of the word $x_1x_2$. For $n = 0$, we are through by Theorem 2 in [5]. Let $n > 0$.

If $x_1 = y_2 = \varepsilon$ ($y_1 = x_2 = \varepsilon$, resp.), Lemma 10 implies our result. Assume, without loss of generality, that $x_1$ and $y_1$ are both nonempty.

If $u_1 = v_1$, then $x_1 u_2^i x_2 u_3^i = y_1 v_2^i y_2 v_3^i$ for $i = 0, 1, 2, 3$. By Theorem 7, $x_1 u_2^i x_2 u_3^i = y_1 v_2^i y_2 v_3^i$ for each $i = 0, 1, 2, 3, \ldots$, implying that $u_1^i x_1 u_2^i u_3^i = v_1^i x_1 v_2^i v_3^i$ for each $i = 0, 1, 2, 3, \ldots$.

Assume, without loss of generality, that $|u_1| > |v_1|$. By Lemma 4(a), all nonempty words in the set \{ $u_1, v_1, x_1, y_1$ \} begin with the same symbol, $b$, say. There then exist
words $u, v$ and $x, y$ such that $bu = u_1 b$, $bv = v_1 b$, $x_1 = bx$ and $y_1 = by$. Now (14) for $i = 0, 1, 2, 3, 4$ implies that
\[ u' x u_2 ^ i x_2 u_3 ^ i = v' y v_2 ^ i y_2 v_3 ^ i \]  
for $i = 0, 1, 2, 3, 4$. By the induction hypothesis, the equality (15) is true for all $i = 1, 2, 3, \ldots$. We deduce that $bu' x u_2 ^ i x_2 u_3 ^ i = v' y v_2 ^ i y_2 v_3 ^ i$ for each $i = 1, 2, 3, \ldots$. 

**Lemma 12.** Let $x, u, v_1, v_2$ and $v_3$ be words such that $x u_3 ^ i = v_1 ^ i v_2 ^ i v_3 ^ i x$. Then $x u_3 ^ i = v_1 ^ i v_2 ^ i v_3 ^ i x$ is true for each $i = 1, 2, 3, \ldots$.

**Proof.** The assumption $x u_3 ^ i = v_1 ^ i v_2 ^ i v_3 ^ i x$ implies that the words $u_3 ^ i$ and $v_1 ^ i v_2 ^ i v_3 ^ i$ are conjugate. By Lemma 1 there exists a (primitive) word $t = t_1 t_2$ and nonnegative integer $r$ such that $u = (t_1 t_2) r$, $v_1 ^ i v_2 ^ i v_3 ^ i = (t_1 t_2) r$ and $x(t_1 t_2) = x(t_1 t_2) r$. Theorem 1 in [3] implies that there exists $r_i \in \mathbb{N}$ such that $v_i = (t_1 t_2) r_i ^ i$ for $i = 1, 2, 3$ and $r_1 + r_2 + r_3 = r$. Certainly $x u_3 ^ i = x(t_1 t_2) r_i ^ i = (t_1 t_2) r_i ^ i x = v_1 ^ i v_2 ^ i v_3 ^ i x$ for all $i = 1, 2, 3, \ldots$.

**Lemma 13.** Let $u_i, v_i$, $i = 1, 2, 3$ and $x$ be words such that
\[ x u_1 ^ i u_2 ^ i u_3 ^ i = v_1 ^ i v_2 ^ i v_3 ^ i x \]  
for $i = 1, 2, 3, 4, 5$. Then (16) is true for each $i = 1, 2, 3, \ldots$.

**Proof.** If $u_i = v_j = v$ for some $i, j \in \{1, 2, 3\}$, Lemma 6 implies the result. Assume, without loss of generality, that $v_j$ is nonempty for $j = 1, 2, 3$. If any two of the words $u_1, u_2, u_3$ are empty, we are done by Lemma 12. Assume, again without loss of generality, that $u_1$ and $u_2$ are both nonempty.

We proceed by induction on the length $n$ of the word $x u_1 u_2 u_3$ ($= v_1 v_2 v_3 x$).

If $|x| = 0$, we are through by Theorem 2 in [5]. Assume that $x \neq v$.

