Dynamic generalization of Stoner–Wohlfarth model

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A dynamic generalization of the Stoner–Wohlfarth (SW) model is presented, in which the applied field component \( h_{az} \) parallel to the particle anisotropy axis is slowly varied, whereas the one in the perpendicular plane, \( h_{a\perp} \), is rotated at the angular frequency \( \omega \). The Landau–Lifshitz–Gilbert (LLG) equation is solved to calculate the system response under the constraint of spatially uniform magnetization at all times. Switching under slowly varying \( h_{az} \) is discussed. Various switching modes can occur in correspondence of different types of bifurcation present in the dynamics. In addition, it is shown that the system response can become quasiperiodic, with spontaneous magnetization oscillations at a frequency definitely lower than \( \omega \).

I. INTRODUCTION

In the classical Stoner–Wohlfarth (SW) model\(^1\)–\(^3\) one describes the magnetic response of a spheroidal particle with uniaxial anisotropy, subject to a slowly varying field of components \( h_{az} \) and \( h_{a\perp} \), parallel and perpendicular to the anisotropy axis, respectively. The basic assumption made in the model is that the magnetization is spatially uniform at all times, which in particular implies that switching will occur by coherent rotation. The switching properties of the model are controlled by the astroid of equation \( h_{az}^{2/3} + h_{a\perp}^{2/3} = 1 \) (if \( h_{az} \) and \( h_{a\perp} \) are normalized to the anisotropy field), representing the locus of the critical fields at which irreversible rotation of the magnetization can occur.

In this article, a dynamic generalization of the SW model is studied, in which the field component \( h_{a\perp} \) is rotated at the angular frequency \( \omega \), i.e., is circularly polarized, and the Landau–Lifshitz–Gilbert (LLG) equation is solved to calculate the system response under the constraint of spatially uniform magnetization at all times. The analogy with the SW model becomes apparent by passing to the rotating frame of reference in which the field is stationary and the dynamics acquire autonomous (i.e., time-independent) form.\(^4\) In fact, the rotating-frame formulation always admits exact stationary solutions, which represent uniform magnetization modes rigidly rotating with the field. These modes are the close analogue of the static equilibrium states of the SW model. In fact, quite similarly to the SW case, only two situations are possible, one with two and one with four rotating modes. The boundary between the field regions where the two situations are realized represents the natural generalization of the SW astroid. An analytical expression can be derived for the \( \omega \)-dependent astroid shape, which approaches the expression \( h_{az}^{2/3} + h_{a\perp}^{2/3} = 1 \) in the limit \( \omega \to 0 \).

Switching can be studied by slowly varying \( h_{az} \) and/or \( h_{a\perp} \) under constant \( \omega \). Contrary to the classical SW model, where switching necessarily involves a saddle-node bifurcation at the astroid boundary, various switching modes associated with different types of bifurcation can occur under circularly polarized field. In addition, one finds that the system response can become quasiperiodic. When this is the case, the magnetization exhibits spontaneous oscillations at a frequency definitely lower than \( \omega \), with a superimposed modulation at the frequency \( \omega \).

II. LLG SOLUTIONS UNDER CIRCULARLY POLARIZED FIELD

We consider a spheroidal particle with symmetry axis along \( z \) and uniaxial anisotropy also along \( z \). The magnetization \( \mathbf{M} \) is spatially uniform and depends on time only. The LLG equation for \( \mathbf{M}(t) \), in dimensionless Gilbert form, reads

\[
\frac{d\mathbf{m}}{dt} - \alpha \mathbf{m} \times \frac{d\mathbf{m}}{dt} = -\mathbf{m} \times \mathbf{h}_{\text{eff}},
\]

