On the Integrability of Quasihomogeneous and Related Planar Vector Fields

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Abstract

In this work we consider planar quasihomogeneous vector fields and we show, among other qualitative properties, how to calculate all the inverse integrating factors of such $C^1$ systems. Additionally, we obtain a necessary condition in order to have analytic inverse integrating factors and first integrals for planar positively semi-quasihomogeneous vector fields which is related with the existence of polynomial inverse integrating factors and first integrals for the quasihomogeneous cut. Examples are given and their relationship with Kovalevskaya exponents is shown.

1 Introduction

We concentrate our attention to the quasihomogeneous planar differential systems, which are also called similarity invariant systems or weighted homogeneous systems. That is, autonomous differential equations in the real affine plane

$$\frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dy}{dt} = \dot{y} = Q(x, y),$$

which are invariant under the similarity transformation

$$(x, y, t) \rightarrow (\alpha^p x, \alpha^q y, \alpha^{-\ell} t)$$

for all $\alpha \in \mathbb{R}$ and where $p$ and $q$ are positive integers. In other words, $P$ and $Q$ are $p - q$ quasihomogeneous functions of weighted degrees $p + \ell$ and $q + \ell$ respectively, i.e.,

$$P(\alpha^p x, \alpha^q y) = \alpha^{p+\ell} P(x, y) \quad \text{and} \quad Q(\alpha^p x, \alpha^q y) = \alpha^{q+\ell} Q(x, y)$$

for all $\alpha \in \mathbb{R}$. We will say that system (1) is $p - q$ quasihomogeneous of weighted degree $\ell$.

Let us notice that if $p$ is even and $q$ and $\ell$ are odd, then the $p - q$ quasihomogeneous systems include some class of time-reversible systems which are invariant under the symmetry $(x, y, t) \rightarrow (x, -y, -t)$. Moreover, in the particular case $p = q = 1$, the

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$p - q$ quasihomogeneous systems reduce to an homogeneous system.

Two of the main open problems of the qualitative theory of planar differential equations are the center-focus problem and the determination of the number of limit cycles and their distribution in the plane. We recall that an isolated periodic orbit in the set of all periodic orbits is called a limit cycle. Recently, several works have been show that an unified method based in the concept of inverse integrating factor can be used to study both problems, see for instance [Chavarriga et.al, 1999] and references therein.

One of the best ways to understand the properties of a given planar differential system is through their inverse integrating factors. Existence of inverse integrating factors gives a lot of information on dynamics, phase space structure and so on, but in general, it is difficult to search inverse integrating factors.

It is known that the existence or non-existence of first integrals of system (1) is related to the Kovalevskaya exponents, first called by [Yoshida, 1983]. Although the main theorem of such work seems powerful, it has an important weak point, namely, it gives some conditions for the Kovalevskaya exponents under the existence of a “non-degenerate” first integral. Later, in [Furta, 1996] the author overcame this weak point although the assertion is a little weak.

The paper is organized as follows: In second section we give some qualitative properties of system (1) related with its singular points and the associated monodromy problem. In the third section we prove one of the main results of the paper, see Theorem 4, which states how to calculate all the inverse integrating factors of $C^1$ systems (1) while section 4 is devoted to show some examples. In the section fifth we recall the concept of Kovalevskaya exponents and, finally the last section tries on another fundamental result of the paper (Theorem 9) which gives a necessary condition in order to have analytic inverse integrating factors and first integrals for planar positively semi-quasihomogeneous vector fields.

2 Some properties of quasihomogeneous vector fields

The point $(x_0, y_0) \in \mathbb{C}^2$ is called a finite singular point of system (1) if $P(x_0, y_0) = Q(x_0, y_0) = 0$. For the subset of real finite singular points of quasihomogeneous systems we have the next result.

**Lemma 1** The polynomial $p-q$ quasihomogeneous system (1) with $P$ and $Q$ coprimes only has the origin as real finite singular point.

**Proof.** Let $(P(x, y), Q(x, y))$ be a quasihomogeneous vector field. Hence taking $\alpha = 0$ into its definition easily follows that $P(0, 0) = Q(0, 0) = 0$ and the origin is always a singular point.

Assume now that $(x_0, y_0) \in \mathbb{R}^2$ is another singular point different from the origin. In consequence, since $P(\alpha^p x_0, \alpha^q y_0) = \alpha^{p+q} P(x_0, y_0) = 0$ and $Q(\alpha^p x_0, \alpha^q y_0) = \alpha^{p+q} Q(x_0, y_0) = 0$, all the points of the curve $\gamma = \{(\alpha^p x_0, \alpha^q y_0) : \alpha \in \mathbb{R}\}$ are singular points also. But this implies that the polynomials $P$ and $Q$ are not coprimes in contradiction with our hypothesis. $\blacksquare$
There is, in general, common particular solutions of special type for similarity invariant systems as we will show. See also [Yoshida, 1983] for the particular case $\ell = 1$.

