The completion of
$L$-topological groups

Fatma Bayoumi* and Ismail Ibedou

Department of Mathematics, Faculty of Sciences, Benha University, Benha 12517, Egypt

Abstract

The target in this paper, is to extend an $L$-topological group to a complete $L$-topological group, and so giving the notion of the completion of an $L$-topological group. In the way, we have introduced the notion of the completion of an $L$-uniform space.

Keywords: $L$-topological groups; complete $L$-topological groups; $L$-uniform spaces; complete $L$-uniform spaces; $U$-cauchy filters; $L$-filters; $L$-topological spaces.

1. Introduction

In this paper, we gave new notions of $L$-filter, $L$-uniform space and $L$-topological group. We defined, in an $L$-uniform space $(X, \mathcal{U})$, a $U$-cauchy filter and have shown when $(X, \mathcal{U})$ to be a complete $L$-uniform space, and also how an $L$-topological group $(G, \tau)$ to be complete. Finally, the completion of an $L$-uniform space and the completion of an $L$-topological group are investigated.

In Section 2, we recall some results of $L$-filters and $L$-neighborhood filters defined by Gähler in [11, 13, 14]. Also, we have defined the product of two $L$-sets and the product of two $L$-filters.

In Section 3, we have defined in an $L$-uniform space $(X, \mathcal{U})$, a new notion of $L$-filter called $U$-cauchy filter. We showed that any convergent $L$-filter is a $U$-cauchy filter and the converse holds in the complete $L$-uniform spaces.

Section 4 is devoted to show how to extend an $L$-uniform space to a complete $L$-uniform space, and so the completion of an $L$-uniform space here is given as a reduced extension $L$-uniform space with a complete $L$-uniform structure.

In Section 5, using the $L$-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ defined on the $L$-topological group $(G, \tau)$ which are compatible with $\tau$ as in [8], we shall define the notion of complete $L$-topological group. A complete separated $L$-topological group $(H, \sigma)$ in which $(G, \tau)$ is a dense subgroup will be called a completion of $(G, \tau)$.

*Corresponding author: e-mail: fatma_bayoumi@hotmail.com
2. On $L$-filters

In this section, we recall and show some results concerning $L$-filters needed in the paper. Denote by $L^X$ the set of all $L$-subsets of a non-empty set $X$, where $L$ is a complete chain with different least and greatest elements 0 and 1, respectively [19]. For each $L$-set $\lambda \in L^X$, let $\lambda'$ denote the complement of $\lambda$, defined by $\lambda'(x) = \lambda(x)'$ for all $x \in X$. For all $x \in X$ and $\alpha \in L_0$, the $L$-subset $x_\alpha$ of $X$ whose value $\alpha$ at $x$ and 0 otherwise is called an $L$-point in $X$ and the constant $L$-subset of $X$ with value $\alpha$ will be denoted by $\overline{\alpha}$.

$L$-filters. By an $L$-filter on a non-empty set $X$ we mean [13] a mapping $\mathcal{M} : L^X \to L$ such that $\mathcal{M}(\overline{\alpha}) \leq \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$, and also $\mathcal{M}(\lambda \wedge \mu) = \mathcal{M}(\lambda) \wedge \mathcal{M}(\mu)$ for all $\lambda, \mu \in L^X$. $\mathcal{M}$ is called homogeneous [11] if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are $L$-filters on $X$, $\mathcal{M}$ is called finer than $\mathcal{N}$, denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(\lambda) \geq \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$.

Let $\mathcal{F}_L X$ denote the set of all $L$-filters on $X$, $f : X \to Y$ a mapping and $\mathcal{M}_i, \mathcal{N}_i$ are $L$-filters on $X, Y$, respectively. Then the image of $\mathcal{M}$ and the preimage of $\mathcal{N}$ with respect to the mapping $f$ are the $L$-filters $\mathcal{F}_L f(\mathcal{M})$ on $Y$ and $\mathcal{F}_L^{-1} f(\mathcal{N})$ on $X$ defined by $\mathcal{F}_L f(\mathcal{M})(\mu) = \mathcal{M}(\mu \circ f)$ for all $\mu \in L^Y$ and $\mathcal{F}_L^{-1} f(\mathcal{N})(\lambda) = \bigvee_{\mu \circ f \leq \lambda} \mathcal{N}(\mu)$ for all $\lambda \in L^X$, respectively. For each mapping $f : X \to Y$ and each $L$-filter $\mathcal{N}$ on $Y$, for which the preimage $\mathcal{F}_L^{-1} f(\mathcal{N})$ exists, we have $\mathcal{F}_L f(\mathcal{F}_L^{-1} f(\mathcal{N})) \leq \mathcal{N}$. Moreover, for each $L$-filter $\mathcal{M}$ on $X$, the inequality $\mathcal{M} \leq \mathcal{F}_L^{-1} f(\mathcal{F}_L f(\mathcal{M}))$ holds [13].

For each non-empty set $A$ of $L$-filters on $X$, the supremum $\bigvee_{\mathcal{M} \in A} \mathcal{M}$ with respect to the finer relation of $L$-filters exists and we have

$$ ( \bigvee_{\mathcal{M} \in A} \mathcal{M})(f) = \bigwedge_{\mathcal{M} \in A} \mathcal{M}(f) $$

for all $f \in L^X$. The supremum $\bigwedge_{\mathcal{M} \in A} \mathcal{M}$ of $A$ exists if and only if for each non-empty finite subset $\{M_1, \ldots, M_n\}$ of $A$ we have $M_1(\lambda_1) \wedge \cdots \wedge M_n(\lambda_n) \leq \sup(\lambda_1 \wedge \cdots \wedge \lambda_n)$ for all $\lambda_1, \ldots, \lambda_n \in L^X$ [11]. If the infimum of $A$ exists, then for each $\lambda \in L^X$ and $n$ as a positive integer we have

$$ ( \bigwedge_{\mathcal{M} \in A} \mathcal{M})(\lambda) = \bigvee_{\lambda_1 \wedge \cdots \wedge \lambda_n \leq \lambda, M_1, \ldots, M_n \in A} (M_1(\lambda_1) \wedge \cdots \wedge M_n(\lambda_n)). $$

By a filter on $X$ we mean a non-empty subset $\mathcal{F}$ of $L^X$ which does not contain $\overline{0}$ and closed under finite infima and super sets [17]. For each $L$-filter $\mathcal{M}$ on $X$, the subset $\alpha$-pr $\mathcal{M}$ of $L^X$ defined by: $\alpha$-pr $\mathcal{M} = \{ \lambda \in L^X \mid \mathcal{M}(\lambda) \geq \alpha \}$ is a filter on $X$.

A family $(B_\alpha)_{\alpha \in L_0}$ of non-empty subsets of $L^X$ is called valued $L$-filter base on $X$ [13] if the following conditions are fulfilled:

(V1) $\lambda \in B_\alpha$ implies $\alpha \leq \sup \lambda$.

(V2) For all $\alpha, \beta \in L_0$ and all $L$-sets $\lambda \in B_\alpha$ and $\mu \in B_\beta$, if even $\alpha \wedge \beta > 0$ holds, then there are a $\gamma \geq \alpha \wedge \beta$ and an $L$-set $\nu \leq \lambda \wedge \mu$ such that $\nu \in B_\gamma$. 


Each valued $L$-filter base $(B_{\alpha})_{\alpha \in L_0}$ on a set $X$ defines an $L$-filter $\mathcal{M}$ on $X$ by: $\mathcal{M}(\lambda) = \bigvee_{\mu \in B_{\alpha}, \mu \leq \lambda} \alpha$ for all $\lambda \in L^X$. On the other hand, each $L$-filter $\mathcal{M}$ can be generated by many valued $L$-filter bases, and among them the greatest one $(\alpha \text{-pr} \mathcal{M})_{\alpha \in L_0}$.

**Proposition 2.1** [13] There is a one-to-one correspondence between the $L$- filters $\mathcal{M}$ on $X$ and the families $(\mathcal{M}_\alpha)_{\alpha \in L_0}$ of prefilters on $X$ which fulfill the following conditions:

1. $f \in \mathcal{M}_\alpha$ implies $\alpha \leq \sup f$.
2. $0 < \alpha \leq \beta$ implies $\mathcal{M}_\alpha \supseteq \mathcal{M}_\beta$.
3. For each $\alpha \in L_0$ with $\bigvee_{0 < \beta < \alpha} \beta = \alpha$ we have $\bigcap_{0 < \beta < \alpha} \mathcal{M}_\beta = \mathcal{M}_\alpha$.

This correspondence is given by $\mathcal{M}_\alpha = \alpha \text{-pr} \mathcal{M}$ for all $\alpha \in L_0$ and $\mathcal{M}(f) = \bigvee_{g \in \mathcal{M}_\alpha, g \leq f} \alpha$ for all $f \in L^X$.

