Proof of Berge’s strong path partition conjecture for $k = 2$

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Abstract

Berge’s strong path partition conjecture from 1982 generalizes and extends Dilworth’s theorem and the Greene–Kleitman theorem which are well known for partially ordered sets. The conjecture is known to be true for all digraphs only for $k = 1$ (by the Gallai–Milgram theorem) and for $k \geq \lambda$ (where $\lambda$ is the cardinality of the longest path in the graph). The attempts made, so far, to prove the conjecture for other values of $k$ have yielded proofs for acyclic digraphs, but not for general digraphs. In this paper, we prove the conjecture for $k = 2$ for all digraphs. The proof is constructive and it extends the proof for $k = 1$.

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The trivial path partition, where every path is a trivial path, is an example of a path partition. For each non-negative integer \( k \), the \( k \)-norm \( |\mathcal{P}|_k \) of a path partition \( \mathcal{P} = \{P_1, \ldots, P_m\} \) is defined by

\[
|\mathcal{P}|_k := \sum_{i=1}^{m} \min\{|P_i|, k\}.
\]  

(1)

A partition which minimizes \( |\mathcal{P}|_k \) is called \( k \)-optimal. Note that a 1-optimal path partition is a partition that contains a minimum number of paths and \( |\mathcal{P}|_1 = |\mathcal{P}| \). Denote by \( \mathcal{P}^{\geq k} \) the set of paths in \( \mathcal{P} \) of cardinality at least \( k \) (which we also call long paths), and by \( \mathcal{P}^{< k} \) the set of paths in \( \mathcal{P} \) of cardinality less than \( k \) (called short paths). Then Eq. (1) can be alternatively written as

\[
|\mathcal{P}|_k = \sum_{i=1}^{m} \min\{|P_i|, k\} = k|\mathcal{P}^{\geq k}| + |\mathcal{P}^{< k}|.
\]

Given a path partition \( \mathcal{P} \), denote by \( x^+_P \), or for short, \( x^+ \) the vertex (if it exists) which succeeds \( x \) on \( \mathcal{P} \). Similarly, \( x^-_P \) (or for short, \( x^- \)) is the vertex (if it exists) which precedes \( x \) on \( \mathcal{P} \). For a subset \( X \subseteq V \), define \( X^+ = \{x^+; x \in X\} \) and \( X^- = \{x^-; x \in X\} \).

For convenience, we draw all paths in a path partition vertically downwards, i.e. \( x^- \) is drawn above \( x \) (see Fig. 1).

A \( k \)-colouring is a family \( \mathcal{C}^k = \{C_1, C_2, \ldots, C_k\} \) of \( k \) disjoint independent sets called colour classes. (Some of the colour classes may be empty.) The cardinality of a \( k \)-colouring is the sum of the sizes of the colour classes, i.e., \( |\mathcal{C}^k| = \sum_{i=1}^{k} |C_i| \) and \( \mathcal{C}^k \) is said to be optimal if \( |\mathcal{C}^k| \) is as large as possible. A path partition \( \mathcal{P} \) and a \( k \)-colouring \( \mathcal{C}^k \) are orthogonal if every path \( P_i \) in \( \mathcal{P} \) meets \( \min\{|P_i|, k\} \) different colour classes of \( \mathcal{C}^k \). Note that this is the maximum number of different colour classes that a path can intersect in a \( k \)-colouring.

Conjecture 1.1 (Berge’s Strong Path Partition Conjecture [3]). Let \( G \) be a digraph and let \( k \) be a positive integer. Then for every \( k \)-optimal path partition \( \mathcal{P} \) there exists a \( k \)-colouring orthogonal to it.

The conjecture holds for \( k = 1 \) for all digraphs by the Gallai–Milgram theorem [6]. Berge’s strong path partition conjecture has also been proved for acyclic digraphs (see [4,12,2,1,9]). It is
not difficult to see, as was shown in [3], that the conjecture is true when \( k \geq \lambda \), where \( \lambda \) is the cardinality of the longest path in \( G \), and when the \( k \)-optimal path partition contains only short paths (i.e., paths of cardinality less than \( k \)). In [1] it was proved that the conjecture holds also in the case that the given \( k \)-optimal path partition contains only long paths, i.e., \( P = P^\geq_k \). For a survey of Berge’s conjecture and related problems see [8]. See also [5,13,14] for related results.