**Lemma 2** The $p-q$ quasihomogeneous system (1) of weighted degree $\ell \neq 0$ possesses the particular solution $x(t) = c_1 t^{-p/\ell}$, $y(t) = c_2 t^{-q/\ell}$ where the constant vector $(c_1, c_2)$ belongs to the set of real solutions of the algebraic equations

$$P(c_1, c_2) + \frac{p}{\ell}c_1 = 0, \quad Q(c_1, c_2) + \frac{q}{\ell}c_2 = 0.$$  

(3)

**Proof.** Keeping in mind the $p-q$ quasihomogeneous structure of the system, straightforward computation gives the result. \[\square\]

A real finite singular point is called **monodromic** if there are no orbits tending or leaving it with a certain angle. When $P$ and $Q$ are analytic, a monodromic singular point is always either a **center** or a **focus**, i.e., either the singular point is surrounded by closed periodic orbits or each trajectory in a neighborhood of the singular point is a spiral winding around it.

Let $X = (P, Q)$ be the analytic vector field associated to a non necessary weighted homogeneous system (1). A singular point $(x_0, y_0)$ is called **degenerate** if the differential matrix $DX(x_0, y_0)$ associated to it is degenerate, that is, the jacobian $\text{det} DX(x_0, y_0) = 0$. Otherwise, we will say that the singular point $(x_0, y_0)$ is **nondegenerate**. If $DX(x_0, y_0)$ has only one eigenvalue equal to zero then $(x_0, y_0)$ is an **elementary degenerate** singular point and, according to [Andronov et al., 1973] it cannot be monodromic. If zero is a double eigenvalue of $DX(x_0, y_0)$ but $DX(x_0, y_0)$ is not identically zero then the degenerate singular point $(x_0, y_0)$ is called **nilpotent**. For nilpotent singular points, the **monodromy problem**, which consists in to determine under what conditions such point is monodromic, was solved in [Andreev, 1958]. Finally, for degenerate singular points $(x_0, y_0)$ with $DX(x_0, y_0)$ identically zero the monodromy problem is poorly-understood. In fact, two different types of behaviour are possible depending on the existence or not of the so-called **characteristic directions** of $(x_0, y_0)$. Let $P(x, y) = \sum_{j \geq 1} P_j(x, y)$ and $Q(x, y) = \sum_{j \geq 1} Q_k(x, y)$ where $P_j$ and $Q_j$ are homogeneous polynomials of degree $j$ and $k \geq 2$. Hence $(x_0, y_0) = (0, 0)$ is a degenerate singular point with associated Jacobian matrix null. Taking polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, we say that $\theta = \theta^*$ is a characteristic direction of the origin if $\cos \theta^* Q_k(\cos \theta^*, \sin \theta^*) - \sin \theta^* P_k(\cos \theta^*, \sin \theta^*) = 0$. It is well known that if there exists an orbit tending or leaving the origin at a certain angle, this angle is a characteristic direction.

**Corollary 3** Assume that the polynomial $p-q$ quasihomogeneous system (1) of weighted degree $\ell \neq 0$ and $P$ and $Q$ coprimes possesses either a monodromic singular point at the origin or a limit cycle. Then equations (3) do not have any nontrivial real solution.

**Proof.** From Lemma 1 it follows that $P(0, 0) = Q(0, 0) = 0$ and therefore equations (3) have the trivial solution $(0, 0)$. Let us assume that there exists a real nontrivial solution $(c_1, c_2) \neq (0, 0)$ of (3). Hence the particular solution $S(t) = (x(t), y(t))$ described in Lemma 2 goes through the singular point $(0, 0)$ and therefore it cannot be a center of system (1). In addition, since such particular solutions are not spirals the singular point $(0, 0)$ neither can be a focus. Therefore the origin is not monodromic.
On the other hand, let us suppose that system (1) has a limit cycle $\Gamma$. Of course, $\Gamma$ must surround at least one singular point and so, from Lemma 1, $\Gamma$ only surrounds the origin. Finally, since $S(t)$ is not bounded we have $S \cap \Gamma \neq \emptyset$ which is impossible from the definition of limit cycle.

