EXTENSION OF LINEAR SELECTORS OF LINEAR FUZZY MULTIVALUED OPERATORS

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Abstract. We prove an extension theorem for linear selectors of linear fuzzy multivalued operators.

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1. Introduction

One of the most famous and well cited results in mathematics is the classical Hahn-Banach theorem regarding the extension of linear operators (see Buskes [8]). Linear multifunctions/multivalued operators made their first appearance in functional analysis in Neumann [14], motivated by the need to consider adjoints of nondensely defined linear differential operators. The theory of linear multivalued operators would seem to have the potential for contributing to the enrichment and clarification of many aspects of operator theory. During the past six decades the theory of linear multivalued operators has developed rapidly and different aspects of the theory of linear multivalued operators were investigated extensively (see [2], [3], [10], [12], [16], [17], [20] and references therein). Study of selectors of multivalued operators is an active area of research with significant applications (see [9], [11], [13], [15], [18], [19]). Recently Albricht and Maltoka [1], Tsiporkova-Hristoskova, De Baets and Kerre [21] and Beg [5] started the study of fuzzy multifunctions. Beg [4] proved existence of an extension for fuzzy positive linear
operators and generalized classical Hahn-Banach theorem. Subsequently Beg [6] has obtained results for existence of selectors of fuzzy multifunctions, fuzzy analogue of the results of Knaster [13], Tarski [19] and Smithson [18]. The main aim of this paper is to study the extension of selectors of linear fuzzy multivalued operators defined on vector spaces.

2. Preliminaries

Let $X$ be an arbitrary (nonempty) set. A fuzzy set (in $X$) is a function with domain $X$ and values in $[0,1]$. If $A$ is a fuzzy set and $x \in X$, the function value $A(x)$ is called the grade of membership of $x$ in $A$. A fuzzy set $A$ is called nonempty if $A(x) = 0$ for all $x$ in $X$. Let $A$ and $B$ be fuzzy sets in $X$. We write $A \subseteq B$ if $A(x) \leq B(x)$ for each $x \in X$. For any family $\{A_i\}_{i \in I}$ of fuzzy sets in $X$, we define

$$\bigcap_{i \in I} A_i(x) = \inf_{i \in I} A_i(x).$$

Definition 2.1. A family $B$ of fuzzy subsets of a set $X$ is said to have property (I): every subfamily $B_0$ having the property that $A \cap B \neq \emptyset$ for all $A, B \in B_0$, has a nonempty intersection.

A fuzzy multifunction $F$ from a set $X$ into a set $Y$ assigns to each $x$ in $X$ a fuzzy subset $F(x)$ of $Y$. We denote it by $F : X \rightarrow Y$. We can also identify $F$ with a fuzzy subset $F$ of $X \times Y$ with $F(x,y) = [F(x)](y)$. If $A$ is a fuzzy subset of $X$, then the fuzzy set $F(A)$ in $Y$ is defined by

$$[f(A)](y) = \sup_{x \in X} [F(x,y) \wedge A(x)].$$

If $F$ is a fuzzy multifunction between $X$ and $Y$, then the set $D_F = \{x \in X : F(x)$ is nonempty$\}$ is called the domain of $F$. If $D_F = X$, then $F$ is called a fuzzy multifunction from $X$ into $Y$. If $\text{Domain}(F) \neq X$, then $F$ is called a fuzzy multifunction between $X$ and $Y$. $F$ is called a fuzzy function if $F(x,y) = F(x,z) \neq 0$ implies $y = z$. In particular if $F$ is a fuzzy function and $[F(x)](y) \neq 0$, then we may write $F(x) = y$. We use capital letters for fuzzy multifunctions and small ones for (single valued) fuzzy functions.

Let $F$ and $T$ be fuzzy multifunctions from $X$ into $Y$ such that $T \subset F$, or equivalently $T(x) \subset F(x)$ for all $x \in X$, then $T$ is a selector fuzzy function or a selector fuzzy multifunction of $F$. The selector fuzzy multifunction of the restriction $F|_Z = F \cap (Z \times Y)$ of $F$ to a subset $Z$ of $X$ is called partial fuzzy selector of $F$. For further details we refer to [1], [5], [10], [21], [22].

Let $X$ be an additive group (with $+$ as binary operation) and $A, B$ fuzzy subsets of $X$ then

$$A + B = \{x + y : x \in A, y \in B\},$$
with, 
\[(A + B)(z) = \sup_{z=x+y} \inf \{A(x), B(y)\}.\]

Also, 
\[-A = \{-x : A(x) \neq 0\},\]

with 
\[-A(-x) = A(x)\]

and 
\[A - B = A + (-B).\]

If \(X\) is a vector space (over \(\Gamma\)) then we write \(\lambda A = \{\lambda x : A(x) \neq 0\}\) for all \(\lambda \in \Gamma\) with \(\lambda A(\lambda x) = Ax\).

