Belief Aggregation in Fuzzy Framework

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Abstract:
We explore how belief aggregation in the fuzzy framework can be molded into an optimization problem which helps avoid paradoxical outcomes without the fear of indecision. We further illustrate that depending on the choice of \( t \)-norm and fuzzy implication, we can find aggregation functions that produce collectively rational outcome without compromising on systematicity.

Keywords:
Belief Aggregation; triangular norms; triangular implications.

1. Introduction

There are large bodies of work on judgment aggregation motivated by List and Petit’s initial contribution. Using classical propositional logic, List and Petit [10] formalized judgment aggregation and proved the first social choice theoretic impossibility result similar to those of Arrow [1] and Sen’s [13] impossibility theorems. Subsequently, several impossibility theorems were proved concluding that there is no non-dictatorial aggregation function that satisfies Collective Rationality, universal Domain, Systematicity and Anonymity simultaneously. Pigozzi [11] using distance based approach in belief merging framework attempted to highlight all possible interpretations which had the least distance from a profile. That is, the interpretations that best represented the choices of the individuals. This method lead to indecision in cases where more than one interpretations qualified to have the least distance from the profile. Even in cases where collective rationality was achieved, it was at the expense of systematicity; since certain propositions known as integrity constraints were given preference over other propositions in the agenda. The crux of the work done in classical propositional logic represents the fact that Doctrinal Paradox persists to exist and in the covet of attaining a collectively rational outcome, systemsaticity or anonymity has to be sacrificed.

In the framework of classical two-valued logic individuals are restricted to opt for a Yes or a No even when they do not completely agree or disagree with a proposition representing a particular idea. Van Hees [14] specified several generalizations of the
paradox using multivalued logic. He further showed that allowing degrees of belief using multivalued logic does not ensure collective rationality. Duddy and Piggins [8] presented a general model in which judgments on propositions were not binary but came in degrees. Triangular norms were used to define totally blocked agendas along with some results. Beg and Butt [2] presented impossibility theorems in fuzzy framework where fuzzy aggregation operator known as LAMA operator was used which satisfied Arrow’s axioms of social choice but failed to ensure collective rationality. Dietrich and List [7] presented a general theory of propositional attitude aggregation and proved two theorems which are of considerable importance. Non-binary beliefs however were represented by probability functions.

The restriction of classical propositional calculus to a two-valued logic has created many paradoxes. For example, the Optimist’s conclusion (is the glass half-full or half-empty when the volume is at 500 milliliters). Is the liter-full glass still full if we remove one millimeter of water, 2, 3 or hundred milliliters? Unfortunately no single milliliter of liquid provides for a transition between full and empty glass. This transition is gradual so that as each milliliter of water is removed, the truth value of the glass being full gradually diminishes from a value 1 at 1000 milliliters to 0 at 0 milliliters.

In most decision making problems propositions representing certain situations are vague. A fuzzy proposition is a statement involving some concept without clearly defined boundaries; statements that tend to express ideas that can be interpreted differently by various individuals. In this paper, using a fuzzy logic framework individuals give degree of truthfulness on crisp or fuzzy propositions. This degree of truthfulness is said to be the belief of the decision maker. We present certain conditions under which allowing degrees of truth values to individuals using fuzzy framework ensures collective rationality without having to compromise on systematicity or anonymity. The structure of paper is as follows: Section 2 defines preliminaries. Section 3 presents an example of the doctrinal paradox and its reformulation in the fuzzy framework. Section 4 illustrates how the paradox is avoided by molding the problem into an optimization problem where a unique optimal fuzzy solution is achieved using a distance based approach in the fuzzy framework. Section 5 uses the notion of implication preservation and explains how using a specific class of t-norms and fuzzy implications, collective rationality is achieved without violating systematicity or anonymity. Section 6 summarizes the topic and presents some direction of future interests.

2. Preliminaries

We assume that decision makers are rational and they have the freedom to express their opinions on a proposition with which they do not completely agree or disagree. So any number in the interval [0,1] that best represents their opinions can be opted; 0 representing complete disagreement and 1 representing full agreement with the notion expressed by the proposition. Let $N = \{1, 2, \ldots, n\}$ denote a finite set of individual decision makers where $n \geq 1$. Let $\phi$ be a finite set of atomic propositions $p$, $q$ etc. the set of all propositions $\phi_0$ is obtained by closing $\phi$ under the fuzzy connective $t$-norm ($\Delta$) and fuzzy negation ($\eta$). Thus $\phi \subseteq \phi_0$ and $\forall p, q \in \phi: \eta(p) \in \phi, p \Delta q \in \phi$. Let $X$
be a non-empty subset of \( \phi_0 \) and contains all propositions about which the decision makers have to make a decision.