**$L$-neighborhood filters.** In the following, the topology in sense of [10, 16] will be used which will be called $L$-topology. $\text{int}_\tau$ and $\text{cl}_\tau$ denote the interior and the closure operators with respect to the $L$-topology $\tau$, respectively. For each $L$-topological space $(X, \tau)$ and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \to L$ defined by: $\mathcal{N}(x)(\lambda) = \text{int}_\tau \lambda(x)$ for all $\lambda \in L^X$ is an $L$-filter on $X$, called the $L$-neighborhood filter of the space $(X, \tau)$ at $x$, and for short is called a $\tau$-neighborhood filter at $x$. The mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\lambda) = \lambda(x)$ for all $\lambda \in L^X$ is a homogeneous $L$-filter on $X$. Let $(X, \tau)$ and $(Y, \sigma)$ be two $L$-topological spaces. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ is called $L$-continuous (or $(\tau, \sigma)$-continuous) provided $\text{int}_\tau \mu \circ f \leq \text{int}_\tau (\mu \circ f)$ for all $\mu \in L^Y$. An $L$-filter $\mathcal{M}$ is said to converge to $x \in X$, denoted by $\mathcal{M} \underset{\tau}{\to} x$, if $\mathcal{M} \leq \mathcal{N}(x)$ [14]. The $L$-neighborhood filter $\mathcal{N}(F)$ at an ordinary subset $F$ of $X$ is the $L$-filter on $X$ defined, by the authors in [3], by means of $\mathcal{N}(x), x \in F$ as:

$$\mathcal{N}(F) = \bigvee_{x \in F} \mathcal{N}(x).$$

The $L$-filter $\hat{F}$ is defined by $\hat{F} = \bigvee_{x \in F} \dot{x}$. $\hat{F} \leq \mathcal{N}(F)$ holds for all $F \subseteq X$.

**Lemma 2.1** [14] Let $(X, \tau)$ and $(Y, \sigma)$ be two $L$-topological spaces and $\mathcal{M}$ an $L$-filter on $X$, and let $f : X \to Y$ be a $(\tau, \sigma)$-continuous mapping. Then $\mathcal{M} \underset{\tau}{\to} x$ implies that $\mathcal{F}_L f(\mathcal{M}) \underset{\sigma}{\to} f(x)$.

Firstly, let us give this important definition.

For $\lambda, \mu \in L^X$, let $\lambda \times \mu : X \times X \to L$ be the $L$-set defined as follows:

$$(\lambda \times \mu)(x, y) = \lambda(x) \wedge \mu(y) \quad (2.1)$$

for all $x, y \in X$. 

3
Remark 2.1 For all $\lambda, \mu, \xi, \eta \in L^X$, we have

$$(\lambda \wedge \mu) \times (\xi \wedge \eta) = (\lambda \times \xi) \wedge (\mu \times \eta) = (\lambda \times \eta) \wedge (\mu \times \xi).$$

Proposition 2.2 For any two $L$-filters $\mathcal{L}, \mathcal{M}$ on $X$, the mapping $\mathcal{L} \times \mathcal{M} : L^{X \times X} \to L$ defined by

$$(\mathcal{L} \times \mathcal{M})(u) = \bigvee_{\lambda \times \mu \leq u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu))$$

for all $u \in L^{X \times X}$ is an $L$-filter on $X \times X$.

Proof. From (2.1) and that $\mathcal{L}, \mathcal{M}$ are $L$-filters, we get that

$$(\mathcal{L} \times \mathcal{M})(\tilde{\alpha}) = \bigvee_{\lambda \times \mu \leq \tilde{\alpha}} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \leq \alpha.$$ 

Moreover, $(\mathcal{L} \times \mathcal{M})(\tilde{1}) = 1$.

From Remark 2.1 and for all $u, v \in L^{X \times X}$, we get that

$$(\mathcal{L} \times \mathcal{M})(u) \wedge (\mathcal{L} \times \mathcal{M})(v) = \bigvee_{\lambda \times \mu \leq u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \wedge \bigvee_{\xi \times \eta \leq v} (\mathcal{L}(\xi) \wedge \mathcal{M}(\eta))$$

$$= \bigvee_{\lambda \times \mu \leq u, \xi \times \eta \leq v} (\mathcal{L}(\lambda \wedge \xi) \wedge \mathcal{M}(\mu \wedge \eta))$$

$$\leq \bigvee_{(\lambda \wedge \xi) \times (\mu \wedge \eta) \leq u \wedge v} (\mathcal{L}(\lambda \wedge \xi) \wedge \mathcal{M}(\mu \wedge \eta))$$

$$= (\mathcal{L} \times \mathcal{M})(u \wedge v).$$

Also,

$$(\mathcal{L} \times \mathcal{M})(u \wedge v) = \bigvee_{\lambda \times \mu \leq u \wedge v} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu))$$

$$\leq \bigvee_{\lambda \times \mu \leq u, \lambda \times \mu \leq v} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu))$$

$$= \bigvee_{\lambda \times \mu \leq u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \wedge \bigvee_{\lambda \times \mu \leq v} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu))$$

$$= (\mathcal{L} \times \mathcal{M})(u) \wedge (\mathcal{L} \times \mathcal{M})(v).$$

Hence, $(\mathcal{L} \times \mathcal{M})$ is an $L$-filter on $X \times X$. □

Here, we prove the following result.

Lemma 2.2 Let $\mathcal{L}$ and $\mathcal{M}$ be $L$-filters on $X$, and let $(\mathcal{L}_\alpha)_{\alpha \in L_0}$ and $(\mathcal{M}_\alpha)_{\alpha \in L_0}$ be the families of prefilters on $X$ correspond, according to Proposition 2.1, $\mathcal{L}$ and $\mathcal{M}$, respectively. Then the family $(\mathcal{K}_\alpha)_{\alpha \in L_0}$ of subsets $\mathcal{K}_\alpha$ of $L^{X \times X}$, where

$$\mathcal{K}_\alpha = \{\lambda \times \mu \mid \lambda \in \mathcal{L}_\alpha, \mu \in \mathcal{M}_\alpha\},$$

is a family of prefilters on $X \times X$ corresponds the $L$-filter $\mathcal{L} \times \mathcal{M}$. 4
Proof. Firstly, we show that, for all $\alpha \in L_0$, $K_\alpha$ is a prefilter on $X \times X$. For any $\alpha \in L_0$, we have $K_\alpha = \{ \lambda \times \mu \mid \lambda \in L_\alpha, \mu \in M_\alpha \}$ is non-empty, where $L_\alpha$ and $M_\alpha$ are non-empty for all $\alpha \in L_0$. Also, $0$ does not exist in $L_\alpha$ or $M_\alpha$ implies that $\emptyset \notin K_\alpha$ for all $\alpha \in L_0$. From Remark 2.1 and from that $L_\alpha$ and $M_\alpha$ are prefilters, we get for all $u, v \in K_\alpha$ and $w \geq v$ that $u \cup v \in K_\alpha$ and $w \in K_\alpha$ for all $\alpha \in L_0$. That is, $K_\alpha$, for all $\alpha \in L_0$, is a prefilter on $X \times X$.

Let $u \in K_\alpha$. Then $u = \lambda \times \mu$, where $\lambda \in L_\alpha$ and $\mu \in M_\alpha$, which implies that $\alpha \leq \sup \lambda$, $\alpha \leq \sup \mu$, and $\alpha \leq \sup (\lambda \times \mu) = \sup \mu u$, that is, condition (1) of Proposition 2.1 holds.

Let $0 < \alpha \leq \beta$ and $u \in K_\beta$. Then $u = \lambda \times \mu$, where $\lambda \in L_\beta$ and $\mu \in M_\beta$, which implies, from $L_\alpha \supseteq L_\beta$ and $M_\alpha \supseteq M_\beta$, that $\lambda \in L_\alpha$ and $\mu \in M_\alpha$, that is, $u \in K_\alpha$ and condition (2) of Proposition 2.1 is fulfilled.

Since $\bigcap_{0 < \beta < \alpha} L_\beta = L_\alpha$ and $\bigcap_{0 < \beta < \alpha} M_\beta = M_\alpha$, we get that

$$\bigcap_{0 < \beta < \alpha} K_\beta = \bigcap_{0 < \beta < \alpha} \{ \lambda \times \mu \mid \lambda \in L_\beta, \mu \in M_\beta \} = \{ \lambda \times \mu \mid \lambda \in \bigcap_{0 < \beta < \alpha} L_\beta, \mu \in \bigcap_{0 < \beta < \alpha} M_\beta \} = \{ \lambda \times \mu \mid \lambda \in L_\alpha, \mu \in M_\alpha \} = K_\alpha,$$

which means that condition (3) of Proposition 2.1 holds.