Suppose that $|x| < |v_1 v_2|$. By Lemma 4(b), the words $x$ and $v_1$ and each nonempty word in $\{v_2, v_3\}$ begin with the same symbol, $c$, say. There thus exist words $y, w_1, w_2$ and $w_3$ such that $x = c y, v_j c = c w_j, j = 1, 2, 3$. Then (16) for $i = 1, 2, 3, 4, 5$ implies
\[ y u_1 ^ i u_2 ^ i u_3 ^ i = w_1 ^ i w_2 ^ i w_3 ^ i y \]  
for $i = 1, 2, 3, 4, 5$. By the induction hypothesis, (17) holds for all $i = 1, 2, 3, \ldots$ which allows us to deduce that $x u_1 ^ i u_2 ^ i u_3 ^ i = x y u_1 ^ i u_2 ^ i u_3 ^ i = x w_1 ^ i w_2 ^ i w_3 ^ i y = v_1 ^ i v_2 ^ i v_3 ^ i x$ for each positive integer $i$.

Let $|x| \leq |v_1 v_2|$. Assume for a while that there exist a (primitive) word $t = t_1 t_2$ and positive integers $r$ and $s$ such that $x(t_1 t_2) = (t_1 t_2) r$ and $v_i = (t_1 t_2) r_i ^ i$. Then (16) for $i = 1, 2, 3, 4, 5$ implies
\[ (t_1 t_2) r_i ^ i x u_1 ^ i u_2 ^ i u_3 ^ i = (t_1 t_2) r_i ^ i v_1 ^ i v_2 ^ i v_3 ^ i x \]  
(18)
for \( i = 1,2,3,4,5 \). Suppose that \( r \geq s \). Then (18) implies that
\[
x(t_2 t_1)^{(r-s)k} u_2^i u_3^j = v_2^i v_3^j x
\]
for \( i = 1,2,3,4,5 \). By the induction hypothesis, the equality (19) holds for all \( i = 1,2,3,\ldots \). Certainly \( x u_1^i u_2^i u_3^j = x(t_2 t_1)^{(r-s)k} u_2^i u_3^j = v_2^i v_3^j x \) for \( i = 1,2,3,\ldots \). Let \( r < s \). Then (18) for \( i = 1,2,3,4,5 \) implies
\[
x u_2^i v_2^i = (t_1 t_2)^{(s-r)k} v_2^i v_3^j x
\]
for \( i = 1,2,3,4,5 \). Again by the induction hypothesis, (20) is true for each \( i = 1,2,3,\ldots \). We deduce that \( x u_1^i u_2^i u_3^j = (t_1 t_2)^{(s-r)k} v_2^i v_3^j x \) for \( i = 1,2,3,\ldots \).

The basic cases in the length considerations that follow are (A) \( |u_1| > |v_1| \) and (B) \( |u_1| \leq |v_1| \).

Consider first case (A). We shall show that \( x u_1^i \in \text{pref}(v_1^r) \). If \( |x u_1^i| \leq |v_1^r| \), we are done. Assume that \( |x u_1^i| > |v_1^r| \). Since \( |x u_1^i| \leq |v_1^r| \), we have \( |x u_1^i| \leq |v_1^r| \). The equality (16) for \( i = 4 \) implies that \( x u_1^4 = v_1^4 d \) where \( d \in \text{pref}(v_1^4) \). Equality (16) for \( i = 3 \) guarantees that \( v_1 d = dv \) for some word \( v \). Certainly \( d \in \text{pref}(v_1^4) \), so \( x u_1^4 \in \text{pref}(v_1^4) \). Since \( |u_1| > |v_1| \), Lemma 3 implies that \( u_1 \) and \( v_1 \) are conjugate. Moreover, there exist a (primitive) word \( t = t_1 t_2 \) and positive integers \( r \) and \( s \), with \( r > s \), such that \( x(t_2 t_1)^r = (t_1 t_2)^r \).

Let us turn to the case (B).