A fuzzy global valuation is a function \( v' : \phi_0 \to [0,1] \) that satisfies the following conditions:

1. \( v'(\eta(\phi)) = \eta(v'(\phi)) = 1 - v'(\phi) \)
2. \( v'(\phi \Delta \psi) = \Delta(v'(\phi), v'(\psi)) \)

Let \( V' \) represent the set of all fuzzy global valuations. A valuation is a function \( v : X \to [0,1] \) for which there is some \( v' : \phi_0 \to [0,1] \) in \( V' \) such that the restriction of \( v' \) to \( X \) is \( v \). Let \( V \) represent the set of all valuations. A fuzzy aggregation function \( A : V^N \to V \) returns for each \( n \)-tuple of valuations \((v_1, v_2, \cdots, v_n)\) an aggregated valuation \( A(v_1, v_2, \cdots, v_n) \). A decision method \( D \) is a function from the set \([0,1]_n\) to the set \([0,1]\).

**Definition 2.1** (Minimal Agenda Richness). The agenda \( X \) contains at least two distinct propositions \( p, q \) as well as \( p \Delta q \) and \( \eta(p \Delta q) \).

**Definition 2.2** (Anonymity). For any permutation \( f : N \to N \), any valuation profile \((v_1, v_2, \cdots, v_n) \in V^n \) and any proposition \( p \in X \),

\[
A(v_1, v_2, \cdots, v_n)(p) = A(v_{f(1)}, v_{f(2)}, \cdots, v_{f(n)})(p).
\]

**Definition 2.3** (Systematicity). A fuzzy aggregation function \( A : V^n \to V \) is systematic if and only if there is a decision method \( D : [0,1]^n \to [0,1] \) such that for all \((v_1, v_2, \cdots, v_n) \in V^n \) and for all \( p \in X \),

\[
A(v_1, v_2, \cdots, v_n)(p) = D(v_1(p), v_2(p), \cdots, v_n(p)).
\]

**Definition 2.4** (Majority voting). Given an agenda \( X \) which is closed under negation, an aggregation function \( F \) is a majority aggregation function if it assigns to a profile of individual judgment sets \( \{A_1, A_2, \cdots, A_n\} \), \( (n \geq 3) \), a collective judgment set \( A \) which contains propositions from the agenda that are accepted by at least half of the members.

\[
F(A_1, A_2, \cdots, A_n) = \left\{ p : p \in A_k \text{ where } k \geq \frac{n}{2} \right\}.
\]

**Definition 2.5** (Dictatorship). There is some fixed \( i \in N \) such that for all \((v_1, v_2, \cdots, v_n) \in V^n \),

\[
A(v_1, v_2, \cdots, v_n)(p) = v_i(p).
\]
**Definition 2.6** (Triangular Norm). A triangular norm ($\circ$-norm) is a binary operation $\Delta: [0,1] \times [0,1] \rightarrow [0,1]$ satisfying;

1. $\Delta(x,x) = x$. (boundary condition)
2. $\Delta(y,x) = \Delta(x,y)$. (commutativity)
3. $\Delta((x\Delta y), z) = (x\Delta y)\Delta z$. (associativity)
4. If $w \leq x$ and $y \leq z$ then $w\Delta y \leq x\Delta z$. (monotonicity)

**Definition 2.7** (Fuzzy implication). Fuzzy implication is a function $\zeta: [0,1] \times [0,1] \rightarrow [0,1]$. From axiomatic point of view, the following properties are considered as axioms for a fuzzy implication.

1. $\zeta(x, y) = \zeta(1,y,1-x)$. (contraposition)
2. $\zeta(x, \zeta(y, z)) = \zeta(y, \zeta(x, z))$. (exchange property)
3. $x \leq y \Leftrightarrow \zeta(x, y) = 1$ for all $x, y \in [0,1]$. (boundary condition)
4. $\zeta(1,x) = x$, for all $x \in [0,1]$. (neutrality of truth)
5. $\zeta$ is continuous. (continuity)

**Definition 2.8.** Consider a $\circ$-norm $\Delta$ and its corresponding $\circ$-conorm $\nabla$. Then the mappings: $\xi_i: [0,1] \times [0,1] \rightarrow [0,1]$ defined by:

1. $\xi_1^\Delta(x,y) = \Delta(x,y), 1-x$, \quad (Klee-Dienes implication)
2. $\xi_2^\Delta(x,y) = \Delta(1-x,1-y), y$, \quad (Lukasiewicz implication)
3. $\xi_3^\Delta(x,y) = \Delta(1-x, y)$, \quad (Reichbach implication)
4. $\xi_4^\Delta(x,y) = \sup \{z | z \in [0,1] \text{ and } \Delta(x,z) \leq y\}$, \quad (Zadeh implication)
5. $\xi_5^\Delta = \sup \{z | z \in [0,1] \text{ and } \nabla(1-y,z) \leq 1-x\}$, are implicator operators.