Hence, there is a one - to - one correspondence between the family $(K_\alpha)_{\alpha \in L_0}$ of the prefilters on $X \times X$, defined by (2.3), and the $L$ - filter $L \times M$ on $X \times X$, according to Proposition 2.1, where

$$(L \times M)(u) = \bigvee_{v \in K_\alpha, v \leq u} \alpha \text{ and } \alpha - \text{pr} (L \times M) = K_\alpha$$

for all $u \in L^{X \times X}$ and for all $\alpha \in L_0$. □

3. $U$ - cauchy filters

This section is devoted to speak of the cauchy filters in the $L$ - uniform spaces defined in [15].

$L$ - uniform spaces. An $L$ - filter $U$ on $X \times X$ is called $L$ - uniform structure on $X$ [15] if the following conditions are fulfilled:

(U1) $(x, x) \leq U$ for all $x \in X$;
(U2) $U = U^{-1}$;
(U3) $U \circ U \leq U$.

Where $(x, x) (u) = u(x, x)$, $U^{-1}(u) = U(u^{-1})$ and $(U \circ U)(u) = \bigvee_{v \leq u} U(v \wedge w)$ for all $u \in L^{X \times X}$, and $u^{-1}(x, y) = u(y, x)$ and $(v \circ w)(x, y) = \bigvee_{z \in X} (w(x, z) \wedge v(z, y))$ for all $x, y \in X$. 

5
A set $X$ equipped with an $L$-uniform structure $\mathcal{U}$ is called an $L$-uniform space.

To each $L$-uniform structure $\mathcal{U}$ on $X$ is associated a stratified $L$-topology $\tau_\mathcal{U}$. The related interior operator $\text{int}_\mathcal{U}$ is given by:

$$\text{int}_\mathcal{U}(\lambda)(x) = \mathcal{U}[[x]](\lambda)$$

for all $x \in X$ and all $\lambda \in L^X$, where $\mathcal{U}[[x]](\lambda) = \bigvee_{\mu | \lambda} (\mathcal{U}(\mu) \wedge \mu(x))$ and $u[\mu](x) = \bigvee_{y \in X} (\mu(y) \wedge u(y, x))$. For all $x \in X$ we have

$$\mathcal{U}[[x]] = \mathcal{N}(x)$$

where $\mathcal{N}(x)$ is the $L$-neighborhood filter of the space $(X, \tau_\mathcal{U})$ at $x$. That is, an $L$-filter $\mathcal{M}$ in an $L$-uniform space $(X, \mathcal{U})$ is said to converge to $x \in X$ if $\mathcal{M} \leq \mathcal{U}[[x]]$.

Let $\mathcal{U}$ be an $L$-uniform structure on a set $X$. Then $u \in L^{X \times X}$ is called a surrounding provided $\mathcal{U}(u) \geq \alpha$ for some $\alpha \in L_0$ and $u = u^{-1}$ [8].

A subset $A \subseteq X$, for a surrounding $u$ in $(X, \mathcal{U})$, is called small of order $u$ if $u(x, y) \geq \alpha$ for all $x, y \in A$ and for some $\alpha \in L_0$.

**Definition 3.1** In an $L$-uniform space $(X, \mathcal{U})$, an $L$-filter $\mathcal{M}$ on $X$ is said to be a $\mathcal{U}$-cauchy filter provided for any surrounding $u$, there exists a set $B \subseteq X$ such that $\mathcal{M} \leq \hat{B}$ and $B$ is small of order $u$.

Now, we have the following expected result for the convergent $L$-filters.

**Proposition 3.1** Every convergent $L$-filter in an $L$-uniform space $(X, \mathcal{U})$ is a $\mathcal{U}$-cauchy filter.

**Proof.** Let $\mathcal{M}$ be an $L$-filter on $X$ which converges to $x \in X$. Since $\mathcal{M} \leq \mathcal{U}[[x]]$, then we can choose a set $B \subseteq X$ such that $\mathcal{M} \leq B = \mathcal{U}[[x]]$, that is,

$$\mathcal{M}(\lambda) \geq \bigvee_{\mu | \lambda} (\mathcal{U}(\mu) \wedge \mu(x)) = \bigwedge_{y \in B} \lambda(y) = \hat{B}(\lambda)$$

for all $\lambda \in L^X$. Since $(x, x)^* \leq \mathcal{U}$ for all $x \in X$, then $u(x, x) \geq \mathcal{U}(u) \geq \alpha$ for any surrounding $u$ and for some $\alpha \in L_0$, that is, $u(x, x) \geq \alpha$ for all $x \in X$ and for some $\alpha \in L_0$. Now, $x \in B$ where $\hat{x} \leq \mathcal{U}[[x]] = \hat{B}$. Also, for any $y \in B$ we get that $\bigwedge_{\mu | \lambda} (\alpha \wedge \mu(x)) \leq \lambda(y)$, for which $\bigwedge_{\mu | \lambda} (u(z, y) \wedge \mu(z)) \leq \lambda(y)$, and so $\alpha \wedge \mu(x) \leq u(x, y) \wedge \mu(x) \leq \lambda(y)$, and thus for all $x, y \in B$, we have $u(x, y) \geq \alpha$ for some $\alpha \in L_0$ and $\mathcal{M} \leq \hat{B}$. Hence, there is a set $B \subseteq X$ small of order any surrounding $u$ in $(X, \mathcal{U})$ and $\mathcal{M} \leq \hat{B}$, and therefore $\mathcal{M}$ is a $\mathcal{U}$-cauchy filter on $X$. \qed

Let $A$ be a subset of a set $X$, $\mathcal{U}$ an $L$-uniform structure on $X$ and $i : A \hookrightarrow X$ the inclusion mapping of $A$ into $X$. Then the initial $L$-uniform structure $\mathcal{F}_i(L)(i \times i)(\mathcal{U})$ of $\mathcal{U}$ with respect to $i$, denoted by $\mathcal{U}_A$, is called an $L$-uniform substructure of $\mathcal{U}$ and $(A, \mathcal{U}_A)$ an $L$-uniform subspace of $(X, \mathcal{U})$ [4].

In particular, we have the following result.
**Lemma 3.1** Let \((X, U)\) be an \(L\)-uniform space and \(A\) a non-empty subset of \(X\). Then an \(L\)-filter on \(A\) is a \(U_A\)-cauchy filter if and only if it is a \(U\)-cauchy filter.

**Proof.** Let \(\mathcal{M}\) be a \(U_A\)-cauchy filter on \(A\), then there exists \(B \subseteq A\) with \(\mathcal{M} \leq \hat{B}\) and \(B\) is small of order any surrounding \(u_A\) in \((A, U_A)\), which means that there is \(\hat{B} \subseteq A \subseteq X\) such that \(\mathcal{M} \leq \hat{B}\) and \(u_A(x, y) \geq \alpha\) for all \(x, y \in B\) and for some \(\alpha \in L_0\), that is, for any surrounding \(u\) in \((X, U)\),

\[
u(x, y) = (u \circ (i \times i))(x, y) = u_A(x, y) \geq \alpha
\]

for all \(x, y \in B\) and for some \(\alpha \in L_0\), and then \(\mathcal{M} \leq \hat{B}\) and \(B \subseteq X\) is small of order any surrounding \(u\) in \((X, U)\). Hence, \(\mathcal{M}\) is a \(U\)-cauchy filter.

Conversely, there exists \(B \subseteq A \subseteq X\) with \(\mathcal{M} \leq \hat{B}\) and \(B\) is small of order any surrounding \(u\) in \((X, U)\), that is, \(u(x, y) \geq \alpha\) for all \(x, y \in B\) and for some \(\alpha \in L_0\), which means that, for every surrounding \(u_A\) in \((A, U_A)\),

\[
u(x, y) = (u \circ (i \times i))(x, y) = u(x, y) \geq \alpha
\]

for all \(x, y \in B\) and for some \(\alpha \in L_0\). Hence, \(\mathcal{M} \leq \hat{B}\) and \(B \subseteq A\) is small of order any surrounding \(u_A\) in \((A, U_A)\), and thus \(\mathcal{M}\) is a \(U_A\)-cauchy filter. \(\square\)

A mapping \(f : (X, U) \rightarrow (Y, V)\) between \(L\)-uniform spaces \((X, U)\) and \((Y, V)\) is said to be \(L\)-uniformly continuous (or \((U, V)\)-continuous) provided

\[
\mathcal{F}_L(f \times f)(U) \leq V
\]

holds.

We shall use this result.

**Lemma 3.2** Let \((X, U)\) and \((Y, V)\) be \(L\)-uniform spaces and \(f : X \rightarrow Y\) a \((U, V)\)-continuous mapping. If \(\mathcal{M}\) is a \(U\)-cauchy filter, then \(\mathcal{F}_L f(\mathcal{M})\) is a \(V\)-cauchy filter.