where \( \mathbf{h}_{\text{eff}} = \mathbf{H}_{\text{eff}}/M_s \) is the normalized effective field, \( \mathbf{m} = M/M_s \) is the unit magnetization vector, \( \alpha \) is the damping constant. The effective field includes the applied, demagnetizing and anisotropy fields. The applied field has a dc component \( h_{az} \) along \( z \) and a circularly polarized component of amplitude \( h_{a\perp} \) rotating at the angular frequency \( \omega \) in the \( x-y \) plane. Let us pass to the rotating frame of reference in which the applied field is stationary, and let us introduce spherical coordinates for \( \mathbf{m} \). In other words, let us set \( m_x = \sin \theta \cos(\omega t - \phi), m_y = \sin \theta \sin(\omega t - \phi), m_z = \cos \theta \), where \( \phi \) represents the magnetization lag with respect to the rotating field. In terms of \( (\theta, \phi) \), Eq. (1) becomes

\[
\frac{d\theta}{dt} - \alpha \sin \theta \frac{d\phi}{dt} = \kappa_{\text{eff}} [h_{a\perp} \sin \phi - \Omega \sin \theta],
\]

\[
\frac{d\phi}{dt} = \kappa_{\text{eff}} [h_{a\perp} \sin \phi - \Omega \sin \theta],
\]

\[\begin{align*}
\frac{\partial F}{\partial \mu} &= 0, \\
\frac{\partial F}{\partial \lambda} &= 0,
\end{align*}\]

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\end{align*}\]
\[
\alpha \frac{d \theta}{dt} + \sin \theta \frac{d \phi}{dt} = \kappa_{\text{eff}} b_\perp \cos \phi \cos \theta
\]
\[
-(b_z + \cos \theta) \sin \theta,
\]
where \( b_z = (h_{az} - \omega) / \kappa_{\text{eff}}, \quad b_\perp = h_{a\perp} / \kappa_{\text{eff}}, \quad \Omega = \alpha \omega / \kappa_{\text{eff}}, \quad \kappa_{\text{eff}} = 2 K_1 / \mu_0 M_s^2 + N_1 - N_2, \) and \((N_1, N_2)\) are the particle demagnetizing factors. Differences between systems are measured by \((\alpha, \kappa_{\text{eff}})\), differences in the excitation conditions by \((b_z, b_\perp, \Omega)\) or, equivalently, by \((h_{az}, h_{a\perp}, \omega)\).

Equations (2) and (3) describe an autonomous dynamical system on the unit sphere. It is known that any autonomous dynamics on the sphere always admits stationary solutions (fixed points), and that the difference between the number of node-type (or focus-type) fixed points and the number of saddle-type fixed points is equal to two (Poincaré–index theorem).

Therefore, Eqs. (2) and (3) always admit at least two fixed points in the rotating frame, which, in the laboratory frame, result in magnetization modes rigidly rotating with the field. These modes will be termed \( P \) modes.

Interestingly, the number of \( P \) modes is limited to two or four. In fact, by setting \( d \theta / dt = d \phi / dt = 0 \) in Eqs. (2) and (3) and by eliminating \( \phi \) from the ensuing equations, one finds that \( P \) modes must satisfy the fourth-order polynomial equation in \( m_z \):

\[
\frac{b_z^2}{1-m_z^2} - \frac{(b_z + m_z)^2}{m_z^2} = \Omega^2. \tag{4}
\]

Equation (4) has at most four real roots in the interval \(-1 \leq m_z \leq 1\). When combined with Poincaré index theorem, this conclusion shows that only two \( P \)-mode configurations are possible: (i) two nodes or foci; (ii) three nodes or foci plus one saddle.