Eq. (16) implies for \( i = 2 \) that there exists a word \( d \) such that \( x u_1^i d = v_1^i v_2^i \). Then the equation with index \( i \) in (16) guarantees the existence of a word \( d_i \) such that
\[
x u_1^i u_2^i u_3^j = v_1^{i-2} x u_1 d d_i
\]
for \( i = 3,4,5 \). There clearly exists a word \( w \) for which \( x w = v_1 x \). Thus, by denoting \( w_i = u_1 d d_i \), we deduce from (21) that
\[
u_1^i u_2^i u_3^j = w^{i-2} w_i
\]
for \( i = 3,4,5 \).

Assume that \( u_3 = e \). If \( 4|u_1| \geq |w| + |u_1| \), then by Lemma 3, the words \( u_1 \) and \( w \) are powers of the same (primitive) word. Let \( |w| > 3|u_1| \). The equality (22) for \( i = 3 \) implies that \( w = u_1^3 d_1 \) for some nonempty word \( d_1 \in \text{pref}(u_1^3) \). Now (22) for \( i = 4 \) implies that \( u_1 d_1 = d_1 u_1 \), which means that \( d_1, u_1 \) and \( w \) are all powers of the same primitive word. In both of the cases above, \( u_1 \) and \( v_1 \) are conjugate. Moreover, there exist a (primitive) word \( t = t_1 t_2 \) and positive integers \( r \) and \( s \), such that \( x(t_2 t_1)^r = (t_1 t_2)^r \).

Note that the lemma is now proved in the case that either \( u_j = e \) or \( v_j = e \) for some \( j \in \{1,2,3\} \). Even more can be said: only the assumptions that (16) is true for each \( i = 1,2,3,4 \) are necessary.

Let \( u_3 \neq e \). Suppose that \( 5|u_1| \geq |w| + |u_1| \). Reasoning exactly as in the case \( u_2 = e \) and \( 4|u_1| \geq |w| + |u_1| \), one can deduce that (16) is true for all \( i \) in \( \mathbb{N}_+ \).
Assume that $|w| > 4|u_1|$. Then $w = u_1^4 p$ for some word $p$. From (22) we deduce
\[ u_1^3 u_2^3 = u_1 p w_1, \]
\[ u_2^4 u_3^4 = pu_1^4 p w_2. \] (23)
\[ u_1 u_2^4 u_3^4 = pu_1^4 pu_1^4 p w_3. \]
The last equality of (23) implies the existence of a word $q$ such that $pu_1 = u_1 q$. Then (23) gives
\[ u_2^4 u_3^4 = pu_1^4 p w_2, \]
\[ u_2^4 u_3^4 = qu_1^4 pu_1^4 p w_3. \] (24)
If $|p| < 4|u_2|$, then $p = q$, the words $u_1$ and $p$ as well as $u_1$ and $w$ have the same primitive root and we are through as above.

Suppose that $|p| > 4|u_2|$. Then $w = u_1^4 u_2^4 z$ for a word $z \in \text{pref}(u_1^4)$. The third equality of (23) implies the existence of a word $u_{12}$ such that $u_1 u_{12} = u_2 u_{12}$. Clearly, by the first and third equalities of (23), both $u_1$ and $u_{12}$ are prefixes of words starting with $u_2^4$. If $|u_1| < 3|u_2|$, then $u_1 = u_{12}$, and, by Lemma 2, the words $u_1, u_2$ are powers of the same (primitive) word. Then the words $u_1$ and $u_2$ can be unified to form only one loop and (16) is true for each positive integer $i$, since the case where $u_j = e$ for some $j \in \{1, 2, 3\}$ was considered above.

Assume that $|u_1| > 3|u_2|$. Then $u_1 = u_2^3 y$ for some word $y \in \text{pref}(u_2^3)$. The substitution of $w = u_1^4 u_2^4 z$ and $u_1 = u_2^3 y$ into (22) for $i = 5$ implies that
\[ y u_2^4 z = u_2 z u_2^3 y u_2 \] (25)
when appropriate prefixes on both side of the equality are considered. Now, since $z \in \text{pref}(u_2^3)$ and $y \in \text{pref}(u_2^3)$, either $z$ is a prefix of $y$ or vice versa. Assume that $z = y z_1$ for some word $z_1$. Obviously (25) implies that $yu_2 = u_2 y$, so $u_2$ and $y$ commute. Let then $y = z y_1$ for some word $y_1$. Again, by (25), $u_2 z = z u$ for some word $u$ and, moreover, $yz u_2^4 = u_2 z u_2^3 y u_2$, so $u = u_2$. We now have $yu_2^4 z = u_2^4 z y$. Since $u_2$ and $z$ commute, $u_2$ and $y$ commute, too. In both cases $u_2 y = y u_2$, which implies, by Lemma 2, that $u_2$ and $y$ as well as $u_1$ and $u_2$, are powers of the same (primitive) word. As above, the words $u_1$ and $u_2$ can be unified to form only one loop. Once more, equality (16) is true for each $i = 1, 2, 3, \ldots$. The proof is now complete. \(\square\)