3 About the integrability of quasihomogeneous systems

Let $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ be a $C^1$ vector field defined in an open subset $U \subset \mathbb{R}^2$. We call inverse integrating factor for such vector field to a $C^1(U)$ solution $V(x, y)$ of the linear partial differential equation

$$P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) V. \quad (4)$$

Observe that $1/V$ is an integrating factor of the vector field $(P, Q)$ in $U \setminus \{ V = 0 \}$. In [Chavarriga et al., 1999], the authors show that the inverse integrating factor exists and it is unique, except for a multiplicative constant factor, in an open neighborhood of generic singular points.

If $p = q = 1$, then the $p - q$ quasihomogeneous vector field $(P, Q)$ reduces to an homogeneous vector field and it is well known that it possesses the homogeneous inverse integrating factor $V = xQ - yP$ provided that $V \neq 0$. Now we state a theorem which generalizes the above comment to $p - q$ quasihomogeneous vector fields.

**Theorem 4** Let $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ be a $C^1$ $p - q$ quasihomogeneous vector field. Hence any inverse integrating factor of it must be a $p - q$ quasihomogeneous function. Moreover, if $\alpha(u) = Q(1, u) - \frac{2}{p} u P(1, u) \neq 0$ then $V(x, y) = x^{m/p} \int (1, y/x^{q/p})$ is a $p - q$ quasihomogeneous inverse integrating factor of weighted degree $m$, where

$$f(1, u) = \exp \left( - \int \frac{\beta(u)}{\alpha(u)} \, du \right), \quad (5)$$

and $\beta(u) = \frac{q}{p} P(1, u) - \partial P/\partial x(1, u) - \partial Q/\partial y(1, u)$.

**Proof.** Let $V(x, y)$ be an inverse integrating factor of the $p - q$ quasihomogeneous vector field $(P, Q)$, that is, $V$ satisfies equation (4). Since $P$, $Q$ and $(\partial P/\partial x + \partial Q/\partial y)$ are $p - q$ quasihomogeneous functions of weighted degrees $p + \ell$, $q + \ell$ and $\ell$ respectively, it is easy to see by straightforward computation that equation (4) is invariant under the change of variables $(x, y) \to (\alpha^p x, \alpha^q y)$. In consequence their solutions are also invariants, i.e., $V(\alpha^p x, \alpha^q y) = V(x, y)$ or equivalently $V(\alpha^p x, \alpha^q y) = \alpha^m V(x, y)$.

Let $V(x, y)$ be a $p - q$ quasihomogeneous function of weighted degree $m$, that is $V(\alpha^p x, \alpha^q y) = \alpha^m V(x, y)$. If we impose that $V$ be an inverse integrating factor of the $p - q$ quasihomogeneous system (1), the following equation is verified

$$P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} = \text{div}(P, Q)V, \quad (6)$$

where $\text{div}(P, Q)(x, y) = (\partial P/\partial x + \partial Q/\partial y)(x, y)$ is the divergence of the vector field $(P, Q)$. Observe that $P$, $Q$, $\partial V/\partial x$, $\partial V/\partial y$, $V$ and $\text{div}(P, Q)$ are $p - q$ quasihomogeneous functions of weighted degrees $p + \ell$, $q + \ell$, $m - p$, $m - q$, $m$ and $\ell$ respectively.
Now, we make the blow-up \((x, y) \rightarrow (w, u)\) where \(w = x\) and \(u = y/x^{\ell/p}\) into the above partial differential equation \((6)\). Taking into account the weighted quasihomogeneity of the involved functions in such equations we have \(P(w, uw^{\ell/p}) = w^{p+q}P(1, u),\)
\(Q(w, uw^{\ell/p}) = w^{\frac{2\ell}{p}}Q(1, u)\) and \(\div(P, Q)(w, uw^{\ell/p}) = w^{\frac{q}{p}}\div(P, Q)(1, u)\). On the other hand, by the chain rule we have that \(\partial/\partial x = \partial/\partial w - \frac{2}{p} \frac{\partial}{\partial u}\) and \(\partial/\partial y = w^{-\frac{q}{p}}\partial/\partial u\). Hence in the new variables \((w, u)\), equation \((6)\) becomes

\[
\left[Q(1, u) - \frac{q}{p} u P(1, u)\right] \frac{\partial \tilde{V}}{\partial u} + w P(1, u) \frac{\partial \tilde{V}}{\partial w} = \div(P, Q)(1, u) \tilde{V},
\]

where \(\tilde{V}(w, u) = V(w, uw^{\ell/p}) = w^{m/p}V(1, u)\). In consequence \(\partial \tilde{V} / \partial w = \frac{m}{p} w^{\frac{q}{p}-1}V(1, u),\)
\(\partial \tilde{V} / \partial u = w^{m/p}dV(1, u)/du\) and the above partial differential equation reduce to the following first order linear ordinary differential equation

\[
\alpha(u) \frac{dV(1, u)}{du} + \beta(u)V(1, u) = 0,
\]

where the coefficients \(\alpha(u)\) and \(\beta(u)\) are the given in the statement of the theorem.