The implicator listed in (4) is called residual implicator ($R$-implicator) with respect to the $\circ$-norm $\Delta$, while the one listed in (3) is known as an $S$-implicator with respect to the $\circ$-conorm $\nabla$.

**Remark 2.9.** Some important implicators borrowed from [9] are listed below:

1. $\xi_0(x,y) = \max(1-x,y)$, \quad (Klee-Dienes implication)
2. $\xi_1(x,y) = \min(1-x+y,1)$, \quad (Lukasiewicz implication)
3. $\xi_2(x,y) = 1-x+y$, \quad (Reichbach implication)
4. $\xi_3(x,y) = \max(1-x,\min(x,y))$, \quad (Zadeh implication)
5. $\xi_4(x,y) = \min(\max(1-x,y),\max(x,1-x),\max(y,1-y))$. \quad
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(6) \( \zeta_{G}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases} \) (Godel implication)

The class of \( R \)-implicators is important in our work, we specifically use Lukasiewicz implicator.

**Lemma 2.10** [3]. If \( \Delta \) is left-continuous, and \( \zeta \) is the \( R \)-Implicator associated with \( \Delta \), then the following hold for all \( x, y, z \in [0,1] \):

1. \( \Delta(x,y) \leq z \iff x \leq \zeta(y,z) \)
2. \( x \leq y \Rightarrow \zeta(x,y) = 1 \)
3. \( \Delta(\zeta(x,y), \zeta(y,z)) \leq \zeta(x,z) \)
4. \( \zeta(1,y) = y \)
5. \( \Delta(x, \zeta(x,y)) \leq y \)

**Definition 2.11** (Linear Aggregation rule). An aggregation rule is linear if for every profile \( \{A_1, A_2, \ldots, A_n\} \) and every proposition \( p \) in the agenda \( X \), the collective belief on \( p \) is the weighted average of the individual opinions on it, i-e

\[
A(p) = \omega_1 A_1(p) + \omega_2 A_2(p) + \cdots + \omega_n A_n(p),
\]

where \( \omega_i \geq 0 \) and \( \sum_{i=1}^{n} \omega_i = 1 \).

**Definition 2.12** (Implication Preservation). For all propositions \( p \) and \( q \) in \( X \) and all admissible profiles, if all individuals assign a value 1 to the fuzzy implication \( \zeta(p,q) \), then so does the collective belief function representing the beliefs of the group of individuals. (The choice of fuzzy implication in a problem is context dependent. For implication preservation we choose any implication from the class of \( R \)-implications).

Elementary Properties of a fuzzy aggregation function;

An aggregation is a function \( F \) which assigns to each profile of individual belief sets, a collective belief set.

1. If 0 and 1 are the extremal values, then \( F(0, \ldots, 0) = 0 \), \( F(1, \ldots, 1) = 1 \). \( F(a, a, \ldots, a) = a \) for all \( a \in [0,1] \). (Idempotence)
2. \( F \) is a continuous function of \( a_1, a_2, \ldots, a_n \). (Continuity)
3. \( F \) is monotonically non-decreasing with respect to each argument if \( a'_i > a_i \) implies that \( F(a_1, \ldots, a'_i, \ldots, a_n) \geq F(a_1, \ldots, a_i, \ldots, a_n) \). (Monotonicity)
4. \( \min \{a_i\} \leq F(a_1, \ldots, a_i, \ldots, a_n) \leq \max \{a_i\} \). (Compensativeness)

3. The doctrinal paradox

Given several logically interconnected propositions, Doctrinal Paradox can emerge when individual judgments on these propositions are combined into a collective decision which represents the judgments of the group of individuals as a whole. For example,
consider a set of propositions, where some propositions (the premises) are taken to be equivalent to another proposition (the conclusion) in a logical truth functional way. When majority voting is applied to premises it may give a different outcome than majority voting applied to the conclusion.

**Example 3.1** Suppose three policy makers have to pass their judgment on the following propositions:

- \( P \) : Poverty rate is low.
- \( Q \) : Literacy rate is high.
- \( R \) : Crime rate is low.