**Proof.** \(\mathcal{M}\) is a \(U\)-cauchy filter on \(X\) means that there exists \(B \subseteq X\) such that \(\mathcal{M} \leq \hat{B}\) and \(B\) is small of order any surrounding \(u\) in \((X, U)\), that is, \(\mathcal{M} \leq \hat{B}\) and \(u(x, y) \geq \alpha\) for all \(x, y \in B\) and for some \(\alpha \in L_0\), which implies that,

\[
\mathcal{F}_L f(\mathcal{M}) \leq \mathcal{F}_L f(\hat{B}) = (f(\hat{B}))
\]

for the set \(f(B) \subseteq Y\). Let \(v\) be a surrounding in \((Y, V)\), then from being \(f\) is \((U, V)\)-continuous, we have

\[
\alpha \leq V(v) \leq U(v \circ (f \times f)) = \mathcal{F}_L(f \times f)(U)(v)
\]

for some \(\alpha \in L_0\), and \(v = v^{-1}\) implies that \((v \circ (f \times f))^{-1} = v^{-1} \circ (f \times f) = v \circ (f \times f)\), that is, \(u = v \circ (f \times f)\) is a surrounding in \((X, U)\), which means that

\[
\alpha \leq u(x, y) = (v \circ (f \times f))(x, y) = v(f(x), f(y))
\]

for all \(f(x), f(y) \in f(B)\) and for some \(\alpha \in L_0\). Hence, \(\mathcal{F}_L f(\mathcal{M}) \leq (f(\hat{B}))\) for the set \(f(B) \subseteq Y\) and \(f(B)\) is small of order every surrounding in \((Y, V)\), and thus \(\mathcal{F}_L f(\mathcal{M})\) is a \(V\)-cauchy filter. \(\square\)
4. The completion of $L$-uniform spaces

Firstly, we give these general notes.

If $(Y, \sigma)$ is an $L$-topological space and $X$ is a non-empty subset of $Y$, then the initial $L$-topology of $\sigma$, with respect to the inclusion mapping $i : X \hookrightarrow Y$, is the $L$-topology $i^{-1}(\sigma) = \{i^{-1}(\lambda) \mid \lambda \in \sigma\}$ on $X$ and is denoted by $\sigma_X$.

An $L$-topological space $(Y, \sigma)$ is called an extension of the $L$-topological space $(X, \tau)$ if $X \subseteq Y$, $\tau = \sigma_X$ and $X$ is $\sigma$-dense in $Y$.

The extension $(Y, \sigma)$ of $(X, \tau)$ is called reduced if for any $x \neq y$ in $Y$ and $x \in Y \setminus X$, we have $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$, where $\mathcal{N}_\sigma(x)$ denotes the $L$-neighborhood filter of $(Y, \sigma)$ at a point $x \in Y$.

In [2, 3, 7, 8], we have introduced and studied the notion of $GT_i$-spaces for all $i = 0, 1, 2, 3, \frac{3}{2}, 4$.

$GT_i$-spaces. An $L$-topological space $(X, \tau)$ is called [2, 3, 7]:

(1) $GT_0$ if for all $x, y \in X$ with $x \neq y$ we have $\bar{x} \not\subseteq \mathcal{N}(y)$ or $\bar{y} \not\subseteq \mathcal{N}(x)$.

(2) $GT_1$ if for all $x, y \in X$ with $x \neq y$ we have $\bar{x} \not\subseteq \mathcal{N}(y)$ and $\bar{y} \not\subseteq \mathcal{N}(x)$.

(3) $GT_2$ if for all $x, y \in X$ with $x \neq y$, we have $\mathcal{M} \not\subseteq \mathcal{N}(x)$ or $\mathcal{M} \not\subseteq \mathcal{N}(y)$ for all $L$-filters $\mathcal{M}$ on $X$.

(4) regular if for all $x \notin F$ and $F = \text{cl}_\tau F$, we have $\mathcal{N}(x) \land \mathcal{N}(F)$ does not exist.

(5) $GT_3$ if it is $GT_1$ and regular.

(6) completely regular if for all $x \notin F \in \tau'$, there exists a $L$- continuous mapping $f : (X, \tau) \to (I_L, 3)$ such that $f(x) = \bar{1}$ and $f(y) = \bar{5}$ for all $y \in F$.

(7) $GT_{3\frac{1}{2}}$ (or $L$-Tychonoff) if it is $GT_1$ and completely regular.

Denote by $GT_i$-space the $L$-topological space which is $GT_i$, $i = 0, 1, 2, 3, \frac{3}{2}$.

**Proposition 4.1** [2, 3, 7] Every $GT_i$-space is $GT_{i-1}$-space for each $i = 1, 2, 3$, and every $GT_{3\frac{1}{2}}$-space is a $GT_3$-space.

**Lemma 4.1** If the extension $(Y, \sigma)$ of $(X, \tau)$ is a $GT_0$-space, then $(Y, \sigma)$ is a reduced extension of $(X, \tau)$.

**Proof.** Clear. □

**Lemma 4.2** For a $GT_0$-space $(X, \tau)$, the reduced extension $(Y, \sigma)$ also is a $GT_0$-space.

**Proof.** For all $x \neq y$ in $Y \setminus X$, we have $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$. Also for all $x \neq y$ in $X$, we have $\mathcal{N}_\mathcal{\sigma}(x) \neq \mathcal{N}_\mathcal{\sigma}(y)$. Hence, for all $x \neq y$ in $Y$ we get that $\mathcal{N}_\mathcal{\sigma}(x) \neq \mathcal{N}_\mathcal{\sigma}(y)$, and thus $(Y, \sigma)$ is a $GT_0$-space. □
Remark 4.1 Let \((X, \tau)\) be an L-topological space and \(X \subseteq Y\). If we succeed in defining an L-topology \(\sigma\) on \(Y\) such that \((Y, \sigma)\) is an extension of \((X, \tau)\), then \(X\) is a \(\sigma\)-dense in \(Y\) implies that every \(\sigma\)-neighborhood of each \(y \in Y\) intersects \(X\), hence the infimum \(N_\sigma(y) \land \bar{X}\) exists where, for all \(f, g \in L^X\), \(\text{int}_\sigma f(y) = f(x)\) for some \(x \in X\) implies \(\text{int}_\sigma f(y) \land \bigwedge_{x \in X} g(x) \leq f(x)\) for some \(x \in X\) and also \(\text{int}_\sigma f(y) \land \bigwedge_{x \in X} g(x) \leq g(x)\) for all \(x \in X\), and thus \(\text{int}_\sigma f(y) \land \bigwedge_{x \in X} g(x) \leq \sup(f \land g)\) for all \(f, g \in L^X\).

Definition 4.1 Let \((X, \tau), (Y, \sigma)\) be two L-topological spaces and \((Y, \sigma)\) an extension of \((X, \tau)\). Then the L-filter \(N_\sigma(x) \land \bar{X}\) on \(X\), denoted by \(M_x\), will be called a \textit{trace filter at} \(x \in Y\) into \(Y\) and \(M_x = N_\tau(x)\) whenever \(x \in X\). Clearly, \(M_x \not\rightarrow x\).

Definition 4.2 Let \((X, \tau)\) and \((Y, \sigma)\) be two L-topological spaces, \((X', \tau^*)\) an extension of \((X, \tau)\) and let \(f : X \rightarrow Y\) be a \((\tau, \sigma)\)-continuous mapping. Then the restriction mapping \(g|_X\) on \(X\) of the \((\tau^*, \sigma)\)-continuous mapping \(g : X' \rightarrow Y\), which coincides with \(f\), is called a \textit{continuous extension of} \(f\) into \(X'\).

Remark 4.2 Let \((X, \tau)\) and \((Y, \sigma)\) be two L-topological spaces, \((X', \tau^*)\) an extension of \((X, \tau)\), \(f : X \rightarrow Y\) a mapping and \(M_x = N_\tau(x) \land \bar{X}\) a trace filter on \(Y\) at \(x \in X'\). For the existence of a continuous extension \(g : X' \rightarrow Y\), it is necessary that \(f\) is \((\tau, \sigma)\)-continuous and \(F_L f(M_x) \not\rightarrow x\) for a trace filter \(M_x\) at \(x \in X'\). If \((Y, \sigma)\) is a regular space, then these conditions also are sufficient. It is clear that \(M_x \not\rightarrow x\).

Lemma 4.3 With the notations in Remark 4.2, let \(g_1 : X' \rightarrow Y\) and \(g_2 : X' \rightarrow Y\) be \((\tau^*, \sigma)\)-continuous, \((Y, \sigma)\) is a GT$_2$-space and \(g_1|_X = g_2|_X = f\). Then \(g_1 = g_2\).

Proof. Let \(x \in X'\) be arbitrary and \(M_x \not\rightarrow x\). From Lemma 2.1, we get that \(F_L g_1(M_x) \not\rightarrow g_1(x)\) and \(F_L g_2(M_x) \not\rightarrow g_2(x)\), and also we have \(F_L g_1(M_x) = F_L g_2(M_x) = F_L f(M_x)\) an L-filter on \(Y\), and since \((Y, \sigma)\) is a GT$_2$-space, then \(g_1(x) = g_2(x)\). Thus \(g_1 = g_2\). □

Lemma 4.4 An extension \((Y, \sigma)\) of \((X, \tau)\) is reduced if and only if \(M_x \neq M_y\) for all \(x \neq y\) in \(Y\) and \(x \in Y \setminus X\).