Provided that \(\alpha(u) \neq 0\), the general solution of equation \((7)\) adopts the form \((5)\) except for a multiplicative arbitrary constant which we take the unity because such constant does not affect to our computation. Finally we undoing the blow-up and the theorem is proved.

**Remark 1.** We can use Theorem 4 to get a family of inverse integrating factors \(V_m\) of weighted degree \(m\) and next compute a first integral \(H\) either from \(H = V_{m_1}/V_{m_2}\) with \(m_1 \neq m_2\) or from \(H = \int P/V_m \ dy + f(x)\) satisfying \(\partial H / \partial x = -Q/V_m\).

## 4 Some examples

Take the \(p - q\) quasihomogeneous vector field \(\dot{x} = P(x, y) = \sum_{i=0}^{k} f_{p+\ell-iq}(x)y^i, \dot{y} = Q(x, y) = \sum_{i=0}^{k} g_{q+i-lq}(x)y^i\) where \(f_j\) and \(g_j\) are \(p - q\) quasihomogeneous functions of weighted degree \(j\). The next example leads with the particular case \(k = 2\) and \(f\) and \(g\) potential functions.

**Example 5** The \(p - q\) quasihomogeneous system of weighted degree \(\ell\)

\[
\begin{align*}
\dot{x} &= P(x, y) = a_0 x^{p+\ell} + a_1 x^{p+\ell-q} y + a_2 x^{p+\ell-2q} y^2, \\
\dot{y} &= Q(x, y) = b_0 x^{p+q} + b_1 x^{\ell} y + b_2 x^{\ell-2} y^2, 
\end{align*}
\]

has the inverse integrating factor

\[
V(x, y) = \begin{cases} 
\frac{y x^{p+\ell-2q}}{x^{p+q}} ([b_1 p - a_0 q] x^{\ell} y + [b_2 p - a_1 q] x^{\ell} y - a_2 q y^2) & \text{if } b_0 = 0, \\
\frac{b_0 p x^{2\ell}}{x^{p+q}} ([b_1 p - a_0 q] x^{\ell} y + [b_2 p - a_1 q] y^2) & \text{if } a_2 = 0.
\end{cases}
\]

**Proof.** For system \((8)\) we have \(\alpha(u) = [b_0 p + (b_1 p - a_0 q) u + (b_2 p - a_1 q) u^2 - a_2 q u^3] / p\) and \(\beta(u) = \{a_0 (m - \ell - p) - b_1 p + (a_1 (m + q - \ell - p) - 2 b_2 p) u + a_2 (m + 2q - \ell - p) u^2\} / p\) from Theorem 4. In consequence:
• If $b_0 = 0$ then taking $m = p + q + \ell$ we have $f(1, u) = -u[b_1p + a_0q + (a_1q - b_2p)u + a_2qu^2]$. Undoing the blow-up we obtain $V(x, y) = x^{\frac{p+q+\ell}{p+q} - \frac{a}{c}} f(1, y/x^\frac{a}{c}) = yx^{\frac{a}{c} - \frac{2a}{c}}[(b_1p - a_0q)x^\frac{2a}{c} + (b_2p - a_1q)x^\frac{a}{c}y - a_2xy^2]$, which is an inverse integrating factor of system (8).

• If $a_2 = 0$ then, taking $m = p + q + \ell$, we have $f(1, u) = b_0p + (b_1p - a_0q)u + (b_2p - a_1q)u^2$. In the initial coordinates we obtain $V(x, y) = x^{\frac{p+q+\ell}{p+q} - \frac{1}{p+q}} f(1, y/x^\frac{1}{p+q})$, which gives, for system (8), the inverse integrating factor $V(x, y) = x^{\frac{a}{c} - \frac{2a}{c}}[b_0px^\frac{a}{c} + (b_1p - a_0q)x^\frac{a}{c}y + (b_2p - a_1q)y^2]$. □

The $p-q$ homogeneous systems with $p$ even and $q$ and $\ell$ odd are included in the class of time-reversible systems. For the polynomial time-reversible $2n-1$ quasihomogeneous vector fields of weighted degree 1 we have the next example.