<table>
<thead>
<tr>
<th>Policymaker 1</th>
<th>( P )</th>
<th>( Q )</th>
<th>( P \land Q \rightarrow R )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Policymaker 2</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Policymaker 3</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Majority</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes/No</td>
</tr>
</tbody>
</table>

Table 1

Each policymaker assigns a binary truth value to the propositions \( P, Q \) and \( R \). The paradox is precisely the fact that when majority vote is taken on the premises it gives a contradictory outcome as compared to when majority is calculated on the conclusion. It is assumed that the decision makers are rational that is their judgment sets are complete and consistent. Arguably in some decision problems propositions are “vague” and can have truth values between true and false. Let us reformulate the entire problem in fuzzy logic framework where individuals can take on values between 0 and 1. “\( \land \)” is replaced by the fuzzy \( t \)-norm \( \Delta \) and implication “\( \rightarrow \)” is replaced by fuzzy implication \( \zeta \). The choice of the fuzzy connectives used in a problem are context dependent. Moreover, if an implication is formed with the help of both \( t \)-norms and \( t \)-conorms, they are not randomly selected in fact they are dual of each other with respect to the fuzzy negation [12]. Note that corresponding to each proposition there is a fuzzy set. When for instance a decision maker assigns a truth value to the proposition \( Q \) in the above example, she basically has a fuzzy set \( A' \) of say “countries with high literacy rate” in her mind and she expresses the degree of membership of this specific country in the fuzzy set \( A' \). Degree of truth of proposition \( Q : x \in A' \) is equal to the membership grade of \( x \) in the fuzzy set \( A' \). Whenever a decision maker is assigning truth values to a proposition, the above argument will hold but it will not be mentioned.

4. **Distance based approach in the fuzzy framework: The formal model**

   Given a finite set of \( n \) individuals (not necessarily odd) and a finite set \( X \) of propositions over which individuals have to express their beliefs. A belief set of an individual \( i \) denoted by \( A_i \) is a function \( A_i : P \rightarrow [0,1] \). A profile \( K = (A_1, A_2, \ldots, A_n) \) is an \( n \)-tuple of individual belief sets. An aggregation is a function \( F \) that maps to
each profile a collective belief set $A$ such that $F(A_1, A_2, \cdots, A_n) = A$. Here $A_i(p)$ is the truth value of a proposition $p$ in the belief set of individual $i$.

The formal model consists of a prepositional language $L$ built from a finite set of propositions $P$ and the fuzzy connectives $(\eta, \Delta, \lor, \Rightarrow, \Leftarrow)$ namely negation, $t$-norm, $t$-conorm, fuzzy implication and fuzzy bi-implication respectively [12]. Belief set whose elements are the integrity constraints is denoted by $IC$. The $IC$s are propositions that should be satisfied by the merged base. These constraints are dependent on the choice of fuzzy $t$-norm and implication in our case. $\psi$ maps $K$ and IC into a collective belief set denoted $\psi_{IC}(K)$. This belief set is a result of merging the individual belief sets into one which best represents the beliefs of the individuals.

An interpretation is a function $w : P \rightarrow [0,1]$. Let $W$ denote the set of all interpretations. A distance between interpretations is a real valued function $d : W \times W \rightarrow R$ such that for all $w, w', w'' \in W$:

1. $d(w, w') \geq 0$.
2. $d(w, w') = 0$ if and only if $w = w'$.
3. $d(w, w') = d(w', w)$.
4. $d(w, w'') \leq d(w, w') + d(w', w'')$.

We choose Euclidean metric as our distance function

$$d^*(w, w') = \left(\sum_{x \in P} |w(x) - w'(x)|^2\right)^{1/2}.$$  

Euclidean metric helps us find a distance between possibly conflicting interpretations. Our goal is to find an interpretation $w \in W$ which has the least distance from the profile of belief sets $K$. This interpretation is required to satisfy the integrity constraint which varies according to our choice of $t$-norm and implication. The belief merging operator used to find the distance between an interpretation and a profile is defined as

$$D^*(w, A^*) = \sum_{i \in N} d(w, A_i).$$

Distance based approach in the fuzzy framework guarantees an outcome without the fear of ending up with indecision. We concede that a wide variety of distance measures exist. Also, the choice of fuzzy connectives to be used in a specific problem are context dependent. We opt for Lukasiewicz $t$-norm and implication in our next example 4.1 only for the sake of illustration.

We can formulate any aggregation problem as an optimization problem where constraints would vary according to the choice of fuzzy connectives employed. Let $w$ be any arbitrary interpretation. In this case $w(p) = \{\theta_1, \theta_2, \cdots, \theta_{|P|}\}$ where $|P|$ is the cardinality of $P$. The optimization problem can now be stated as follows:

Minimize $D^*(w, A^*)$ subject to the fuzzy IC. Here $w(p) = \{\theta_1, \theta_2, \cdots, \theta_{|P|}\}$, $A_i(j) \leq \theta_j \leq \max A_i(j)$ for $i \in \{1, 2, \cdots, n\}$, $j \in \{1, 2, \cdots, |P|\}$ and $\theta_j \in [0,1]$.
The above optimization problem helps us find a unique optimal fuzzy aggregation function. We say that an aggregation function is optimal if the collective belief set is as close as possible to the individual belief sets.