Proof. The proof comes from that

\[ M_x = N_\sigma(x) \land \bar{X} \neq N_\sigma(y) \land \bar{X} = M_y \]

if and only if \(N_\sigma(x) \neq N_\sigma(y)\). □

Definition 4.3 An L-uniform space \((Y, U^*)\) is called an \textit{extension} of the L-uniform space \((X, U)\) if \(X \subseteq Y\), \(U = U_X^*\) and \(X\) is a \(\tau_U^*\)-dense in \(Y\).

Definition 4.4 An L-uniform space \((Y, U^*)\) is called a \textit{reduced extension} of the L-uniform space \((X, U)\) if \((Y, \tau_U^*)\) is a reduced extension of \((X, \tau_U)\).
An $L$-uniform structure $\mathcal{U}$ on a set $X$ is called separated [5] if for all $x, y \in X$ with $x \neq y$ there is $u \in L^{X \times X}$ such that $\mathcal{U}(u) = 1$ and $u(x, y) = 0$. The space $(X, \mathcal{U})$ is called separated $L$-uniform space.

**Proposition 4.2** [5] Let $X$ be a set, $\mathcal{U}$ an $L$-uniform structure on $X$ and $\tau_\mathcal{U}$ the $L$-topology associated with $\mathcal{U}$. Then $(X, \mathcal{U})$ is separated if and only if $(X, \tau_\mathcal{U})$ is $GT_0$-space.

**Lemma 4.5** If $(X, \mathcal{U})$ is a separated $L$-uniform space and $(Y, \mathcal{U}^*)$ is a reduced extension of $(X, \mathcal{U})$, then $(Y, \mathcal{U}^*)$ is separated as well.

**Proof.** From Proposition 4.2, we get that $(X, \tau_\mathcal{U})$ is a $GT_0$-space and since $(Y, \tau_\mathcal{U}^*)$ is a reduced extension of $(X, \tau_\mathcal{U})$, then by Lemma 4.2 we have $(Y, \tau_\mathcal{U}^*)$ is a $GT_0$-space. Again by Proposition 4.2, we get that $(Y, \mathcal{U}^*)$ is separated. □

Now, we give this definition.

**Definition 4.5** An $L$-uniform space $(X, \mathcal{U})$ is called complete if every $\mathcal{U}$-cauchy filter $\mathcal{M}$ on $X$ is convergent.

**Definition 4.6** An $L$-uniform space $(Y, \mathcal{U}^*)$ is called a completion of the $L$-uniform space $(X, \mathcal{U})$ if it is a reduced extension of $(X, \mathcal{U})$ and $\mathcal{U}^*$ is complete.

**Lemma 4.6** The completion of a separated $L$-uniform space is separated as well.

**Proof.** The proof comes from Lemma 4.5. □

### 5. The completion of $L$-topological groups

In this section, we introduce the main notion of this paper, that the completion of $L$-topological groups using the completion of $L$-uniform spaces.

**$L$-topological groups.** Let $G$ be a multiplicative group. We denote, as usual, the identity element of $G$ by $e$ and the inverse of an element $a$ of $G$ by $a^{-1}$.

**Definition 5.1** [1, 6] Let $G$ be a group and $\tau$ an $L$-topology on $G$. Then $(G, \tau)$ will be called an $L$-topological group if the mappings

$$\pi : (G \times G, \tau \times \tau) \to (G, \tau)$$

defined by $\pi(a, b) = ab$ for all $a, b \in G$

and

$$i : (G, \tau) \to (G, \tau)$$

defined by $i(a) = a^{-1}$ for all $a \in G$

are $L$-continuous. $\pi$ and $i$ are the binary operation and the unary operation of the inverse on $G$, respectively.
For all \( \lambda \in L^G \), denote by \( \lambda^i \) the L-set \( \lambda \circ i \) in \( G \), that is, \( \lambda^i(x) = \lambda(x^{-1}) \) for all \( x \in G \). We also denote \( \mathcal{F}_L \pi (\mathcal{L} \times \mathcal{M}) \) by \( \mathcal{LM} \) and \( \mathcal{F}_L i(\mathcal{M}) \) by \( \mathcal{M} \), which means that \( \mathcal{LM}(\lambda) = \mathcal{L} \times \mathcal{M}(\lambda \circ \pi) \), and \( \mathcal{M}(\lambda^i) = \mathcal{M}(\lambda^t) \) for all L-filters \( \mathcal{L}, \mathcal{M} \) on \( G \) and all L-sets \( \lambda \in L^G \).

A surrounding \( u \in L^{X \times X} \) is called \textit{left (right) invariant} provided

\[
u(ax, ay) = u(x, y) \quad (u(xa, ya) = u(x, y)) \text{ for all } a, x, y \in X.
\]

\( \mathcal{U} \) is called a \textit{left (right) invariant} L-uniform structure if \( \mathcal{U} \) has a valued L-filter base consists of left (right) invariant surroundings \[8\].

**Proposition 5.1** \[8\] Let \((G, \tau)\) be an L-topological group. Then there exist on \( G \) a unique left invariant L-uniform structure \( \mathcal{U}^l \) and a unique right invariant L-uniform structure \( \mathcal{U}^r \) compatible with \( \tau \), constructed using the family \((\alpha \text{-pr} \mathcal{N}(e))_{\alpha \in \mathcal{L}_0}\) of all filters \( \alpha \text{-pr} \mathcal{N}(e) \), where \( \mathcal{N}(e) \) is the L-neighborhood filter at the identity element \( e \) of \((G, \tau)\), as follows:

\[
\mathcal{U}^l(u) = \bigvee_{v \in \mathcal{U}^{\alpha}_u, v \leq u} \alpha \quad \text{and} \quad \mathcal{U}^r(u) = \bigvee_{v \in \mathcal{U}^\alpha_u, v \leq u} \alpha, \tag{5.1}
\]

where

\[
\mathcal{U}^{\alpha}_u = \alpha \text{-pr} \mathcal{U}^l \quad \text{and} \quad \mathcal{U}^{\alpha}_u = \alpha \text{-pr} \mathcal{U}^r \tag{5.2}
\]

are defined by

\[
\mathcal{U}^{\alpha}_u = \{u \in L^{G \times G} \mid u(x, y) = (\lambda \land \lambda^t)(x^{-1}y) \text{ for some } \lambda \in \alpha \text{-pr} \mathcal{N}(e)\} \tag{5.3}
\]

and

\[
\mathcal{U}^{\alpha}_u = \{u \in L^{G \times G} \mid u(x, y) = (\lambda \land \lambda^t)(xy^{-1}) \text{ for some } \lambda \in \alpha \text{-pr} \mathcal{N}(e)\} \tag{5.4}
\]

We should notice that we shall fix the notations \( \mathcal{U}^l, \mathcal{U}^r, \mathcal{U}^{\alpha}_u, \mathcal{U}^{\alpha}_u \) along the paper to be these defined above.

**Definition 5.2** \( \mathcal{U}^b = \mathcal{U}^l \lor \mathcal{U}^r \) is called the \textit{bilateral} L-uniform structure of the L-topological group \((G, \tau)\), where \( \mathcal{U}^l \) and \( \mathcal{U}^r \) are defined in (5.1) - (5.4).

**Remark 5.1** \( \mathcal{M} \) is a \( \mathcal{U}^b \)-cauchy filter if it is \( \mathcal{U}^l \)-cauchy filter and \( \mathcal{U}^r \)-cauchy filter simultaneously.

**Remark 5.2** \( \text{(cf. \[8\])} \) For the L-topological group \((G, \tau)\), the elements of \( \mathcal{U}^{\alpha}_u \) (\( \mathcal{U}^{\alpha}_u \)) are left (right) invariant surroundings. Moreover, \((\mathcal{U}^{\alpha}_u)_{\alpha \in \mathcal{L}_0}\) \((\mathcal{U}^{\alpha}_u)_{\alpha \in \mathcal{L}_0}\) is a valued L-filter base for the left (right) invariant L-uniform structure \( \mathcal{U}^{\lambda}_u \) \((\mathcal{U}^{\lambda}_u) \) defined by (5.1) - (5.4), respectively.

Now, suppose that \((G, \tau)\) has a countable L-neighborhood filter \( \mathcal{N}(e) \) at the identity \( e \). Since any L-topological group, from Proposition 5.1, is uniformizable, then the left and the right invariant L-uniform structures \( \mathcal{U}^l \) and \( \mathcal{U}^r \), constructed also in Proposition 5.1, has, from Remark 5.2, a countable L-filter base \( \mathcal{U}^{\lambda}_n \) \((\mathcal{U}^{\lambda}_n) \) respectively, \( n \in \mathbb{N} \).