The stability of \( P \) modes is controlled by the trace and the determinant of the stability matrix \( A \) obtained by linearizing Eqs. (2) and (3) around a given \( P \)-mode solution. By standard methods\(^7\) one obtains

\[
\text{tr} \ A = -\frac{2 \alpha \kappa_{\text{eff}}}{1+\alpha^2} \left[ \left( \frac{1}{2} + \frac{b_z}{m_z} \right) - \frac{1-m_z^2}{2} + \frac{\Omega m_z}{\alpha} \right], \tag{5}
\]

\[
\det \ A = \frac{\kappa_{\text{eff}}}{1+\alpha^2} \left[ \left( \frac{1}{2} + \frac{b_z}{m_z} \right)^2 - (1-m_z^2) \left( 1 + \frac{b_z}{m_z} \right) + \Omega^2 m_z^2 \right]. \tag{6}
\]

Stable nodes or foci are characterized by \((\text{tr} \ A < 0, \det \ A > 0)\), unstable nodes or foci by \((\text{tr} \ A > 0, \det \ A > 0)\), saddles by \((\text{tr} \ A < 0, \det A < 0)\).\(^7\) The \((b_z, b_\perp)\) regions admitting two and four \( P \) modes are separated by the boundary \( \det A = 0 \), because one of the four \( P \) modes is necessarily a saddle, characterized by \( \det A < 0 \). The curve \( \det A = 0 \) can be expressed in parametric form \( [b_z(m_z), b_\perp(m_z)] \), with \( m_z \) as independent variable. By equating to zero Eq. (6) one finds

\[
b_z(m_z) = -m_z \left[ 1 - \frac{1-m_z^2}{2} \right] \left[ 1 \pm \sqrt{1 - \frac{4\Omega^2 m_z^2}{(1-m_z^2)^2}} \right] \tag{7}
\]

where \( b_\perp(m_z) \) is obtained by inserting Eq. (7) into Eq. (4). The result is shown in Fig. 1. The region \( \det A \leq 0 \) represents the dynamic generalization of SW astroid\(^2\) and reduces to it when \( \omega \to 0 \). In fact, in this limit the two branches of Eq. (7) become \( [b_z(m_z) = -m_z^2, b_\perp(m_z) = (1 - m_z^2)^{3/2}] \) and \( [b_z(m_z) = -m_z, b_\perp(m_z) = 0] \). The first branch is nothing but the astroid, \( b_z^2 + b_\perp^2 = 1 \).

III. Magnetization Switching

Magnetization switching occurs whenever the state occupied by the system loses stability. In the SW model, there is only one situation in which this can occur, i.e., when the state is destroyed by a saddle-node bifurcation at the asteroid boundary. Conversely, several switching modes are possible in the dynamic generalization here proposed. There are two main reasons for this more complex behavior: the fact that stable quasi-oscillatory magnetization modes may be present in addition to \( P \) modes, and the fact that several types of bifurcation can affect the stability of a given state.

Quasi-oscillatory modes result from the fact that the autonomous dynamics on the unit sphere admit not only fixed-point solutions but also limit-cycle solutions (Poincaré–Bendixson theorem\(^3\)). In a limit cycle, \( m \) executes a periodic motion along a closed path on the unit sphere. However, this conclusion holds in the rotating frame. In the laboratory frame, the same motion appears as the combination of the periodic motion along the limit cycle and the rotation of the reference frame. The resulting motion will be quasi-oscillatory whenever the limit cycle and the external-field frequencies are not commensurable. These quasi-oscillatory modes will be termed \( Q \) modes. The physical reasons for the onset of \( Q \) modes can be illustrated by the following example. Let us consider a thin film \((\kappa_{\text{eff}} < 0)\) subject to the dc field \( h_{az} < |\kappa_{\text{eff}}| \) and to no rotating field \( h_{a\perp} = 0 \). Under these conditions, the system admits a continuous set of equilibrium states with \( \cos \theta = h_{a\perp} / |\kappa_{\text{eff}}| \) and arbitrary \( \phi \). In the rotating frame, this set appears as a limit cycle of period \( 2\pi / \omega \). When the small circularly polarized component \( h_{a\perp} \) is applied, the original static equilibrium state is changed into a quasi-periodic motion. This occurs whenever \( h_{a\perp} \) is not large enough to force the magnetization into synchronous rotation. The vector \( m \) follows \( h_{a\perp} \) only for a small part of each rotation period and then periodically falls off synchronism. The result is a \( Q \) mode consisting of a slow precession of \( m \) around the \( z \) axis.
accompanied by a superimposed small nutation of angular frequency \( \omega \). If \( h_{az} \) is made large enough, \( m \) gets locked to \( h_{az} \) and the \( Q \) mode is destroyed in favor of a stable \( P \) mode.