**Example 4.1.** In this example, policy makers have to express their belief on the same propositions as in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>$P394Q \Rightarrow R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Policymaker 1</td>
<td>0.5</td>
<td>0.6</td>
<td>0.4</td>
<td>1</td>
</tr>
<tr>
<td>Policymaker 2</td>
<td>0.3</td>
<td>0.7</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>Policymaker 3</td>
<td>0.8</td>
<td>0.4</td>
<td>0.3</td>
<td>1</td>
</tr>
<tr>
<td>Collective Decision</td>
<td>$\theta_1$</td>
<td>$\theta_2$</td>
<td>$\theta_3$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2

We choose Lukasiewicz $t$-norm and implication which is defined as: $\Delta(x,y) = \max(0, x + y - 1)$ and $\zeta(x,y) = \min(1, 1 - x + y)$. Fuzzy integrity constraint is $(\pi(P) \Delta \pi(Q) \Rightarrow \pi(R))$ which by the choice of Lukasiewicz $t$-norm and implication can be translated as $\theta_3 \geq \max(0, \theta_1 + \theta_2 - 1)$. Finding collective social choice function in Table 2 now becomes an optimization problem. The problem is framed in Matlab. The optimal fuzzy aggregation function gives the solution for Table 2 as $(\theta_1, \theta_2, \theta_3) = (0.4667979, 0.555718, 0.369934)$ (for details see Appendix 1).

Framing the problem of belief merging into an optimization problem works equally well in cases where voters are allowed to express their beliefs on the implication, that is, when implication is part of the agenda and hence policy makers can assign truth values to it. For instance, in our next Example 4.2, we let $P$, $Q$ and $P \Rightarrow Q$ as the propositions in the agenda on which committee members are required to express their beliefs. Implication chosen in the table 3 is Zadeh’s implication defined as $\zeta(x,y) = \max\left[1 - x, \min(x,y)\right]$.

**Example 4.2.**

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>$P \Rightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Policymaker 1</td>
<td>0.6</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Policymaker 2</td>
<td>0.8</td>
<td>0.9</td>
<td>0.8</td>
</tr>
<tr>
<td>Policymaker 3</td>
<td>0.2</td>
<td>0.9</td>
<td>0.8</td>
</tr>
<tr>
<td>Collective Decision</td>
<td>$\theta_1$</td>
<td>$\theta_2$</td>
<td>$\theta_3$</td>
</tr>
</tbody>
</table>

Table 3

The fuzzy integrity constraint $\pi(P) \Delta \pi(Q) \Rightarrow \pi(R)$ is translated as $\theta_3 = \max\left[1 - \theta_1, \min(\theta_2, \theta_3)\right]$. The optimal collective outcome obtained is $(\theta_1, \theta_2, \theta_3) = (0.35, 0.65, 0.65)$. (For details see Appendix 2). This procedure of tackling the belief aggregation problem
as an optimization problem guarantees a unique collective outcome in fuzzy framework but at the expense of systematicity.

5. Abiding by the decision rule

We now restrict ourselves to the problem where decision rule is not a part of the agenda $X$. Our interest now is in the problems where implication is to be preserved. Which means that individuals have to abide by the decision rule and are not allowed to express their beliefs on it. For instance, policy makers in Example 4.1 did not express their beliefs on the decision rule but it was to be followed by them. Molding the problem into an optimization problem helped us achieve a unique outcome but at the expense of systematicity. We now focus on the class of continuous $t$-norms and $R$ implications which help us define aggregation rules that satisfy collective rationality without having to compromise with systematicity.

**Example 5.1.** Let $F$ be an aggregation function such that for every proposition $p \in X$,

$$F(A_1, A_2, \ldots, A_n)(p) = \omega_1 A_1(p) + \omega_2 A_2(p) + \cdots + \omega_n A_n(p).$$

Given that individuals have non-binary beliefs and that fuzzy connectives may be employed according to the nature of the problem, using context dependent $R$-implications guarantee that the linear aggregation function defined above would produce a collective belief set such that our aggregation rule satisfies universal domain, systematicity, anonymity and collective rationality.

| Policymaker 1 | $p_1$ | $q_1$ | 1 | $r_1$ |
| Policymaker 2 | $p_2$ | $q_2$ | 1 | $r_2$ |
| . . . . . . | . . . . . . | . . . . . . | . . . . . . |
| Policymaker $i$ | $p_i$ | $q_i$ | 1 | $r_i$ |
| . . . . . . | . . . . . . | . . . . . . | . . . . . . |
| Policymaker $n$ | $p_n$ | $q_n$ | 1 | $r_n$ |
| Collective Decision | $\theta_1$ | $\theta_2$ | 1 | $\theta_3$ |