**Example 6** The most general planar polynomial $2n-1$ quasihomogeneous of weighted degree 1 time-reversible system is given by

$$
\dot{x} = y(ax + by^2), \quad \dot{y} = cx + dy^2,
$$

if $n = 1$ and by

$$
\dot{x} = y(ax + by^{2n}), \quad \dot{y} = cy^2,
$$

if $n \neq 1$. $V_1(x, y) = -2cx^2 + ay^2x - 2dxy^2 + by^4$ and $V_2(x, y) = y^{2+\frac{a}{c}}$ with $c \neq 0$ are inverse integrating factors of systems (9) and (10) respectively. In addition, both systems (9) and (10) have no limit cycles.

**Proof.** Since $\alpha(u) = [2c + (2d - a)u^2 - bu^4]/2$ and $\beta(u) = u[-2a - 4d + m(a + bu^2)]/2$ for system (9), applying Theorem 4 we have

$$
f(1, u) = (2d - a + \sqrt{\Delta} - 2bu^4)(\frac{m-4}{4\sqrt{\Delta}}(a - 2d + \sqrt{\Delta} + 2bu^2) - \frac{m-4}{4\sqrt{\Delta}}(a + 2d + \sqrt{\Delta} - 2bu^2)),
$$

where $\Delta := a^2 - 4ad + 4(2bc + d^2)$. Taking $m = 4$ and undoing the blow-up we obtain $x^2 f(1, y/\sqrt{x}) = -4b V_1(x, y)$. Therefore $V_1$ is a polynomial inverse integrating factor of system (9).

Now we restrict our attention to the problem of the existence, nature and location in phase plane of limit cycles for system (9). First of all notice that, since $V_1$ is polynomial, it is defined in the whole plane and therefore any limit cycle of the system (if it exists) must be algebraic and contained in the zero level set of $V_1$, see [Giacomini et al., 1996]. We can assume that $c \neq 0$ since otherwise $y = 0$ is an invariant straight line through the origin of the system and hence it does not have limit cycles. Furthermore, assume that $bc - ad = 0$ or equivalently $b = (ad)/c$. In this particular case we can suppose $a + 2d \neq 0$ because otherwise the components of the system are not coprimes and after a time rescaling the system becomes linear and does not have limit cycles. Moreover, equations (3) have the nontrivial solution

$$
(c_1, c_2) = \left( \frac{2a}{c(a + 2d)^2}, \frac{-2}{a + 2d} \right) \in \mathbb{R}^2,
$$

and in consequence there is absence of limit cycles from Corollary 3. We continue with the hypothesis $c(bc - ad) \neq 0$ over the parameters of the system. In this case equations
Finally, if where the prime denotes differentiation with respect to \( p \)

Let us consider the Kovalevskaya exponents for quasihomogeneous vector fields

\[
\begin{align*}
V_1(x, y) &= 0 \text{ with } cd(bc - ad) \neq 0 \text{ and } \Delta < 0. \text{ But if } \Delta < 0 \text{ then } V_1(x, y) = 0 \text{ only has an isolated point in } \mathbb{R}^2 \text{ and system (9) does not have limit cycles.}
\end{align*}
\]

In summary, if system (9) has a limit cycle then it is algebraic, contained into the algebraic curve \( V_1(x, y) = 0 \) with \( cd(bc - ad) \neq 0 \) and \( \Delta < 0 \). Moreover, since \( \Delta \geq 0 \) then \( (\alpha_1, \alpha_2) \in \mathbb{R}^2 \) and there are no limit cycles in the phase plane from Corollary 3.

Let us now study system (10). Since \( \alpha(u) = u^2[2nc - a - bu^{2n+1}]/(2n) \) and \( \beta(u) = u[ma - 2n(a + 2c) + mbu^{2n+1}]/(2n) \), we have

\[
\begin{align*}
f(1, u) &= u^{\frac{ma - 2n(a + 2c)}{2n}} + \frac{1}{2n}(a - 2nc + bu^{2n+1}) \frac{2n[a -(m-2)c]}{(2n)[(n-2)nc]}. \\
\end{align*}
\]

Finally, if \( e \neq 0 \), taking \( m = 2 + a/c \) and undoing the blow-up we obtain \( V_2(x, y) = x^{\frac{2n}{2n}}f(1, y/x^{1/2n}) = y^{2+\frac{a}{c}} \) as inverse integrating factor for system (10). Obviously the system cannot have limit cycles since \( y = 0 \) is an invariant straight line through the origin.