Denote by \( \alpha_k(G) \) the cardinality of an optimal \( k \)-colouring in \( G \), and by \( \pi_k(G) \) the \( k \)-norm of a \( k \)-optimal path partition in \( G \). Conjecture 1.1 implies the following:

**Conjecture 1.2 (Weak Path Partition Conjecture—Linial [10]).** For any digraph \( G \) and positive integer \( k, \alpha_k(G) \geq \pi_k(G) \).

If \( G \) is transitive and acyclic (i.e., the graph of a partially ordered set), then the Green–Kleitman theorem [7] states that \( \alpha_k(G) = \pi_k(G) \), implying Conjecture 1.2.

In this paper we will prove Conjecture 1.1 for \( k = 2 \) for all graphs. In Section 1 we give a proof of the conjecture for \( k = 1 \) which will set the foundations for the proof for \( k = 2 \). From here on, for a path partition \( P \), we denote the set of non-trivial paths as \( P^{>1} \) and the set of trivial paths as \( P^1 \).

### 2. Proof for \( k = 1 \), the Gallai–Milgram theorem

In this section we give a proof of the Gallai–Milgram theorem which will set the groundwork for the proof for \( k = 2 \) and motivate it. We begin with some definitions which will be used in Section 3. We will then continue with the definition of a 1-snapshot (which will be generalized in Section 3 to a 2-snapshot), its properties, and its role in finding either an orthogonal 1-colouring (a 2-colouring in Section 3) or another path partition with an improved first (second, in Section 3) norm. The proof described here for \( k = 1 \) is based on the original inductive proof of the Gallai–Milgram theorem, though it will be described in different terms.

#### 2.1. Preliminary definitions

For an edge \( e = (u, v) \) we define head\((e) := v\), tail\((e) := u\). If \( A \subseteq E \) we define head\((A) = \{\text{head}(e); e \in A\} \) and tail\((A) = \{\text{tail}(e); e \in A\} \).

**Definition 2.1 (Undirected Trail, Forward and Backward Edges).** An undirected trail \( Q \) in \( G \) is a sequence \( Q = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l) \) such that, for each \( 1 \leq i \leq l \), either \( e_i = (v_{i-1}, v_i) \in E(G) \) or \( e_i = (v_i, v_{i-1}) \in E(G) \), and all the edges are distinct. We assign a direction to \( Q \) from \( v_0 \) to \( v_l \) so that if \( e_i = (v_{i-1}, v_i) \) then \( e_i \) is considered a forward edge, and if \( e_i = (v_i, v_{i-1}) \) then \( e_i \) is considered a backward edge. Denote all forward and backward edges of \( Q \) by \( F_Q \) and \( B_Q \), respectively.

Since the graph is simple, no ambiguity arises, and we will denote an undirected trail \( Q = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l) \) also as \( Q = (v_0, v_1, v_2, \ldots, v_l) \).

Given \( Q \) and \( x, y \in V(Q) \) we denote by \( Q_{xy} \) the subtrail of \( Q \) preceding (and including) \( y \), by \( xQ \) the subtrail of \( Q \) following (and including) \( x \), and by \( xQy \) the part of \( Q \) between \( x \) and \( y \) and including them both.

For trails \( Q_1 = (v_0, \ldots, v_s) \) and \( Q_2 = (v_t, \ldots, v_l) \) we denote by \( Q_1 \ast Q_2 = (v_0, \ldots, v_s, \ldots, v_l) \) the concatenation of \( Q_1 \) and \( Q_2 \). Note that \( Q_1 \ast Q_2 \) is a trail only if \( Q_1 \) and \( Q_2 \) are edge disjoint. Otherwise, it is called a walk.
For any path or trail \( Q \) in \( G \) (directed or undirected) we denote the first vertex by in(\( Q \)) and the last vertex by ter(\( Q \)).

For a set of vertex disjoint paths \( \mathcal{P} \), we write \( V[\mathcal{P}] = \cup\{V(P); P \in \mathcal{P}\} \), \( E[\mathcal{P}] = \cup\{E(P); P \in \mathcal{P}\} \), in[\( \mathcal{P} \)] = \{in(P); P \in \mathcal{P}\}, and ter[\( \mathcal{P} \)] = \{ter(P); P \in \mathcal{P}\}.