Table 4

Here $\theta_1 = \sum_{i=1}^{n} w_i p_i / n$, $\theta_2 = \sum_{i=1}^{n} w_i q_i / n$ and $\theta_3 = \sum_{i=1}^{n} w_i r_i / n$. Since we do not want to violate anonymity so equal weights are assigned to all the policymakers. The function as defined above produces collective outcome on each proposition denoted by $\theta_i$'s $1 \leq n$ which satisfies collective rationality and implication preservation.

**Theorem 5.2.** A linear aggregation rule satisfies universal domain, collective rationality, independence and implication preservation.

**Proof.** Assume that $F$ is a linear aggregation rule. Then $F$ satisfies independence and universal domain. If beliefs of all individuals are binary, then the only linear aggre-
The aggregation rule is dictatorial one where a weight of 1 is assigned to a particular individual and the rest are given a weight of 0. Thus, collective rationality and implication preservation are satisfied as well. However, if beliefs are something between a yes or a no and individuals preserve the implication, we need to prove that the collective outcome produced by the linear aggregation rule will also preserve the implication. This is trivial by Lemma 2.10. Since implication is preserved by the individuals. It means that \( A_i (p \Rightarrow q) = 1 \) for all \( i \in \{1, 2, \ldots, n\} \). Thus \( F(A_1, A_2, \ldots, A_n)(p \Rightarrow q) = \omega_1 A_1 (p \Rightarrow q) + \omega_2 A_2(p \Rightarrow q) + \cdots + \omega_n A_n (p \Rightarrow q) \) where \( \sum_{i=1}^{n} \omega_i = 1 \) and \( A_i (p \Rightarrow q) = 1 \forall i \in \{0, 1, \ldots, n\} \). This implies that \( F(A_1, A_2, \ldots, A_n)(p \Rightarrow q) = 1 \) the collective outcome preserves implication and hence collective rationality and implication preservation are satisfied as well.

**Theorem 5.3.** If an aggregation rule \( F \) satisfies universal domain, collective rationality, systematicity and implication preservation, then \( F \) is linear.

**Proof.** Suppose \( F \) satisfies the hypothesis. In cases where beliefs are binary, \( F \) is clearly a dictatorial rule and thus linear. In case where beliefs are non-binary and implication preservation and collective rationality holds, then it means that \( F(A_1, A_2, \ldots, A_n)(p \Rightarrow q) = 1 \) given \( A_i (p \Rightarrow q) = 1 \) for all \( i \in \{1, 2, \ldots, n\} \). It implies that \( \omega_1 A_1 (p \Rightarrow q) + \omega_2 A_2(p \Rightarrow q) + \cdots + \omega_n A_n (p \Rightarrow q) = 1 \). It further implies that \( \omega_1 + \cdots + \omega_n = 1 \).

**Theorem 5.4.** If an aggregation function \( F \) satisfies universal domain, collective rationality and implication preservation then \( F \) is monotonic.

**Proof.** Suppose \( F \) is an aggregation rule that satisfies the hypothesis. It implies that \( F(A_i(p \Rightarrow q)) = 1 \) given \( A_i(p \Rightarrow q) = 1 \). By Lemma 2.10 it further implies that \( F(A_i(p)) \leq F(A_i(q)) \) if \( A_i(p) \leq A_i(q) \).

**Remark 5.5.** An aggregation rule is strategy proof if it is independent and monotonic [6].

Since our linear aggregation rule is both independent and monotonic hence it is strategy proof.

**Theorem 5.6.** Let \( F \) be an aggregation function and for all \( p \in X \) let there be a decision method \( D_p \) such that for all \( (v_1, v_2, \ldots, v_n) \in V^n \),

\[
A(v_1, v_2, \ldots, v_n)(p) = D_p(v_1(p), \ldots, v_n(p)).
\]

Then the following properties hold:

(1) For every literal \( p \) such that \( p, \eta(p) \in X \) and for all \( x \in [0,1]^n \), we have:

\[
D_{\eta(p)}(\eta(x_1), \ldots, \eta(x_n)) = \eta(D_p(x_1, x_2, \ldots, x_n)).
\]
Belief Aggregation in Fuzzy Framework

(2) For all literals \( p \neq q \) and for all \( x, y \in [0,1]^n \), where \( \eta(p) = 1-p \) and \( \Delta \) is the Godel \( t \)-norm or the Lukasiewicz \( t \)-norm (in cases where \( \Delta(x,y) \neq 0 \) such that \( p \Rightarrow q = 1 \), \( \Delta \left( D_p(x_1, \ldots, x_n), D_q(y_1, \ldots, y_n) \right) = D_{\rho\Delta} \left( \Delta(x_1, y_1), \ldots, \Delta(x_n, y_n) \right) \)