5 \ Kovalevskaya exponents for quasihomogeneous vector fields

Let us consider the \( p - q \) quasihomogeneous system (1) of weighted degree \( \ell \neq 0 \) or equivalently its associated vector field \( \mathcal{X} = (P, Q) \). We make the following nonautonomous change of variables

\[
(x, y) \rightarrow (u, v) = \left( \frac{p/\ell}{x}, \frac{q/\ell}{y} - c_2 \right),
\]

which is a variation of system (1) around the particular solution of Lemma 2. If in these new variables we add the logarithmic time change \( \tau = \log t \), then system (1) is expressed like the following autonomous system

\[
\begin{align*}
u' &= \frac{p}{\ell} (u + c_1) + P(u + c_1, v + c_2), \quad v' = \frac{q}{\ell} (v + c_2) + Q(u + c_1, v + c_2),
\end{align*}
\]

where the prime denotes differentiation with respect to \( \tau \). Moreover, since \( P \) and \( Q \) are analytic, we can expand such functions around \( (c_1, c_2) \) and we have \( P(u + c_1, v + c_2) = P(c_1, c_2) + < \nabla P(c_1, c_2), (u, v)> + P(u, v) \) and \( Q(u + c_1, v + c_2) = Q(c_1, c_2) + < \nabla Q(c_1, c_2), (u, v)> + Q(u, v) \) where \( <,> \) stands for the euclidean inner product and \( P \) and \( Q \) denote higher order terms. Finally, taking into account that \( (c_1, c_2) \) verifies from definition \( P(c_1, c_2) + \frac{p}{\ell} c_1 = 0 \) and \( Q(c_1, c_2) + \frac{q}{\ell} c_2 = 0 \), we have

\[
\begin{align*}
\left( \begin{array}{c}
u' \\
v'
\end{array} \right) &= K \left( \begin{array}{c}
u \\
v
\end{array} \right) + \left( \begin{array}{c}P(u, v) \\
Q(u, v)
\end{array} \right), \quad (11)
\end{align*}
\]

where \( K := D\mathcal{X}(c_1, c_2) + \text{diag}\{p/\ell, q/\ell\} \) is the Kovalevskaya matrix and represents the linear part of system (11). The eigenvalues of the Kovalevskaya matrix are called Kovalevskaya exponents. It is not difficult to prove that \( \lambda_1 = -1 \) is always a Kovalevskaya exponent, see [Yoshida, 1983].
6 Semi-quasihomogeneous systems

Let us consider a planar system of differential equations
\[ \dot{x} = X(x, y), \quad \dot{y} = Y(x, y), \tag{12} \]
defined in \( \mathbb{C}^2 \) and analytic in a neighbourhood of the origin \((0, 0)\). If system \((12)\) is a \(p-q\) quasihomogeneous system of weighted degree \(\ell\) then all the terms in the expansions of \(X(x, y) = \sum_{i,j} a_{ij} x^i y^j\) and \(Y(x, y) = \sum_{i,j} b_{ij} x^i y^j\) verify \(ip + jq = p + \ell\) and \(ip + jq = q + \ell\) respectively.

**Definition 7** (Furta, 1996) The analytic system \((12)\) is called \(p-q\) positively semi-quasihomogeneous of weighted degree \(\ell\) if it can be expressed as
\[ \dot{x} = P(x, y) + \tilde{P}(x, y), \quad \dot{y} = Q(x, y) + \tilde{Q}(x, y), \tag{13} \]
where \((P, Q)\) is a \(p-q\) quasihomogeneous vector field of weighted degree \(\ell\) and all the terms in the expansions \(\tilde{P}(x, y) = \sum_{i,j} a_{ij} x^i y^j\) and \(\tilde{Q}(x, y) = \sum_{i,j} b_{ij} x^i y^j\) satisfy \(ip + jq > p + \ell\) and \(ip + jq > q + \ell\) respectively. If the above inequalities are reversed then system \((13)\) is called \(p-q\) negatively semi-quasihomogeneous of weighted degree \(\ell\). Moreover, the truncated system \(\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)\) is termed the quasihomogeneous cut.

In [Furta, 1996], the author proves the following result.

**Theorem 8** (Furta) Let system \((13)\) be semi-quasihomogeneous. If the Kovalevskaya matrix is diagonalizable and its eigenvalues \(\lambda_1, \lambda_2\) do not satisfy any resonant condition
\[ k_1\lambda_1 + k_2\lambda_2 = 0, \quad k_1, k_2 \in \mathbb{N} \cup \{0\}, \quad k_1 + k_2 \geq 1, \]
then system \((13)\) does not have any polynomial first integral. Moreover, if system \((13)\) is positively semi-quasihomogeneous then there exists no smooth first integrals which can be expanded into formal Maclaurin series in a neighbourhood of the origin.