Suppose now that in Lemma 13, the word $u_3 = e$. Then, as mentioned in the proof, to prove the lemma, it is sufficient to assume that $x u_i^j u_2^j = v_i^j v_2^j v_3^j x$ only for $i = 1, 2, 3, 4$. All the results that are referred use at most four equalities. We can thus write

**Corollary 14.** Let $u_i, v_j$, $i = 1, 2$, $j = 1, 2, 3$ and $x$ be words such that
\[ x u_i^j u_2^j = v_i^j v_2^j v_3^j x \]
for $i = 1, 2, 3, 4$. Then $x u_i^j u_2^j = v_i^j v_2^j v_3^j x$ for each $i = 1, 2, 3, \ldots$. \[\square\]
Lemma 15. Let $u_j, v_j, j = 1, 2, 3$, and $x_0, x_1, y_2, y_3$ be words such that

$$x_0u'_1x_1u'_2u'_3 = v'_1v'_2y_2v'_3y_3$$

(26)

for $i = 0, 1, 2, 3, 4, 5$. Then (26) is true for each $i = 0, 1, 2, 3, 4, 5$.

Proof. By induction on the length $n$ of the word $|x_0x_1| (= |y_2y_3|)$.

For $n = 0$, Theorem 2 in [5] implies the result. Let $n > 0$.

We have two cases: (A) $x_1 \neq \varepsilon$ and $y_2 \neq \varepsilon$; and (B) $x_1 = \varepsilon$ or $y_2 = \varepsilon$.

Consider case (A) Suppose first that either $|x_0u'_1| > |v'_1|$ or $|v_3y'_3| > |u'_3|$. Without loss of generality, let $|x_0u'_1| > |v'_1|$. Also assume that $x_0 \neq \varepsilon$. The case $x_0 = \varepsilon$ is fully analogous. By Lemma 4(a), the words $x_0, y_2$ and nonempty words in $\{v'_1, v'_2\}$ begin with the same symbol, $c$, say. Thus $x_0 = cx_1, v'_1 = c w'_1, v'_2 = c w'_2$ and $y_2 = cy$ for some words $x_0, w'_1, w'_2$ and $y$. Then (26) is equivalent with

$$xu'_1x_1u'_2u'_3 = w'_1w'_2v'_3y_3$$

(27)

for $i = 0, 1, 2, 3, 4, 5$. By the induction hypothesis, the equality (27) holds for each natural number $i$ implying that (26) is true for each $i \in \mathbb{N}$.

Assume that $|x_0u'_1| \leq |v'_1|$ and $|v_3y'_3| \leq |u'_3|$. There then exist words $d'_1$ and $d'_2$ such that $v'_1 = x_0u'_1, d'_1$ and $u'_3 = d'_2v_3y'_3$. Then (26) implies

$$u'_1^{-1}x_1u'_2d'_2(v_3y'_3d'_2)^{i-1} = (d'_1x_0u'_1)^{i-1}d'_1v'_2y'_2v'_3^{i-1}$$

(28)

for $i = 1, 2, 3, 4, 5$. Denote $u = v_3y'_3d'_2$ and $v = d'_1x_0u'_1$, so

$$u'_1v'_1u'_2(u'_2d'_2)w'_j = v'(d'_1v'_2)v'_2v'_3^{i-1}$$

(29)

for $j = 0, 1, 2, 3, 4$. Corollary 11 now implies that (29) and thus (26) is true for each $i = 0, 1, 2, 3, 4, 5$.

