Generalized Cofactors and Nonlinear Superposition Principles

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Abstract—It is known from Lie's works that the only ordinary differential equation of first order in which the knowledge of a certain number of particular solutions allows the construction of a fundamental set of solutions is, excepting changes of variables, the Riccati equation. For planar complex polynomial differential systems, the classical Darboux integrability theory exists based on the fact that a sufficient number of invariant algebraic curves permits the construction of a first integral or an inverse integrating factor. In this paper, we present a generalisation of the Darboux integrability theory based on the definition of generalized cofactors. © 2003 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

By definition, a complex (respectively, real) planar polynomial differential system or simply a polynomial system will be a differential system of the form

\[
\frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dy}{dt} = \dot{y} = Q(x, y),
\]

in which \(P, Q \in \mathbb{C}[x, y]\) (respectively, \(\mathbb{R}[x, y]\)) are polynomials in the complex (respectively, real) variables \(x\) and \(y\) and the independent one (the time) \(t\) is real. Throughout this paper, we will denote by \(m = \max\{\deg P, \deg Q\}\) the degree of system (1). Obviously, we can also express system (1) as the differential equation

\[
\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)},
\]

and, moreover, we also associate to system (1) the vector field \(\mathcal{X}\) defined by \(\mathcal{X} = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}\).
We say that \( f(z, y) = 0 \), with \( f \in C^1 \), is an invariant curve of equation (2) or equivalently of system (1) if the orbital derivative \( f = \mathcal{X} f = P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} \) vanishes on \( f = 0 \).

The polynomial system (1) is \textit{integrable} on an open subset \( U \) of \( \mathbb{C}^2 \) (respectively, \( \mathbb{R}^2 \)) if there exists a nonconstant complex (respectively, real) function \( H \) defined in \( U \), called the \textit{first integral} of the system in \( U \), which is constant on all solution curves \( (x(t), y(t)) \) of system (1) on \( U \); i.e., \( H(x(t), y(t)) \) is constant for all values of \( t \) for which the solution \( (x(t), y(t)) \) is defined on \( U \).

Clearly, \( H \in C^1(U) \) is a first integral of (1) on \( U \) if and only if \( \mathcal{X} H = P \frac{\partial H}{\partial x} + Q \frac{\partial H}{\partial y} \equiv 0 \) on \( U \).

Let either \( R : U \to \mathbb{C} \) or \( R : U \to \mathbb{R} \) be a \( C^k(U) \) function with \( k \geq 1 \) which is not identically zero on \( U \). If the vector field \( R P \frac{\partial}{\partial x} + R Q \frac{\partial}{\partial y} \) is divergence free, i.e., it is Hamiltonian, then \( R \) is an integrating factor for the complex (respectively, real) system (1) on \( U \). In other words, \( R \) is an integrating factor if \( \mathcal{X} R = -R \text{div} \mathcal{X} \) where \( \text{div} \mathcal{X} \) is the divergence of the vector field \( \mathcal{X} \). We also define \( V = R^{-1} \) as an \textit{inverse integrating factor}.

One interesting question is whether, with a finite number of particular solutions of system (1), we can obtain another new solution or even a first integral of it. In [1], Jones and Ames introduced the idea of \textit{nonlinear superposition principle} in order to find new solutions of ordinary and partial differential equations. Let \( \Sigma = \{ y = g_1(x), \ldots, y = g_n(x) \} \) be a set of particular solutions of system (1). Then \( F(y, g_1, \ldots, g_n) \) is defined as a \textit{connecting function} for (1), if \( F = 0 \) is also a particular solution. Formally, a nonlinear superposition principle is an operation \( F : C^2 \times F^n \to G \) where \( F \) and \( G \) are function spaces such that the former properties hold. Interesting examples of connecting functions arise in the so-called \( \phi \)-\textit{time-reversible} vector fields which verifies \( \phi_\tau(X) = -X(\phi) \) where \( \phi \) is an involution, see [2]. If \( (x(t), y(t)) \) is a solution of \( \mathcal{X} \), then \( \phi(x(-t), y(-t)) \) is also a solution of \( \mathcal{X} \).

Moreover, we will say that \( \Sigma \) is a \textit{fundamental set of solutions} of system (1) if a connecting function \( F \) exists, such that \( F \) is a first integral of it or equivalently \( F = c \) is the general solution of equation (2) where \( c \) is an arbitrary constant. The standard example of nonlinear first-order differential equation with a fundamental set of solutions is the Riccati equation \( \frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2 \) for which the general solution is given in terms of three particular solutions \( y = g_1(x), y = g_2(x), \) and \( y = g_3(x) \) by the cross ratio

\[
F(y, g_1, g_2, g_3) = \frac{(y - g_1(x))(g_3(x) - g_2(x))}{(y - g_2(x))(g_3(x) - g_1(x))} = c,
\]

with \( c \) an arbitrary constant, see [3], for instance.