**Remark 2.** In fact, Furta’s results in [Furta, 1996] are stated for semi-quasihomogeneous systems in \(\mathbb{C}^n\). Since in this work we study the particular planar case \(n = 2\), taking into account that \(\lambda_1 = -1\), the resonant condition of Theorem 8 reduces to \(\lambda_2 \in \mathbb{Q}\).

Let us give some necessary conditions in order to have either an analytic inverse integrating factor or an analytic first integral for positively semi-quasihomogeneous systems.

**Theorem 9** Assume that a positively semi-quasihomogeneous system \((13)\) has an analytic inverse integrating factor (resp. analytic first integral) in a neighbourhood of the origin. Then, its quasihomogeneous cut possesses a polynomial inverse integrating factor (resp. polynomial first integral).

**Proof.** We only give the proof for the case of having an inverse integrating factor. The other case is totally similar.

Let us define the change of variables \(u = \alpha^p x, \quad v = \alpha^q y\). Under the action of the similarity transformation \((2)\), the positively semi-quasihomogeneous system \((13)\) is transformed into
\[ \dot{u} = P(u, v) + \alpha^{p+\ell} \tilde{P}(\alpha^{-p} u, \alpha^{-q} v), \quad \dot{v} = Q(u, v) + \alpha^{q+\ell} \tilde{Q}(\alpha^{-p} u, \alpha^{-q} v), \tag{14} \]
where $\alpha^{p+\ell}\tilde{P}(\alpha^{-p}u, \alpha^{-q}v)$ and $\alpha^{q+\ell}\tilde{Q}(\alpha^{-p}u, \alpha^{-q}v)$ are formal power series with respect to $1/\alpha$ without any constant term.

On the other hand, let $V(x, y)$ be an analytic inverse integrating factor in a neighbourhood of the origin for system (13). Hence it can be expressed like $V(x, y) = \sum_{i=k}^{\infty} V_i(x, y)$ where $V_i$ are homogeneous polynomials of degree $i$ and $V_k \neq 0$. Let us define $\xi = \min\{p, q\}$. In the new variables, $V$ becomes

$$
\tilde{V}(u, v) = V(\alpha^{-p}u, \alpha^{-q}v) = \sum_{i=k}^{\infty} V_i(\alpha^{-p}u, \alpha^{-q}v) = \sum_{i=k}^{\infty} \alpha^{-i\xi} \tilde{V}_i(u, v; \alpha)
$$

$$
= \alpha^{-k\xi} \sum_{i=0}^{\infty} \frac{\tilde{V}_{k+i}(u, v; \alpha)}{\alpha^{i\xi}}.
$$

Hence

$$
W(u, v; \alpha) = \sum_{i=0}^{\infty} \frac{\tilde{V}_{k+i}(u, v; \alpha)}{\alpha^{i\xi}}
$$

is an inverse integrating factor of system (14).

Finally, system (14) approaches to the quasihomogeneous cut $\dot{u} = P(u, v)$, $\dot{v} = Q(u, v)$ as $\alpha \to \infty$. Moreover $W(u, v; \alpha)$ approaches to $\tilde{V}_k(u, v; \infty)$ as $\alpha \to \infty$. Hence $\tilde{V}_k(u, v; \infty)$ is a polynomial inverse integrating factor of the quasihomogeneous cut and the theorem is proved.

**Remark 3.** In [Chavarriga, Giacomini & Giné, 2000], necessary conditions are given in order to have polynomial inverse integrating factors for polynomial planar vector fields. However, by using Theorem 4 we determine all the inverse integrating factors of the quasihomogeneous cut. Therefore it is easy to check if the quasihomogeneous cut has or does not have a polynomial inverse integrating factor.

**Remark 4.** Let us consider an analytic vector field

$$
\dot{x} = P(x, y) = \sum_{k=m}^{\infty} P_k(x, y), \quad \dot{x} = Q(x, y) = \sum_{k=m}^{\infty} Q_k(x, y),
$$