The role of bifurcations in magnetization switching is rather complex. More detailed results on the bifurcation properties of LLG dynamics are presented elsewhere.\(^8\) The main point relevant to the present discussion is that four types of bifurcation can occur, i.e., saddle-node, Andronov–Hopf, saddle-connection, and semistable-limit-cycle bifurcation.\(^7,9\) In saddle-node bifurcation, a saddle-node couple of \( P \) modes is created or destroyed and the system passes from two to four \( P \) modes or vice versa. This bifurcation occurs for \( \det A = 0 \), i.e., when the field crosses the dynamic astroid in the \((h_{az}, h_{a\perp})\) plane (see Fig. 2). This is the only switching mode present in the SW model. In Andronov–Hopf bifurcation, a nonsaddle \( P \) mode changes from stable to unstable or vice versa, with the simultaneous creation or destruction of a limit cycle. This bifurcation occurs for \( \text{tr} A = 0 \) and \( \det A > 0 \) (dashed line in Fig. 2). Saddle connections (circles in Fig. 2) and semistable-limit-cycle bifurcations (triangles in Fig. 2) are bifurcations of more complicated nature which do not affect the stability of \( P \) modes. Nonetheless, they play a role because they involve the creation or destruction of limit cycles which can cause switching from \( P \)-type to \( Q \)-type response or vice versa.

As an example of hysteresis and switching behavior under circularly polarized field, let us consider a particle with \( \kappa_{\text{eff}} > 0 \) subject to a rotating field of constant amplitude \( h_{a\perp} \). We analyze the response of the system when the field \( h_{az} \) is slowly cycled between opposite large values. The two or four \( P \) modes present in the dynamics are characterized by different values of the magnetization component \( m_z \) along \( h_{az} \), obtained by solving Eq. (4) (see Fig. 3). Past field history will determine which \( m_z \) value is realized under given \( h_{az} \). The condition of constant \( h_{a\perp} \) and slowly varying \( h_{az} \) is described by a horizontal line in the \((h_{az}, h_{a\perp})\) plane (see Fig. 2). The points \( A, B, \) etc. of Fig. 2 where this line crosses the various bifurcation lines of the problem are potential switching points where the system may jump from one \( P \) mode to another or between \( P \)-type and \( Q \)-type response.

Under increasing field, all bifurcations encountered at points \( A, B, \) etc. involve states different from the state occupied by the system. Switching occurs only at point \( E \), where the system becomes unstable because of a Hopf bifurcation. Switching results in the sudden jump of \( m_z \) from negative to positive value shown in Fig. 3. Note that, contrary to what occurs in the SW model, the saddle-node bifurcation at the astroid boundary \( F \) plays no role in this case. Under decreasing field, the situation is more complex. A first switching event occurs at point \( C \), where the stable state occupied by the system is destroyed by the saddle-node bifurcation at the dynamic astroid boundary. The system jumps from \( P \)-type to \( Q \)-type response. The magnetization exhibits spontaneous auto oscillations in the entire interval from \( C \) to \( B \), in which \( m_z \) oscillates between the upper and lower bounds of the triangular region shown in Fig. 3. At point \( B \), the \( Q \) mode is changed into a \( P \) mode through a Hopf bifurcation. No magnetization jump occurs at this point. Finally, switching occurs at point \( A \), because of a second Hopf bifurcation in which \( m_z \) jumps back from positive to negative value (see Fig. 3).

6. G. Bertotti, A. Magni, I. D. Mayergoyz, and C. Serpico (these proceedings).