We may recall that if \((G, \tau)\) is an L-topological group and \( A \) is a subgroup of \( G \), then the L-topological subspace \((A, \tau_A)\) is called an L-topological subgroup \([6]\).
Proposition 5.2 Let \((A, \tau_A)\) be an \(L\)-topological subgroup of an \(L\)-topological group \((G, \tau)\), and further \(U\) be a complete \(L\)-uniform structure on \(G\) compatible with \(\tau\) and \(U_A\) is the \(L\)-uniform structure on \(A\) compatible with \(\tau_A\). Then

(d1) If \(L\) and \(M\) are \(U_A\)-cauchy filters, then \(LM\) is a \(U_A\)-cauchy filter as well,
(d2) If \(M\) is a \(U_A\)-cauchy filter, then \(M^i\) is a \(U_A\)-cauchy filter as well.

Proof. By Lemma 3.1, \(L\) and \(M\) are both \(U\)-cauchy filters too, thus \(U\) is complete implies \(L \Rightarrow x\) and \(M \Rightarrow y\) for some \(x, y \in G\), that is, \(L \leq \mathcal{N}(x)\) and \(M \leq \mathcal{N}(y)\). Now, for each \(\xi \in L^G\) we have

\[
LM(\xi) = F_{L^\pi}(L \times M)(\xi) \\
= L \times M(\xi \circ \pi) \\
= \bigwedge_{\lambda \times \mu \leq \xi \circ \pi} L(\lambda) \wedge M(\mu) \\
\geq \bigwedge_{\lambda \times \mu \leq \xi \circ \pi} \mathcal{N}(x)(\lambda) \wedge \mathcal{N}(y)(\mu) \\
= \bigwedge_{\lambda \times \mu \leq \xi \circ \pi} \text{int}_x \lambda(x) \wedge \text{int}_y \mu(y) \\
\geq \text{int}_x \xi(xy) \\
= \mathcal{N}(xy)(\xi).
\]

That is, \(LM \Rightarrow xy\) and hence, \(LM\) is a \(U\)-cauchy filter and at the same time a \(U_A\)-cauchy filter from Proposition 3.1 and Lemma 3.1.

Similarly, if \(M\) is a \(U_A\)-cauchy filter, and thus a \(U\)-cauchy filter, then \(M \Rightarrow x\), and hence by Lemma 2.1, \(M^i(\lambda) = F_{L^i}(M) \Rightarrow i(x) = x^{-1}\). This means that \(M^i\) is a \(U\)-cauchy filter and also a \(U_A\)-cauchy filter.  

\[\Box\]

Definition 5.3 Let us call an \(L\)-uniform structure \(U\) of an \(L\)-topological group \((G, \tau)\) admissible if \(\tau_U = \tau\) and the conditions (d1) and (d2) in Proposition 5.2 are fulfilled.

Definition 5.4 An \(L\)-topological group \((G, \tau)\) is called complete if its bilateral \(L\)-uniform structure \(U^b\) is complete. \((G, \tau)\) is called left complete (right complete) if it is complete and its left (right) \(L\)-uniform structure \(U^l\) (\(U^r\)) is admissible.

Lemma 5.1 The inverse mapping \(i : (G, \tau) \rightarrow (G, \tau), i(x) = x^{-1}\), of any \(L\)-topological group \((G, \tau)\) is \((U^l, U^r)\)-continuous and \((U^r, U^l)\)-continuous, and moreover \(U^r = F_{L}(i \times i)(U^l), U^l = F_{L}(i \times i)(U^r)\).

Proof. For \(u \in U^l_{0}\) and for some \(\lambda \in \alpha - \text{pr} \mathcal{N}(e)\), we have

\[
(u \circ (i \times i))(x, y) = u(x^{-1}, y^{-1}) = (\lambda \land \lambda^i)(xy^{-1}) = w(x, y)
\]
for some \( w \in \mathcal{U}_a^l \). Since \( \mathcal{F}_L(i \circ i)(\mathcal{U}^l)(u) = \mathcal{U}^l(u \circ (i \circ i)) \) for all \( u \in L^{X \times X} \), then \( \mathcal{F}_L(i \circ i)(\mathcal{U}^l)(u) = \mathcal{U}^l(u) \) for all \( u \in L^{X \times X} \), and hence \( i \) is a \((\mathcal{U}^l, \mathcal{U}^l)\)-continuous. Similarly, we get that \( \mathcal{F}_L(i \circ i)(\mathcal{U}^r) = \mathcal{U}^l \), and thus \( i \) is a \((\mathcal{U}^r, \mathcal{U}^l)\)-continuous. \( \square \)

**Proposition 5.3** If \( \mathcal{M} \) is a \( \mathcal{U}^l \)-cauchy filter in an \( L \)-topological group \((G, \tau)\), then \( \mathcal{M}^l \) is a \( \mathcal{U}^r \)-cauchy filter and the converse.

**Proof.** Since, from Lemma 5.1, the mapping \( i : (G, \mathcal{U}^l) \to (G, \mathcal{U}^r) \) is \((\mathcal{U}^l, \mathcal{U}^r)\)-continuous, then \( \mathcal{M} \) is a \( \mathcal{U}^l \)-cauchy filter implies, from Lemma 3.2, that \( \mathcal{F}_L(i)(\mathcal{M}) = \mathcal{M}^l \) is a \( \mathcal{U}^r \)-cauchy filter. Similarly, the converse follows. \( \square \)

**Proposition 5.4** [15] Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be two \( L \)-uniform spaces and \( f : X \to Y \) a mapping. Then the mapping \( f : (X, \tau_{\mathcal{U}}) \to (Y, \tau_{\mathcal{V}}) \) is \( L \)-continuous if and only if \( f \) is \((\mathcal{U}, \mathcal{V})\)-continuous.

Here, we give this result.

**Lemma 5.2** If \( \mathcal{U} \) and \( \mathcal{V} \) are two \( L \)-uniform structures on an \( L \)-topological group \((G, \tau)\) and both \( \mathcal{L} \) and \( \mathcal{M} \) are \( \mathcal{U} \)- (\( \mathcal{V} \)-)cauchy filters on \( G \), then \( \mathcal{L} \times \mathcal{M} \) is a \( \mathcal{U} \times \mathcal{U} \)- (\( \mathcal{V} \times \mathcal{V} \)-)cauchy filter on \( G \times G \).

**Proof.** From Proposition 2.2, \( \mathcal{L} \times \mathcal{M} \) is an \( L \)-filter on \( G \times G \). Let \( \mathcal{L} \) and \( \mathcal{M} \) be \( \mathcal{U} \)-cauchy filters on \( G \), then there exist \( A, B \subseteq G \) such that \( \mathcal{L} \leq \hat{A} \) and \( \mathcal{M} \leq \hat{B} \) and \( A, B \) are small of order every surrounding \( u \) in \((G, \mathcal{U})\). Now,

\[
(\mathcal{L} \times \mathcal{M})(u) = \bigvee_{\lambda \times \mu \leq u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu))
\]

\[
\geq \bigvee_{\lambda \times \mu \leq u} (\hat{A}(\lambda) \wedge \hat{B}(\mu))
\]

\[
= \bigvee_{\lambda \times \mu \leq u} \left( \bigwedge_{x \in A, y \in B} \lambda(x) \wedge \mu(y) \right)
\]

\[
= \bigvee_{\lambda \times \mu \leq u} \left( \bigwedge_{x \in A, y \in B} \lambda \times \mu(x, y) \right)
\]

\[
= u(A, B)
\]

\[
= (A \times B)(u)
\]

for all \( u \in L^{G \times G} \). That is, there exists \( A \times B \subseteq G \times G \) such that \( \mathcal{L} \times \mathcal{M} \leq (A \times B) \).

Let \( \psi : (G \times G) \times (G \times G) \to L \) be a mapping and \( u \) a surrounding in \((G, \mathcal{U})\), then from Proposition 5.4, \( \pi \) is \((\mathcal{U} \times \mathcal{U}, \mathcal{U})\)-continuous, and then

\[
\alpha \leq \mathcal{U}(u) \leq \mathcal{F}_L(\pi \circ \pi)(\mathcal{U} \times \mathcal{U})(u) = \mathcal{U} \times \mathcal{U}(u \circ (\pi \circ \pi)) = \mathcal{U} \times \mathcal{U}(\psi)
\]

and also, \( u = u^{-1} \) implies that

\[
\psi^{-1} = (u \circ (\pi \times \pi))^{-1} = u^{-1} \circ (\pi \times \pi) = u \circ (\pi \times \pi) = \psi,
\]

13
that is, \( \psi \) is a surrounding in \((G \times G, U \times U)\), and for any surrounding \( \psi \) in \((G \times G, U \times U)\), there exists a surrounding \( u \) in \((G, U)\) such that \( \psi = u \circ (\pi \times \pi) \).