It follows from the work of Lie and Scheffers [4] that real equations (2) with \( n \) particular solutions belonging to a fundamental set of solutions are associated with finite-dimensional Lie algebras of vector fields on \( \mathbb{R} \). In fact, Lie showed that there is a fundamental set of solutions for the differential equation (2) if and only if it can be written in the form

\[
\frac{dy}{dx} = \sum_{i=0}^{s} A_i(x)B_i(y),
\]  

where the vector fields \( \mathcal{X}_i = B_i(y) \frac{\partial}{\partial y}, \) with \( i = 0, 1, \ldots, s \), generate an \( r \)-dimensional Lie algebra with \( s+1 \leq r \leq n \). Unfortunately, no easy way to construct the nonlinear superposition principle is known. The next theorem appears in [4].

**Theorem 1. Lie.** The only ordinary differential equations of form (3) allowing a fundamental set of solutions are the Riccati equation \( \frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2 \) and any equation obtained from it by a change of dependent and independent variables \( \psi = \psi(y), \tau = \tau(x) \).

Equation (3) with \( s = 3 \) and \( B_i(y) = y^i \) is an \textit{Abel equation}. These differential equations appeared in the Abel’s study on the theory of elliptic functions. For more details on such equations, see [5]. Moreover, the Abel equation is closely related with planar polynomial differential systems (1) of the form

\[
\dot{x} = P_1(x, y) + P_m(x, y), \quad \dot{y} = Q_1(x, y) + Q_m(x, y),
\]
where $P_k$ and $Q_k$ are homogeneous polynomials of degree $k$ and $m \geq 2$. In order to be more precise, in polar coordinates $(r, \theta)$ defined by $x = r \cos \theta, y = r \sin \theta$, system (4) becomes

$$\dot{r} = f_1(\theta) r + f_m(\theta) r^m, \quad \dot{\theta} = g_1(\theta) + g_m(\theta) r^{m-1},$$

where $f_k(\theta) = \cos \theta P_k(\cos \theta, \sin \theta) + \sin \theta Q_k(\cos \theta, \sin \theta)$ and $g_k(\theta) = \cos \theta Q_k(\cos \theta, \sin \theta) - \sin \theta P_k(\cos \theta, \sin \theta)$ are homogeneous trigonometric polynomials. In the region $R = \{(r, \theta) : g_1(\theta) + g_m(\theta) r^{m-1} > 0\}$, the differential system (5) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{f_1(\theta) r + f_m(\theta) r^m}{g_1(\theta) + g_m(\theta) r^{m-1}}.$$

Finally, the transformation $(r, \theta) \to (\rho, \theta)$ with $\rho = r^{m-1}/(g_1(\theta) + g_m(\theta) r^{m-1})$ is a diffeomorphism from the region $R$ to its image. As far as we know, Cherkas in [6] was the first to use this transformation. Such diffeomorphism transforms equation (6) into the following particular case of an Abel differential equation $\frac{d\rho}{d\theta} = \sum_{i=1}^3 A_i(\theta) \rho^i$, where

$$A_1(\theta) = (m-1) \frac{f_1(\theta)}{g_1(\theta)} - \frac{g_1'(\theta)}{g_1(\theta)}, \quad A_2(\theta) = (m-1) \left[ f_m(\theta) - \frac{2f_1(\theta)g_m(\theta)}{g_1(\theta)} \right] + \frac{g_1'(\theta)g_m(\theta)}{g_1(\theta)} - g_m'(\theta), \quad A_3(\theta) = (m-1) g_m(\theta) \left[ \frac{g_m(\theta)f_1(\theta)}{g_1(\theta)} - f_m(\theta) \right].$$

2. DARBOUX INTEGRABILITY GENERALIZED THEORY

An invariant curve $f(x, y) = 0$ of system (1) is an invariant algebraic curve when $f \in \mathbb{C}[x,y]$ and it is irreducible. It is clear that the orbital derivative $\mathcal{X} f$ should vanish on the algebraic curve $f(x, y) = 0$. On the other hand, since the ideal $(f)$ is radical, then $\mathcal{X} f \in (f)$, and therefore, there exists a polynomial $K(x, y) \in \mathbb{C}[x,y]$ of degree less than or equal to $m - 1$, called cofactor associated to the invariant algebraic curve $f = 0$ such that $\mathcal{X} f = K f$.