**Definition 2.2 (Alternating Trail Relative to \( \mathcal{P} \), Proper Alternating Trail).** Given a path partition \( \mathcal{P} \), an undirected trail \( Q \) is alternating relative to \( \mathcal{P} \) if the following two conditions hold:

- All forward edges of \( Q \) are in \( E(G) - E[\mathcal{P}] \), all backward edges are in \( E[\mathcal{P}] \), and every forward edge \((u, v)\), where \( v \in V[\mathcal{P}^{>1}] \), is followed by a backward edge, unless \( v \in \text{in}[\mathcal{P}^{>1}] \) and \( v = \text{ter}(Q) \).
- For every vertex \( v \in V(Q) \) there exists at most one forward edge \((u, v) \in E(Q), \text{entering } v \), and at most one forward edge \((v, w) \in E(Q), \text{leaving } v \).

An alternating trail \( Q \) is proper if \( \text{in}(Q) \in \text{ter}[\mathcal{P}] \). An alternating cycle is a proper alternating trail where in(\( Q \)) = ter(\( Q \)).

We denote by \( \mathcal{P} \Delta Q \) the spanning subgraph of \( G \) containing edges in the symmetric difference \( E[\mathcal{P}] \Delta E(Q) = E[\mathcal{P}] \cup E(Q) \setminus (E[\mathcal{P}] \cap E(Q)) \). The following claim is self-evident from the definition above:

**Claim 2.3.** Let \( Q \) be either a proper alternating trail or an alternating cycle relative to \( \mathcal{P} \). Then

1. \( \mathcal{P}' = \mathcal{P} \Delta Q \) is a collection of disjoint paths and cycles in \( G \).
2. If \( \mathcal{P}' \) contains no cycles then \( |\mathcal{P}'| - |\mathcal{P}| = (n - |E(\mathcal{P}')}| - (n - |E(\mathcal{P})|) = |E(\mathcal{P})| - |E(\mathcal{P}'')| = |B_{\mathcal{Q}}| - |F_{\mathcal{Q}}| \).

**Definition 2.4 (1-Transversal).** Let \( \mathcal{P} = \{P_1, \ldots, P_m\} \) be a path partition in \( G \). A set of vertices \( Y \subseteq V \) is a 1-transversal (or a transversal), of \( \mathcal{P} \) if for each \( 1 \leq i \leq m, |Y \cap V(P_i)| \geq 1 \). Denote by first\( \mathcal{P} \)(\( Y \)) the set of vertices in \( Y \) which appear first on each path in \( \mathcal{P} \).

To prove the Gallai–Milgram theorem we shall show that every optimal path partition \( \mathcal{P} \) contains a 1-transversal \( Y \) such that \( X = \text{first}_\mathcal{P}(Y) \) is an independent set. The proof is constructive: Given any path partition \( \mathcal{P} \), either we find a transversal \( Y \) where \( X = \text{first}_\mathcal{P}(Y) \) is an independent set, or we find an alternating trail \( Q \) such that \( \mathcal{P}' = \mathcal{P} \Delta Q \) is a path partition with \( |\mathcal{P}'| < |\mathcal{P}| \).

For ease of notation, if \( Y \subseteq S \), and \( x \in S \), then we use the short notation \( Y + x \) (or \( Y - x \)) to denote \( Y \cup \{x\} \) (or \( Y \setminus \{x\} \)). If \( Y \) is an ordered set, then \( Y + x \) is the set obtained by adding \( x \) as the last element in the set.

### 2.2. Definition of a 1-snapshot

**Definition 2.5 (1-Snapshot).** Let \( \mathcal{P} \) be path partition in \( G \). A 1-snapshot in \( \mathcal{P} \) is a sequence of edges \( \mathcal{A} = (a_1, a_2, \ldots) \) where \( a_i \in E(G) \setminus E[\mathcal{P}] \), for \( i = 1, 2, \ldots \), which satisfies the following axioms:

A1. If \( a_i = (u, v) \) then either \( u \in \text{ter}[\mathcal{P}] \) or there exists a \( j < i \) such that \( u^+ = \text{head}(a_j) \).
A2. For each \( v \in V(G) \) there exists at most one edge \( a_i \) with head(\( a_i \)) = \( v \).
A3. If \( a_i = (u, v) \) and \( a_j = (w, x) \) and \( v \geq \mathcal{P} w \) then \( i > j \).
Given a 1-snapshot $\mathcal{A}$, we define the set $Y_\mathcal{A} = \{u; u \in \text{ter}(\mathcal{P}) \text{ or } u^+ \in \text{head}(\mathcal{A})\}$. We will denote $Y_\mathcal{A}$ by $Y$, when no ambiguity arises. Note that $Y$ is a 1-transversal to $\mathcal{P}$. By A1 and A2 it follows that each $u \in Y$ has a unique predecessor, $p(u) \in Y$, defined as $p(u) := u$ if $u \in \text{ter}(\mathcal{P})$ and $p(u) := w$ where $a_j = (w, u^+) \in \mathcal{A}$.