Now, \( \alpha \leq u(x, y) \) for all \( x, y \in A \) and \( \beta \leq u(r, s) \) for all \( r, s \in B \) and for some \( \alpha, \beta \in L_0 \) imply that \( \psi((x, r), (y, s)) = (u \circ (\pi \times \pi))((x, r), (y, s)) = u(xr, ys) \), and by choosing \( (x, y) = (e, e) \) or \( (r, s) = (e, e) \), we get that \( u(xr, ys) \geq \gamma \) for some \( \gamma \in L_0 \), that is, for all \( (x, r), (y, s) \in A \times B \), we have \( \psi((x, r), (y, s)) \geq \gamma \) for some \( \gamma \in L_0 \), which means that \( A \times B \) is small of order every surrounding in \((G \times G, U \times U)\), and therefore \( L \times M \) is a \( U \times U \)-cauchy filter.

**Proposition 5.5** If \( U_l \) and \( U_r \) are the left and the right \( L \)-uniform structures of an \( L \)-topological group \( (G, \tau) \) and both of \( L \) and \( M \) are \( U_l \)-\( (U_r \)-cauchy filters, then \( LM \) has the same property.

**Proof.** From Lemma 5.2 and Lemma 3.2, we have \( LM = F_L \pi(L \times M) \) is a \( U_l \)-\( (U_r \)-cauchy filter. \( \square \)

Accordingly, the property of being admissible depends for \( U_l \) and \( U_r \) on the fact whether condition (d2) of Proposition 5.2 is fulfilled.

**Proposition 5.6** The following statements are equivalent in any \( L \)-topological group \( (G, \tau) \).

1. Together with \( M \), \( M^i \) is a \( U_l \)-cauchy filter,
2. Together with \( M \), \( M^i \) is a \( U_r \)-cauchy filter,
3. Every \( U_l \)-cauchy filter is a \( U_r \)-cauchy filter,
4. Every \( U_r \)-cauchy filter is a \( U_l \)-cauchy filter,
5. \( U_l \) is admissible,
6. \( U_r \) is admissible.

**Proof.** (1) \( \iff \) (5) and (2) \( \iff \) (6) come from Proposition 5.5.

(1) \( \iff \) (2) follows from Proposition 5.3 and that \( (M^i)^i = M \).

From (1), since \( M \) is a \( U_l \)-cauchy filter implies that \( M^i \) is a \( U_l \)-cauchy filter, and thus \( M \) is a \( U_r \)-cauchy filter according to Proposition 5.3, then (1) \( \implies \) (3); On the other hand, if \( M \) is a \( U_l \)-cauchy filter, then it is a \( U_r \)-cauchy filter and thus \( M^i \) is a \( U_l \)-cauchy filter. That is, (1) \( \iff \) (3).

(2) \( \iff \) (4) is obtained similarly. \( \square \)

**Proposition 5.7** If the left \( L \)-uniform structure \( U_l \) or the right \( L \)-uniform structure \( U_r \) of an \( L \)-topological group \( (G, \tau) \) is complete, then the other one is complete as well and both are admissible.
Proof. If $\mathcal{U}^t$ is complete and $\mathcal{M}$ is a $\mathcal{U}^t$-cauchy filter, then from Proposition 5.3, $\mathcal{M}^t$ is a $\mathcal{U}^t$-cauchy filter, thus $\mathcal{M}^t \overset{\tau}{\rightarrow} x$ in $G$ and then $\mathcal{M} \overset{\tau}{\rightarrow} x^{-1}$. Hence, $\mathcal{U}^t$ is complete, and the completeness of $\mathcal{U}^t$ follows by the same way from the completeness of $\mathcal{U}$.

At last, $\mathcal{M}$ is a $\mathcal{U}^t$-cauchy filter implies that $\mathcal{M}$ converges to $x \in G$, that is, $\mathcal{M} \leq \mathcal{U}^t[\dot{x}]$, and then $\mathcal{M}^t \leq \mathcal{U}^t[\dot{x}^{-1}]$ and from Proposition 3.1, $\mathcal{M}^t$ is a $\mathcal{U}^t$-cauchy filter. Proposition 5.6 implies that both $\mathcal{U}^t$ and $\mathcal{U}$ are admissible. □

Lemma 5.3 If $\mathcal{U}^b$ is the bilateral $L$-uniform structure of an $L$-topological group $(G, \tau)$, then $i$ is $(\mathcal{U}^b, \mathcal{U}^b)$-continuous.

Proof. From that $\mathcal{U}^t \leq \mathcal{U}^b$ and $\mathcal{U} \leq \mathcal{U}^b$, we get that $\mathcal{F}(i \times i) \mathcal{U}^t \leq \mathcal{U}^b$ and $\mathcal{F}(i \times i) \mathcal{U} \leq \mathcal{U}^b$, and thus

$$\mathcal{F}(i \times i) \mathcal{U}^b = \mathcal{F}(i \times i) \mathcal{U}^t \vee \mathcal{F}(i \times i) \mathcal{U} \leq \mathcal{U}^b.$$ 

Hence, $i$ is $(\mathcal{U}^b, \mathcal{U}^b)$-continuous. □

$L$-metric spaces. We use here the notion of $L$-metric space defined by means of the notion of $L$-real numbers in [12]. By an $L$-real number is meant [12] a convex, normal, compactly supported and upper semi-continuous $L$-subset of the set of all real numbers $\mathbb{R}$. The set of all $L$-real numbers is denoted by $\mathbb{R}_L$. $\mathbb{R}$ is canonically embedded into $\mathbb{R}_L$, identifying each real number $a$ with the crisp $L$-number $a^\sim$ defined by $a^\sim(\xi) = 1$ if $\xi = a$ and 0 otherwise. The set of all positive $L$-real numbers is defined and denoted by:

$$\mathbb{R}^*_L = \{ x \in \mathbb{R}_L \mid x(0) = 1 \text{ and } 0^\sim \leq x \}$$

A mapping $\varrho : X \times X \rightarrow \mathbb{R}^*_L$ is called an $L$-metric [12] on $X$ if the following conditions are fulfilled:

1. $\varrho(x, y) = 0^\sim$ if and only if $x = y$
2. $\varrho(x, y) = \varrho(y, x)$
3. $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$.

If $\varrho : X \times X \rightarrow \mathbb{R}^*_L$ satisfied the conditions (2) and (3) and the following condition:

$$(1)' \quad \varrho(x, y) = 0^\sim \text{ if } x = y$$
then it is called an $L$-pseudo-metric on $X$.

A set $X$ equipped with an $L$-pseudo-metric ($L$-metric) $\varrho$ on $X$ is called an $L$-pseudo-metric ($L$-metric) space.

To each $L$-pseudo-metric ($L$-metric) $\varrho$ on a set $X$ is generated canonically a stratified $L$-topology $\tau_\varrho$ on $X$ which has $\{ \varepsilon \circ \varrho_x \mid \varepsilon \in \mathcal{E}, \ x \in X \}$ as a base, where $\varrho_x : X \rightarrow \mathbb{R}^*_L$ is the mapping defined by $\varrho_x(y) = \varrho(x, y)$ and

$$\mathcal{E} = \{ \overline{\alpha} \wedge R^\delta | \mathbb{R}^*_L \mid \delta > 0, \ \alpha \in L \} \cup \{ \overline{\alpha} \mid \alpha \in L \},$$

here $\overline{\alpha}$ has $\mathbb{R}^*_L$ as domain.

15
An $L$-topological space $(X, \tau)$ is called pseudo-metrizable (metrizable) if there is an $L$-pseudo-metric ($L$-metric) $\varrho$ on $X$ inducing $\tau$, that is, $\tau = \tau_0$.

An $L$-pseudo-metric $\varrho$ is called left (right) invariant if

\[
\varrho(x, y) = \varrho(ax, ay) \quad (\varrho(x, y) = \varrho(xa, ya)) \quad \text{for all } a, x, y \in X.
\]

An $L$-topological group $(G, \tau)$ is called separated if for the identity element $e$, we have

\[
\bigwedge_{\lambda \in \alpha-\mu N(e)} \lambda(e) \geq \alpha, \quad \text{and} \quad \bigwedge_{\lambda \in \alpha-\mu N(e)} \lambda(x) < \alpha \quad \text{for all } x \in G \text{ with } x \neq e \text{ and for all } \alpha \in L_0 [8].
\]

**Proposition 5.8** [9] Let $(G, \tau)$ be a (separated) $L$-topological group. Then the following statements are equivalent.

1. $\tau$ is pseudo-metrizable (metrizable);
2. $e$ has a countable $L$-neighborhood filter $N(e)$;
3. $\tau$ can be induced by a left invariant $L$-pseudo-metric ($L$-metric);
4. $\tau$ can be induced by a right invariant $L$-pseudo-metric ($L$-metric).