Proof. (1) Let \( x = (x_1, x_2, \ldots, x_n) \in [0,1]^n \) and \( y \in [0,1]^n \). For any \( x_i \) such that \( v_i(p) = x_i \). Then \( v_i(\eta(p)) = \eta(x_i) \). Therefore, \( D_{\rho}(\eta(x_1), \ldots, \eta(x_n)) = \eta(D_{\rho}(v_1(p)), \ldots, v_n(\eta(p))) = \eta(A(v_1, \ldots, v_n)(\eta(p))) = \eta(D_p(x_1, \ldots, x_n)) \).

(2) Consider any \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \). By the same argument used in the proof of the first claim, for each propositions \( p \) and \( q \) there exists a corresponding \( v_i \) such that \( v_i(p) = x_i \) and \( v_i(q) = y_i \). We have \( \Delta(D_p(x), D_q(y)) = \Delta(D_p(v_1(p)), \ldots, v_n(p)), D_q(v_1(q)), \ldots, v_n(q)) = \Delta(A(v_1, \ldots, v_n)(p), A(v_1, \ldots, v_n)(q)) = A(v_1, \ldots, v_n)(p) = D_{\rho\Delta}(\Delta(x_1, y_1), \ldots, \Delta(x_n, y_n)) \).

Example 5.7. In Table 5, Lukasiewicz \( t \)-norm and implication is used:

<table>
<thead>
<tr>
<th>Policymaker 1</th>
<th>( p )</th>
<th>( q )</th>
<th>( p \Rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( y_1 )</td>
<td>( 1 )</td>
<td></td>
</tr>
<tr>
<td>Policymaker 2</td>
<td>( x_2 )</td>
<td>( y_2 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>Policymaker 3</td>
<td>( x_3 )</td>
<td>( y_3 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>Collective Decision</td>
<td>( \theta_1 )</td>
<td>( \theta_2 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

Table 5

Let \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) be the \( n \)-tuple representing the truth assignments of \( n \) individuals on the propositions \( p, q \in X \) such that \( p \Rightarrow q = 1 \) where \( x_i, y_i \in [0,1] \) and \( \Delta(x,y) \neq 0 \). Let us define the decision rule \( D \) by \( D_p(x_1, \ldots, x_n) = \left( \sum_{i=1}^{n} x_i \right) / n \).

1. \( D_{\rho}(\eta(x_1), \ldots, \eta(x_n)) = \left( (1-x_1) + (1-x_2) + (1-x_3) \right) / 3 = 1 - (x_1 + x_2 + x_3) / 3 = \eta(D_p(x_1, x_2, x_3)) \).

2. Let us consider Lukasiewicz \( t \)-norm \( \Delta(x,y) = \max(0, x+y-1) \) such that \( \Delta(x,y) \neq 0 \). \( \Delta(D_p(x_1, x_2, x_3), D_q(y_1, y_2, y_3)) = \Delta((x_1 + x_2 + x_3) / 3, (y_1 + y_2 + y_3) / 3) = \max(0, (x_1 + x_2 + x_3) / 3 + (y_1 + y_2 + y_3) / 3 - 1) = (x_1 + x_2 + x_3) / 3 + (y_1 + y_2 + y_3) / 3 - 1 \).

On the other hand \( D_{\rho\Delta}(\max(0, x_1 + y_1 - 1), \max(0, x_2 + y_2 - 1), \max(0, x_3 + y_3 - 1)) = D_{\rho\Delta}(x_1 - y_1 + 2, y_2 - 1, x_3 + y_3 - 1) = (x_1 + y_1 - 1 + x_2 + y_2 - 1 + x_3 + y_3 - 1) / 3 = (x_1 + x_2 + x_3) / 3 + (y_1 + y_2 + y_3) / 3 - 1 \). Moreover if Godel \( t \)-norm \( \Delta(x,y) = \min(x,y) \) is used
= \min\left(\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)\right) = \left(\frac{x_1 + x_2 + x_3}{3}\right), \text{ using Lemma 2.10 (2)}.\]
Considering the other side
\[D_{\rho D}(\min(x_1, y_1), \min(x_2, y_2), \min(x_3, y_3)) = D_{\rho D}(x_1, x_2, x_3)\]
\[= \left(\frac{x_1 + x_2 + x_3}{3}\right), \text{ by using Lemma 2.10 (2)}.\]