A fuzzy multifunction \(F\) between groups \(X\) and \(Y\) is called superadditive (resp. subadditive) if

\[F(x) + F(y) \subset F(x + y) \quad (\text{resp. } F(x + y) \subset F(x) + F(y)),\]

for all \(x, y\) in \(X\), and additive if it is both superadditive and subadditive. Moreover, \(F\) is said to be odd if \(F(-x) = -F(x)\) for all \(x\) in \(X\).

**Definition 2.2.** Let \(X\) and \(Y\) be two groups and \(Z\) be a subgroup of \(X\). If \(F : X \rightrightarrows Y\) is a fuzzy multifunction and \(T : Z \rightrightarrows Y\) then the fuzzy multifunction \(F * T\) defined by,

\[(F * T)(x) = \bigcap_{z \in Z} [F(x - z) + T(z)],\]

for all \(x \in X\), is called the intersection convolution of \(F\) and \(T\).

The fuzzy multifunction \([F + T(0)] : X \rightrightarrows Y\) is defined by,

\[(F + T(0))(x) = F(x) + T(0).\]

Similarly, \([F(0) + T] : Z \rightrightarrows Y\) is defined by

\[(F(0) + T)(z) = F(0) + T(z).\]

A fuzzy multifunction \(F\) between vector spaces \(X\) and \(Y\) (over \(\Gamma\)) is called fuzzy multivalued operator. The fuzzy multivalued operator \(F\) is said to be homogeneous if \(F(\lambda x) = \lambda F(x)\) for all nonzero \(\lambda \in \Gamma\) and \(x \in X\). The fuzzy multivalued operator \(F\) is homogeneous if and only if \(\lambda F(x) \subset F(\lambda x)\) (or equivalently \(F(\lambda x) \subset \lambda F(x)\)) for all nonzero \(\lambda \in \Gamma\) and \(x \in X\). A homogeneous fuzzy multivalued operator \(F\) is called superlinear, sublinear and linear if it is superadditive, subadditive and additive respectively.

We now state following theorems (slightly modified) due to Beg [7] for subsequent use in section 3.
Theorem 2.3. Let $X$ and $Y$ be two groups and $Z$ be a subgroup of $X$. Let $F : X \leadsto Y$ be a fuzzy multifunction, $T$ a subadditive selector fuzzy multifunction of $F|_Z$ and $T$ can be extended to a subadditive selector fuzzy multifunction of $F$ then $(F * T)(x)$ is nonempty for all $x \in X$.

Theorem 2.4. Let $X$ and $Y$ be two groups, $Z$ be a subgroup of $X$ and $F : X \leadsto Y$ be a superadditive fuzzy multifunction. If $T$ is a selector fuzzy multifunction of $F|_Z$ with $0 \in T(0)$, then $F = F + T(0)$.

Theorem 2.5. Let $X$ and $Y$ be two groups, $Z$ be a subgroup of $X$ and $F : X \leadsto Y$ be a superadditive fuzzy multifunction. If $T$ is a subadditive selector fuzzy multifunction of $F|_Z$ and $0 \in T(0)$ then $F = F * T$.

Theorem 2.6. Let $X$ and $Y$ be two vector spaces (over $\Gamma$) and $Z$ be a subspace of $X$. Let $F : X \leadsto Y$ be a homogeneous fuzzy multivalued operator, $T$ be a linear selector fuzzy multivalued operator of $F|_Z$ and $G \subset F + T(0)$ be a linear fuzzy multivalued operator with $T = G|_Z$. If $(F * G)(x)$ is nonempty for all $x \notin D_G$ then there exists a linear selector fuzzy multivalued operator $S$ of $F + T(0)$ with $T = S|_Z$.

3. Extension of linear selectors

Throughout this section $X$ and $Y$ denote vector spaces (over $\Gamma$) and $Z$ a subspace of $X$.

Theorem 3.1. Let $F : X \leadsto Y$ be a linear fuzzy multivalued operator and $T$ be a linear selector fuzzy multivalued operator of $F|_Z$ then there exists a linear selector fuzzy multivalued operator $S$ of $F$ such that $T = S|_Z$.

Proof. Since $0 \in T(0)$ therefore Theorem 2.4 implies $F = F + T(0)$. It further implies that $F(x)$ is nonempty. Thus if $G \subset F + T(0)$ and $G$ is a linear fuzzy multivalued operator then by Theorem 2.5 we obtain, $(F * G)(x) = F(x) \neq \phi$ for all $x \in X$. By Theorem 2.6, there exists a linear selector fuzzy multivalued operator $S$ of $F + T(0)$ with $T = S|_Z$.