**Definition 5.5** An $L$-uniform structure $\mathcal{U}$ on a set $X$ is called pseudo-metrizable (metrizable) if there exists a countable $L$-uniform base for $\mathcal{U}$ (and $\mathcal{U}$ is separated).

**Proposition 5.9** [8] Let $(G, \tau)$ be an $L$-topological group. Then there exist on $G$ a unique left invariant $L$-uniform structure $\mathcal{U}^l$ and a unique right invariant $L$-uniform structure $\mathcal{U}^r$ compatible with $\tau$, constructed with (5.1) - (5.4).

**Proposition 5.10** For any (separated) $L$-topological group $(G, \tau)$, The $L$-uniform structures $\mathcal{U}^l, \mathcal{U}^r$ and $\mathcal{U}^b$ constructed in (5.1) - (5.4) are pseudo-metrizable (metrizable).

**Proof.** From Proposition 5.8, $\tau = \tau_{e_1} = \tau_{e_2}$ where $\varrho_1$ is a left, $\varrho_2$ is a right invariant $L$-pseudo-metric ($L$-metric) on $G$, and then $U_{e_1}$ is left invariant and $U_{e_2}$ is right invariant. From Proposition 5.9, $\mathcal{U}^l$ and $\mathcal{U}^r$ are unique, that is, $U_{e_1} = \mathcal{U}^l$, $U_{e_2} = \mathcal{U}^r$ and $\mathcal{U}^l$, $\mathcal{U}^r$ are pseudo-metrizable (metrizable). Moreover, $\tau_{U^b} = \tau_{\mathcal{U}^l \cup \mathcal{U}^r} = \tau_{\mathcal{U}^l} \lor \tau_{\mathcal{U}^r} = \tau$. Hence, $\mathcal{U}^b$ is pseudo-metrizable (metrizable) as well. □

**Proposition 5.11** [4] Let $(X, \mathcal{U})$ be an $L$-uniform space, $(A, \mathcal{U}_A)$ an $L$-uniform subspace of $(X, \mathcal{U})$ and $(\tau_{\mathcal{U}})_A$ the $L$-subtopology of the $L$-topology $\tau_{\mathcal{U}}$ associated with $\mathcal{U}$. Then the $L$-topology associated to $\mathcal{U}_A$ coincides with $(\tau_{\mathcal{U}})_A$, that is, $\tau_{(\mathcal{U}_A)} = (\tau_{\mathcal{U}})_A$.

**Lemma 5.4** Let $(A, \tau_A)$ be an $L$-topological subgroup of an $L$-topological subgroup $(G, \tau)$, and $\mathcal{U}^l_A, \mathcal{U}^r_A$ and $\mathcal{U}^b_A$ the left, the right and the bilateral $L$-uniform structures of $(G, \tau)$. Then the corresponding $L$-uniform structures of $(A, \tau_A)$ are $(\mathcal{U}^l)_A, (\mathcal{U}^r)_A$ and $(\mathcal{U}^b)_A$, respectively.
Proof. From Proposition 5.11, we have \( \tau(U^l)_A = (\tau(U^l))_A = \tau_A \) and, together with \( U^l \), \( (U^l)_A \) is left invariant as well, and hence \( (U^l)_A \) is the left invariant \( L \)-uniform structure of \((A, \tau_A)\). By the same \( (U^r)_A \) is the right invariant \( L \)-uniform structure of \((A, \tau_A)\) as well. Moreover,

\[
\tau(\xi)_A = \tau(U^l_A \cup U^r_A) = \tau(U^l_A) \vee \tau(U^r_A) = (\tau(U^l))_A \vee (\tau(U^r))_A = (\tau(U^b))_A = \tau_A.
\]

Here, we give the essential result in this section.

**Definition 5.6** For a separated \( L \)-topological group \((G, \tau)\), let us call \((H, \sigma)\) a completion of \((G, \tau)\) if it is complete separated \( L \)-topological group and in which \((G, \tau)\) is a dense subgroup.

In the following we need this result.

**Proposition 5.12** [8] Let \((G, \tau)\) be an \( L \)-topological group. Then the following statements are equivalent.

1. The \( L \)-topology \( \tau \) is \( GT_0 \).
2. The \( L \)-topology \( \tau \) is \( GT_2 \).
3. The \( L \)-topological group \((G, \tau)\) is separated.

**Proposition 5.13** Let \((G, \tau)\) be a separated \( L \)-topological group, \( U \) an admissible \( L \)-uniform structure on \( G \), and \((H, \nu)\) the completion of \((G, U)\). Then an operation \( \pi' : H \times H \to H \) can be defined on \( H \) in a unique way so that \( H \) equipped with \( \pi' \) is a group, and \((H, \nu)\) is an \( L \)-topological group of which \( G \) is a subgroup.

Proof. Let \( \sigma = \tau_V \). If \( \pi' : H \times H \to H \) is defined by \( \pi'(y, z) = yz \) for all \( y, z \in H \), then \( \pi'|_{G \times G} = \pi \). Now, let \( L_x \) and \( M_y \) be two trace filters on \( H \) at \( x \) and \( y \) into \( H \), respectively. Since \( L_x \xrightarrow{\sigma} x \) and \( M_y \xrightarrow{\sigma} y \), that is, \( L_x(\lambda) \geq \text{int}_\sigma \lambda(x) \) and \( M_y(\mu) \geq \text{int}_\sigma \mu(y) \), then

\[
L_x M_y(\xi) = \mathcal{F}_{L} \pi' (L_x \times M_y)(\xi) = L_x \times M_y(\xi \circ \pi') \geq \bigvee_{\lambda \times \mu \leq \xi} \text{int}_\sigma \lambda(x) \wedge \text{int}_\sigma \mu(y) \geq \text{int}_\sigma \xi(xy) = N_\sigma(xy)(\xi),
\]

and then \( L_x M_y \xrightarrow{\sigma} xy \). From that \( U \) is separated and from Lemma 4.6 and Proposition 5.12, we get \((H, \sigma)\) is a \( GT_2 \)-space, and therefore these properties, using Lemma 4.3
and Remark 4.2, define \( \pi' \) in a unique way as the only continuous extension of \( \pi \) into \( H \times H \). Also, if \( i': H \to H \) is defined by \( i'(y) = y^{-1} \) for all \( y \in H \), then \( i'|_G = i \) and \( \mathcal{F}_L i'(\mathcal{L}_x) = \mathcal{L}_x^L \xrightarrow{\sigma} x^{-1} \) for any trace filter \( \mathcal{L}_x \) on \( H \), and \( i' \) is \((\sigma, \sigma)\)-continuous, that is, as in before, \( i' \) is a continuous extension of \( i \) defined in a unique manner.

Hence, \( \pi' \) is \((\sigma \times \sigma, \sigma)\)-continuous and \( i' \) is \((\sigma, \sigma)\)-continuous imply that \((H, \sigma)\) is an \( L \)-topological group in which \((G, \tau)\) is an \( L \)-topological subgroup. \( \square \)

**Proposition 5.14** Under the hypothesis of Proposition 5.13, if the left, the right or the bilateral \( L \)-uniform structure of \((H, \tau_U)\) is \( \U^{\ell}, \U^{\ell^*}, \) or \( \U^{\ell^*} \) respectively, then the corresponding \( L \)-uniform structures of \((G, \tau)\) are \( (\U^{\ell^*})_G, (\U^{\ell^*})_G, \) or \( (\U^{\ell^*})_G. \)

**Proof.** It is a consequence of Lemma 5.4. \( \square \)

**Proposition 5.15** Let \((G, \tau)\) be a separated \( L \)-topological group, \( \U^b \) its bilateral \( L \)-uniform structure, and \((H, \sigma = \tau_V)\) the \( L \)-topological group constructed in Proposition 5.13 with the choice \( \mathcal{V} = \mathcal{V}^b \). Then \((H, \sigma)\) is a completion of \((G, \tau)\).

**Proof.** If \( \mathcal{U} = \U^b \), then Proposition 5.13 can be applied and \( \U^b \) is admissible where both of \( \U^\ell \) and \( \U^r \) are admissible. Also, \( \mathcal{V} \) is a complete separated \( L \)-uniform structure such that \( \sigma = \tau_V, G \) is \( \sigma \)-dense in \( H \) and \( (\mathcal{V})_G = \U^b \). On the other hand, by Proposition 5.14, for the bilateral \( L \)-uniform structure \( \mathcal{V}^b \) of the \( L \)-topological group \((H, \sigma)\) we have \( \sigma = \tau_{(\mathcal{V}^b)} \) and \( (\mathcal{V}^b)_G = \U^b \). Therefore, the bilateral \( L \)-uniform structure \( \mathcal{V}^b \) of \((H, \sigma)\) is complete and \((H, \sigma)\) is a completion of \((G, \tau)\). \( \square \)

**References**