In 1878, Darboux [7] showed how first integrals of polynomial systems possessing sufficient invariant algebraic curves can be constructed. In short, he proved that if a polynomial system of degree $m$ has at least $q = m(m+1)/2$ invariant algebraic curves, then it has a first integral or an integrating factor of the form $\prod_{i=1}^q f_i^n(x, y)$ for suitable $\lambda_i \in \mathbb{C}$ not all zero and $f_i(x, y) = 0$ invariant algebraic curves for $i = 1, \ldots, q$. Later, in 1979, Jouanolou [8] showed that if $q = m(m+1)/2 + 2$, then the polynomial system has a rational first integral, and consequently, all its invariant curves are algebraic. In 1983, Prelle and Singer [9] proved that if a polynomial system has an elementary first integral, then this first integral can be computed by using the invariant algebraic curves of the system.

Let $h, g \in \mathbb{C}[x,y]$ be coprimes in the ring $\mathbb{C}[x,y]$. Then the function $\exp(g/h)$ is called an exponential factor of the polynomial system (1), if, for some polynomial $K \in \mathbb{C}[x,y]$ of degree at most $m - 1$, it satisfies $\mathcal{X}(\exp(g/h)) = K \exp(g/h)$. As before, we say that $K$ is the cofactor of the exponential factor $\exp(g/h)$. If $\exp(g/h)$ is an exponential factor for the polynomial system (1) and $h$ is not a constant polynomial, then $h = 0$ is an invariant algebraic curve, see [10].

In 1992, Singer [11] proved that, if a polynomial system has a Liouvillian first integral, then it can be computed by using the invariant algebraic curves and the exponential factors of the system. In fact, the system has a Darboux generalized inverse integrating factor $V$, i.e., $V(x, y) = \prod_{i=1}^n f_i^n(x, y) \prod_{j=1}^m [\exp(g_j/h_j)]^{\mu_j}$, where $\lambda_i$ and $\mu_j$ are complex numbers. For a complete exposition of the Darboux theory of integration, see [10], where the version presented improves Darboux’s one essentially because it is taking into account the exponential factors and the independent singular points, see also [12].
3. INTEGRABILITY THEORY WITH GENERALIZED COFACTORS

In the previous section, we have observed that invariant algebraic curves and exponential factors are fundamental ingredients in order to obtain a first integral of a polynomial system. In fact, from the results of [11], it follows that the Darboux integrability generalized theory finds all Liouvillian first integrals of system (1). But we want to stress that there are polynomial systems (1) with non-Liouvillian first integral, see, for instance, [13].

A first approach to a more general integrability theory using transcendental invariant curves to find polynomial systems with non-Liouvillian first integral is based on the next definition.

**Definition 2.** A generalized cofactor $K(x, y)$ associated to a transcendental invariant curve $f(x, y) = 0$ of system (1) is a polynomial in $\mathbb{C}[x, y]$ of degree less than or equal to $m - 1$ verifying $\mathcal{X}f = Kf$.

The above Definition 2 is quite natural since even a formal invariant curve $f(x, y) = 0$ of system (1) must always verify an equation $\mathcal{X}f = Lf$ where $L(x, y)$ is also a formal power series, see [14,15].

**Theorem 3.** Suppose that a polynomial system (1) of degree $m$ admits $q$ invariant curves $f_i(x, y) = 0$ with generalized cofactors $K_i(x, y)$ for $i = 1, \ldots, q$. If we have $q = m(m + 1)/2$, then system (1) has a first integral or an integrating factor of the form $F = \prod_{i=1}^{q} f_i^{\lambda_i}(x, y)$ where $\lambda_i$ are complex numbers not all zero.

**Proof.** Let $K_0 = \text{div} \mathcal{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ be the divergence of the vector field $\mathcal{X}$. Clearly, $\deg K_0 \leq m - 1$. From Definition 2, $\deg K_i \leq m - 1$ for $i = 1, \ldots, q$, and therefore, it follows that each polynomial $K_i$ has at most $m(m + 1)/2$ coefficients. But since $q = m(m + 1)/2$, we have that the set $\{K_0, K_1, \ldots, K_q\}$ is linearly dependent, and therefore, there are complex numbers $\lambda_i$ not all zero such that $\sum_{i=0}^{q} \lambda_i K_i \equiv 0$. Now taking the function $F = \prod_{i=1}^{q} f_i^{\lambda_i}$, one has

$$\mathcal{X}F = F \sum_{i=1}^{q} \lambda_i \frac{\mathcal{X}f_i}{f_i} = F \left[ \sum_{i=1}^{q} \lambda_i K_i \right] \equiv \begin{cases} 0, & \text{if } \lambda_0 = 0, \\ -\lambda_0 K_0 F, & \text{if } \lambda_0 \neq 0. \end{cases}$$

We conclude that if $\lambda_0 = 0$ then $F$ is a first integral of system (1). Otherwise, if $\lambda_0 \neq 0$, then $F$ is an integrating factor of (1).