Let us now turn to case (B). If $x_1 = y_2 = \varepsilon$, we are through by Lemma 13.

Assume, without loss of generality, that $x_1 = \varepsilon$ and $y_2 \neq \varepsilon$.

If $|x_0u'_1| > |v'_1|$, we proceed exactly as in (27) above and deduce by the induction hypothesis that (26) is true for each natural number $i$.

Let $|x_0u'_1| \leq |v'_1|$. Then $v'_1 = x_0u'_1, d$ for some word $d$ and (26) implies that there exist a word $w$ such that $w = d_xu'_1$ and

$$u'_1^{-1}u'_2u'_3 = w'^{-1}d v'_2y'_2v'_3^{i-1}$$

(30)

for $i = 1, 2, 3, 4, 5$. Let $z_1 = d, z_2 = v'_2y'_2$ and $z_3 = v'_3y'_3$. Then we have

$$u'_1u'_2z_1z_2z_3u'_3 = w'^{i}z_1v'_2z_2v'_3^{i-1}z_3$$

(31)

for $i = 0, 1, 2, 3, 4$. It is easy to see that $z_3u'_3 = w'_3z_3$ for some word $w'_3$ and (31) implies

$$u'_1u'_2z_1z_2w'_3 = w'^{i}z_1v'_2z_2v'_3^{i}$$

(32)

for $i = 0, 1, 2, 3, 4$. Corollary 11 now implies that equality (32) and thus equality (26) is true for each $i \in \mathbb{N}$.
Theorem 16. Let $u_j, v_j$, $j = 1, 2, 3$, and $x_k, y_k$, $k = 0, 1, 2, 3$, be words such that
\[
x_0 u'_1 x_1 u'_2 x_2 u'_3 x_3 = y_0 v'_1 y_1 v'_2 y_2 v'_3 y_3
\]
for $i = 0, 1, 2, 3, 4, 5$. Then (33) is true for each $i = 0, 1, 2, \ldots$.

Proof. By induction on the length $n$ of the word $x_0 x_1 x_2 x_3 (= y_0 y_1 y_2 y_3)$.

For $n = 0$, Theorem 2 in [5] is once more applied. Let $n > 0$.

Suppose first that $x_0 x_1 \neq \varepsilon$ and $y_0 y_1 \neq \varepsilon$.

If $x_0$ and $y_0$ are both nonempty, the induction is trivially extended. Assume, without loss of generality, that $y_0 = \varepsilon$. Thus $y_1 \neq \varepsilon$.

If $x_0$ is empty, techniques of Corollary 11 can be applied to shorten the word $x_0 x_1$.

Assume that $x_0 \neq \varepsilon$. There then exist a letter $b$ and words $u, v$ and $y$ such that $x_0 = bx$, $y_1 = by$ and $v_1 b = bv$. Then (33) implies
\[
x u'_1 x_1 u'_2 x_2 u'_3 x_3 = v' y v'_1 y_2 v'_3 y_3
\]
for $i = 0, 1, 2, 3, 4, 5$. By the induction hypothesis, Eq. (34), and thus (33) is true for all natural numbers.

We can thus make the assumption that either $x_0 x_1$ or $y_0 y_1$ ($x_2 x_3$ or $y_2 y_3$, resp.) is empty.

Suppose, without loss of generality, that $x_2 x_3 = y_0 y_1 = \varepsilon$. The claim is now true by Lemma 15. \(\square\)

4. Conclusions

The system of word equations (**) was shown to be equivalent with its finite sub-system
\[
x_0 u'_1 x_1 u'_2 x_2 u'_3 x_3 = y_0 v'_1 y_1 v'_2 y_2 v'_3 y_3 \quad (i = 0, 1, 2, 3, 4, 5).
\]

We do not know whether or not the four or five first equations of (†) are sufficient to imply (**). Certainly at least the three first are necessary (see the note after Lemma 6).

The structure of solutions of (**) (or (*)) is not very complicated. This question as well as the question concerning the applications of the previous results to the test sets are answered in forthcoming papers.

When studying (*) in the general case, more advanced techniques than those applied above should be derived. The research work on the topic is continued in the future.

References


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