where $P_k$ and $Q_k$ are homogeneous polynomials of degree $k$ and $m \geq 1$. Obviously systems (15) are positively semi-quasihomogeneous with $p = q = 1$ and $\ell = m - 1$. In fact, for this particular case the necessary condition given by Theorem 9 is verified trivially as we will see next. Firstly, observe that $V = xQ_m - yP_m$ is always a polynomial inverse integrating factor of the homogeneous cut $\dot{x} = P_m(x, y)$, $\dot{y} = Q_m(x, y)$, provided that $V \neq 0$. Moreover, let $H(x, y) = \sum_{k=n}^{\infty} H_k(x, y)$ be an analytic first integral in a neighbourhood $U$ of the origin for system (15). Here $H_k$ are homogeneous polynomials of degree $k$. Then, $H$ must verify $P\partial H/\partial x + Q\partial H/\partial y \equiv 0$ on $U$. Equaling to zero the homogeneous polynomials of same degree in the previous expression $P_m\partial H_n/\partial x + Q_m\partial H_n/\partial y \equiv 0$ holds on $U$. Therefore $H_n(x, y)$ is a polynomial first integral of the homogeneous cut.

In the next example, let us consider any $2n - 1$ positively semi-quasihomogeneous system of weighted degree 1 where its quasihomogeneous cut is given by the $2n - 1$ quasihomogeneous vector field (10) of weighted degree 1 of Example 6.
Example 10  For the $2n - 1$ positively semi-quasihomogeneous system of weighted degree 1
\[ \dot{x} = y(ax + by^{2n}) + \bar{P}(x, y), \quad \dot{y} = cy^2 + \bar{Q}(x, y), \tag{16} \]
where $n \in \mathbb{N} \setminus \{0, 1\}$ and all the terms in the expansions $\bar{P}(x, y) = \sum_{i,j} a_{ij} x^i y^j$ and $\bar{Q}(x, y) = \sum_{i,j} b_{ij} x^i y^j$ satisfy $2ni + j > p + 1$ and $2ni + j > q + 1$ respectively, the following holds:

(i) If $c(2cn - a) \neq 0$ and $a/c \notin \mathbb{Q}$ then system (16) does not have any polynomial first integral.

(ii) If $bc(2cn - a) \neq 0$ and $a/c \notin \mathbb{N} \cup \{-2, -1, 0\}$ then system (16) does not have any analytic inverse integrating factor defined in a neighborhood of the origin.

Proof. For the quasihomogeneous cut $(P(x, y), Q(x, y)) = (y(ax + by^{2n}), cy^2)$ of system (16), equations (3) have the nontrivial solution
\[ \left(\frac{b/c^{2n}}{2cn - a}, -\frac{1}{c}\right), \]
when $a \neq 2cn$ and $c \neq 0$. Furthermore the Kovalevskaya matrix associated to such solution is
\[ K = \left( \begin{array}{cc} 2n - \frac{a}{c} & \frac{2bn_i[(2n_i+1) - a]}{c^{2n}(2cn - a)} \\ 0 & -1 \end{array} \right), \]
which eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2n - a/c$. Therefore, from Theorem 8, we conclude with statement (i) of Example 10.

The results of this paper are applied in the following way. Let us notice that if $b = 0$ then the quasihomogeneous cut $(P, Q)$ of system (16) reduces to an homogeneous vector field and therefore $V = xQ - yP \neq 0$ is a polynomial inverse integrating factor for it when $a \neq c$. Otherwise, if $b = 0$ and $a = c$ then $V = x^3$ is a polynomial inverse integrating factor.

We continue without loost of generality by assuming $b \neq 0$. From Theorem 4 we obtain for the vector field $(P, Q)$ the most general inverse integrating factor
\[ V(x, y) = \begin{cases} x^{n + (2 - m)/2n} y^{(m - 2n)/2n} \left( a - 2cn + \frac{by^{2n}}{2} \right) \frac{a + (2 - m) - c(2n)}{2cn - a} & \text{if } a \neq 2cn, \\ y^n \exp \left( \frac{c(2n + 2 - m)}{6} \right) & \text{if } a = 2cn, \end{cases} \]
which is a quasihomogeneous function of degree $m$. In order to have $V$ polynomial we have the next possibilities:

- If $a \neq 2cn$ and $c = 0$ then $V = ax + by^{2n} \in \mathbb{R}$ is a polynomial inverse integrating factor. Otherwise, i.e., if $a \neq 2cn$ and $c \neq 0$ then a necessary condition in order to have a polynomial inverse integrating factor is $a + c(2 - m) = 0$. From this relation we have $m = 2 + a/c$ and $V$ becomes $V = y^{2 + a/c}$. Then $V$ is polynomial if and only if $a/c \in \mathbb{N} \cup \{-2, -1, 0\}$.

- Put $a = 2cn$. In this case, taking $m = 2n + 2$ the exponential term of $V$ vanishes and moreover $V$ becomes polynomial.

Finally, applying Theorem 9, we have proved statement (ii) of Example 10. $\square$
References