**Example.** Let us consider the following polynomial Liénard system:

$$\begin{align*}
\dot{x} &= P(x, y) = -y + x^4, \\
\dot{y} &= Q(x, y) = x,
\end{align*}$$

which is, in addition, $\varphi_0$-time-reversible with respect to the involution $\varphi_0(x, y) = (-x, y)$, and consequently, it has a center at the origin. From Odani's result [16], it follows easily that system (7) does not have any invariant algebraic curve. This implies that the only possible Darboux generalized inverse integrating factor, if it exists, is an exponential factor of the form $\exp(h)$ with $h \in \mathbb{C}[x, y]$. But from the definition of inverse integrating factor, $\mathcal{X} \exp(h) = \text{div} \mathcal{X} \exp(h)$, that is,

$$(-y + x^4) \frac{\partial h}{\partial x} + x \frac{\partial h}{\partial y} = 4x^3,$$

where we have simplified the common factor $\exp(h)$. Let $h(x, y) = \sum_{i=0}^{N} h_i(y)x^i$, where $h_i(y) \in \mathbb{C}[y]$ with $h_N(y) \neq 0$. Equating the highest degree terms in both members of (8) gives $N h_N(y) x^{N+2} = 0$. This implies $N = 0$ which gives a contradiction with equation (8). Therefore, system (7) does not have any Liouvillian first integral.

On the other hand, performing into $\frac{d y}{d x} = x/(-y + x^4)$ the change of the dependent and independent variables $\frac{d w}{d y} = -2w^2w$ and $z = 4^{1/3}y$ leads to an Airy differential equation $w'' = z w$ where the prime denotes derivation with respect to $z$. Therefore, see [17], the general solution of
it is given by \(w(z) = c_1 A_i(z) + c_2 B_i(z)\), where \(c_1\) and \(c_2\) are arbitrary constants and \(A_i(z)\) and \(B_i(z)\) are a pair of linearly independent solutions of the Airy equation which have the next integral representation \(A_i(z) = \pi^{-1} \int_0^\infty \cos[t^3/3 + zt] \, dt\) and \(B_i(z) = \pi^{-1} \int_0^\infty \exp[-t^3/3 + zt] + \sin[t^3/3 + zt] \, dt\). Finally, from \(2x^2w + \delta w = 0\), going back through the changes, we obtain that system (7) possesses the non-Liouville first integral \(H(x, y) = f_1 f_2^{-1}\) where \(f_1(x, y) = 2x^2 A_i(4^{1/3} y) + A_i'(4^{1/3} y)\) and \(f_2(x, y) = 2x^2 B_i(4^{1/3} y) + B_i'(4^{1/3} y) = 0\) are invariant curves with associated generalized cofactors \(K_1 = K_2 = 2x^3\). The associated inverse integrating factor \(\tilde{V}\) to the above first integral \(H\) is given by \(\tilde{V}(x, y) = -x/\partial H/\partial y = f_2^2(x, y)/W[A_i, B_i]\). Taking into account that the Wronskian \(W[A_i, B_i] = \pi^{-1}\), see [17], a non-Liouville inverse integrating factor \(V\) of system (7) is \(V(x, y) = f_2^2(x, y)\) with associated generalized cofactor \(\text{div } X = \text{div}(P, Q) = 4x^3\).

Finally, let us notice that system (7) can be written in the form (3) as \(\frac{dy}{dx} = -y/x + x^3\). So, the vector fields \(X_1 = -(1/x) \frac{\partial}{\partial x}\) and \(X_2 = x^3 \frac{\partial}{\partial y}\) generate a three-dimensional Lie algebra since \([X_1, X_2] = X_3 = -4x \frac{\partial}{\partial x}\), \([X_1, X_3] = -8X_1\), and \([X_2, X_3] = 8X_2\). Therefore, system (7) has a fundamental set of solutions with at least three particular solutions. Moreover, recall that Airy equation \(w'' = zw\) is related with the Riccati equation \(W' = z - W^2\) through the change of variables \(W = w'/w\) which agrees with Theorem 1.

REFERENCES