Corollary 3.2. Let $F : X \leadsto Y$ be a linear fuzzy multivalued operator and $t$ be a linear selector (single valued) fuzzy operator of $F|_Z$, then there exists a linear selector (single valued) fuzzy operator $s$ of $F$ such that $t = s|_Z$. In particular each linear fuzzy multivalued operator has a linear selector (single valued) fuzzy operator.

Theorem 3.3. Let $F : X \leadsto Y$ be a homogeneous fuzzy multivalued operator, $t$ be a linear selector fuzzy operator of $F|_Z$ and for every linear fuzzy operator $s \subset F$ such that $t = s|_Z$, we have $(F * s)(x) \neq \phi$ for $x \notin D_s$ then there exists a linear selector fuzzy operator $f$ of $F$ such that $t = f|_Z$. 
Proof. Since $t$ is a linear selector fuzzy operator so $F = F + t(0)$. Hence Theorem 2.6 implies that there exists a linear selector fuzzy multivalued operator $S$ such that $t = S|z$. It further implies that $S$ is necessarily single valued fuzzy operator.

Theorem 2.3 implies the following theorem.

**Theorem 3.4.** Let $F : X \sim Y$ be a homogeneous fuzzy multivalued operator and for every linear fuzzy multivalued operator $T \subset F$ we have a linear selector fuzzy multivalued operator $S$ of $F + T(0)$ such that $T = S|z$. If $s \subset F$ is a linear fuzzy operator then $(F * s)(x)$ is nonempty for all $x \notin D_s$.

**Theorem 3.5.** Let $F : X \sim Y$ be a homogeneous fuzzy multivalued operator and for every linear fuzzy operator $s \subset F$, $(F * s)(x)$ is nonempty for all $x \notin D_s$. If $T \subset F$ is a linear fuzzy multivalued operator then there exists a linear selector fuzzy multivalued operator $S$ of $F + T(0)$ such that $T = S|z$.

**Proof.** Let $T : Z \sim Y$ be a linear fuzzy multivalued operator. Corollary 3.2 implies that there exists a linear selector fuzzy operator $t$ of $T$. Therefore if $T \subset F$ then $(F * t)(x)$ is nonempty for all $x \notin D_t$. Theorem 3.3 further implies that there exists a linear selector operator $s$ of $F$ with $t = s|z$.

Define $G : X \sim Y$ a linear fuzzy multivalued operator by

$$G(x) = (s + T(0))(x).$$

Then,

$$G(x) = (s + T(0))(x) \subset F(x) + T(0) = (F + T(0))(x)$$

for all $x \in X$. And

$$G(z) = s(z) + T(0) = t(z) + T(0) = T(z)$$

for all $z \in Z$. Hence $G$ is the desired linear selector fuzzy multivalued operator.

**Theorem 3.6.** [Fuzzy Hanhn-Banach Thorem] Let $F : X \sim Y$ be a sublinear fuzzy multivalued operator. Let $\mathcal{B}$ be a translation invariant family of fuzzy subsets of $Y$ having property (I), and $F(x) \in \mathcal{B}$ for all $x \in X$. Then for every linear selector fuzzy multivalued operator $T$ of $F|z$ there exists a linear selector fuzzy multivalued operator $S$ of $F + T(0)$ with $T = S|z$.

**Proof.** Let $t \subset F$ be a linear fuzzy operator. If $x \in X$ and $z, w \in Z$, then

$$0 \in (F(w - z) - t(w - z)) = F((x - z) - (x - w)) - t(w - z) \subset (F(x - z) - F(x - w)) - (t(w) - T(z)) = (F(x - z) + t(z)) - (F(x - w) + t(w)).$$
It implies that
\[(F(x - z) + t(z)) \cap (F(x - w) + t(w)) \neq \phi.\]

Property (I) further implies that,
\[\bigcap_{z \in D_T} (F(x - z) + t(z)) \neq \phi.\]

Thus \((F \ast t)(x) \neq \phi.\)

Theorem 3.5 implies that there exists a linear selector fuzzy multivalued operator \(S\) of \(F + T(0)\) with \(T = S|_Z\).

**Corollary 3.7.** If \(F\) and \(B\) are as in Theorem 3.6 and \((F(0))(0) \neq 0\), then there exists a linear selector fuzzy operator \(s\) of \(F\).

**Remark 3.8.** Theorem 3.6 is a fuzzy analogue of generalized classical Hahn-Banach extension theorem (for multifunctions/multivalued operators).

4. Conclusion.

We expect that fuzzy Hahn-Banach Theorem will have wide applicability specially towards proving different equivalent form of Hahn-Banach theorem in fuzzy setting. More over it will also have applications in fuzzy optimization theory and economics.

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**References**


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