6. Conclusion

If the decision makers of a society are restricted to express their beliefs using crisp values, then the merged outcome based on individual beliefs which themselves are not truly representing the individuals, cannot lead to a collective outcome which best represents the society. Majority voting in classical two valued logic results in Doctrinal Paradox. Distance based merging operators used to attain collective rationality results in a situation of indecision or a tie. We allowed the decision makers to opt for values from the interval [0,1] to express their beliefs. We used distance based approach in the fuzzy framework to find an interpretation having the least distance with the profile of individual belief sets. This problems is converted into an optimization problem which helped us avoid the situation of indecision by producing a unique and optimal collective outcome in the fuzzy framework but at the expense of systematicity. Fuzzy framework not only gives freedom of expression to the decision makers but also provides us with a wider range of fuzzy connectives that can be used according to the nature of the problem at hand. We specified the class of $R$-implications which help us find a collective belief using linear aggregation rules such that the aggregated outcome preserves the implication, is collectively rational and does not violate systematicity. Linear aggregation functions work equally well in cases where every individual does not have the same power to influence the final decision. In such a case the aggregated belief set will still satisfy collective rationality and implication preservation will still hold provided that the implication used belongs to the class of $R$-implications. Policy makers could be allowed to opt for triangular fuzzy numbers to express their beliefs better. It would be interesting to find ways to merge their membership functions. Moreover fuzzy optimization techniques could be employed to find an aggregated fuzzy number.

References

Appendices

Both the appendices are written in Mat Lab.

Appendix 1

d = zeros(601,1) ; d2 = zeros(601,1) ; D = zeros(2000,1) ; DD = zeros(601,1) ; d1 = zeros(601,1) ; d2 = zeros(601,1) ; d3 = zeros(601,1) ; pm1 = [0.5,0.6,0.4] ; pm2 = [0.3,0.7,0.2] ; pm3 = [0.8,0.4,0.3] ; o1 = zeros(601,1) ; o2 = zeros(601,1) ; o3 = zeros(601,1) ; o1 = [0.3 : 0.001 : 0.8]' ; o2 = [0.1 : 0.001 : 0.7]' ; o3 = [0.1 : (0.5 – 0.1)/600 : 0.5]' ; Bsmallest = 10 ; for i = 1:length(o1)–1 ; if o3(i) > (max(0,o1(i) + o2(i) – 1)) d1 = sqrt(((o1 – pm1(1)).^2 +(o2 – pm1(2)).^2 +(o3(i) – pm1(3)).^2)) ; d2 = sqrt((o1 – pm2(1)).^2 +(o2 – pm2(2)).^2 +(o3(i) – pm2(3)).^2) ; d3 = sqrt((o1 – pm3(1)).^2 +(o2 – pm3(2)).^2 +(o3(i) – pm3(3)).^2) ; d = (d1 + d2 + d3) ; small = min(d) ; if (Bsmallest small) x = o1(i) ; y = o2(i) ; z = o3(i) ; Bsmallest = small ; end end end

Appendix 2

d = zeros(1,2) ; d2 = zeros(601,1) ; D = zeros(2000,1) ; D = zeros(601,1) ; d1 = zeros(601,1) ; d2 = zeros(601,1) ; d3 = zeros(601,1) ; c1 = [0.6,0.5,0.5] ; c2 = [0.8,0.9,0.8] ; c3 = [0.2,0.9,0.8] ; o1 = zeros(601,1) ; o2 = zeros(601,1) ; o3 = zeros(601,1) ; o1 = [0.2 : 0.0001 : 0.8]' ; o2 = [0.5 : 0.0001 : 0.9]' ; o3 = [0.5 : 0.0001 : 0.8]' ; Bsmallest = 10 ; for i = 1:length(o3)–1 ; if o3(i) = (max(1 – o1(i)),min(o1(i),o2(i)))) d1(j) = sqrt(((o1(i) – c1(1)).^2 +(o2(i) – c1(2)).^2 +(o3(i) – c1(3)).^2)) ; d2(j) = sqrt(((o1(i) – c2(1)).^2 +(o2(i) – c2(2)).^2 +(o3(i) – c2(3)).^2)) ; d3(j) = sqrt((o1(i) –
\[
0_{3}(1)^{2} + (0_{2}(i) - 0_{3}(2))^{2} + (0_{3}(i) - 0_{3}(3))^{2} \]
\[v = (d_{1}(j) + d_{2}(j) + d_{3}(j)) ; d_{1}, 2 = v ;
\]
\[d(j, 1) = i ; j + 1 ; \text{end} \]
\[\min_{a,l, index} - \min(d(2)) ; \text{end}
\]
\[\min_{a,l} (d(index, 1)) 0_{2}(d(index, 1)) 0_{3}(d(index, 1))\]