**Example 2.6.** Let $\mathcal{A} = \emptyset$. Then $\mathcal{A}$ is a 1-snapshot and $Y_\mathcal{A} = \text{ter}(\mathcal{P})$.

For the reader who is familiar with the original inductive proof of the Gallai–Milgram theorem, the edges chosen for the induction step are precisely the edges in $\mathcal{A}$. This is the motivation for the name snapshot, it is a snapshot of the inductive process.

### 2.3. Acyclicity of the auxiliary digraph

Define the auxiliary digraph of $G$ with respect to $\mathcal{P}$ and $\mathcal{A}$ as the graph $H$ induced by $\mathcal{A} \cup E[\mathcal{P}]$.

**Lemma 2.7.** Let $\mathcal{P}$ be a path partition in $G$, and let $\mathcal{A}$ be a 1-snapshot in $\mathcal{P}$. Then the auxiliary digraph $H$ is acyclic; furthermore, for $u, v \in \text{first}_\mathcal{P}(Y)$, the graph induced by $H \cup \{(u, v)\}$ is also acyclic.

**Proof.** Since $u$ is the first vertex in some path $P \in \mathcal{P}$ that belongs to $Y$, no edge of $\mathcal{A}$ can enter $P$ in $u$ or in a vertex that precedes $u$ in $P$. Therefore, no cycle in $\mathcal{A} \cup E[\mathcal{P}] \cup \{(u, v)\}$ can contain $u$.

Assume, by contradiction, that $\mathcal{A} \cup E[\mathcal{P}]$ contains a cycle $C$. Then $C$ is of the form $C = (e_{i_1}P_2e_{i_2}P_3 \cdots e_{i_t}P_1)$, where $t \geq 2$, $P_i$ is a subpath of $P_i \in \mathcal{P}$ ($P_i$ may be either a single vertex, or a non-trivial subpath of $P_i$ or equal to $P_i$), and $e_{i_j} \in \mathcal{A}$ for all $1 \leq j \leq t$. (See Fig. 1.) By A3 of Definition 2.5, it follows that $i_1 > i_2 > \cdots > i_t > i_1$, a contradiction. ■

### 2.4. Updating a 1-snapshot

Let $\mathcal{P}$ be a path partition.

**Initialize** the snapshot: $\mathcal{A} \leftarrow \emptyset$. Let $X := \text{first}_\mathcal{P}(Y_\mathcal{A})$.

**Update:** For $e = (u, v) \in E, u, v \in X$ and $v \notin \text{in}[\mathcal{P}]$, let $\mathcal{A} \leftarrow \mathcal{A} + e$.

**Claim 2.8.** If $\mathcal{A}$ is a 1-snapshot, then the updated snapshot is also a 1-snapshot.

**Proof.** Let $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ be a snapshot. We will show that axioms A1–A3 hold after the above update for $\mathcal{A}' = \mathcal{A} + a_{m+1}$ where $a_{m+1} = e = (u, v)$. Axiom A1 holds since $u \in \text{first}_\mathcal{P}(Y_\mathcal{A})$, implying that either $u \in \text{ter}(\mathcal{P})$ or $u^+ = \text{head}(a_j)$ for some $a_j \in \mathcal{A}$. A2 holds since whenever an edge $e = (u, v)$ is added to $\mathcal{A}$, the vertex $v$ in no longer in $X$, and is replaced by $v^-$ in $X$. This fact also proves A3, since $v^- >_\mathcal{P} w$; hence after the edge $(u, v)$ is added to $\mathcal{A}$, $(w, x)$ cannot be added to $\mathcal{A}$. ■

### 2.5. Backtracking and improving the path partition

By the remark following Definition 2.5, each vertex $u \in Y$ has a unique predecessor $p(u) \in Y$. It is therefore possible to retrieve a trail from $u$ back to $p(u)$, back to $p(p(u))$, etc. Define $p^2(u) := p(p(u))$ and recursively, $p^i(u) := p(p^{i-1}(u))$.

**Lemma 2.9.** For each $u \in Y$, if $p^i(u) = u$ then $p(u) = u$.

**Proof.** It is impossible to have a ‘cycle’ where $p^i(u) = p^{i+i}(u)$ for some $i > 1$ since this
would contradict axiom A1 which claims that if \( a_{i_1} = (u, v) \) and \( a_{i_2} = (p(u), u^+) \) then \( i_2 < i_1 \). Hence, the lemma follows.

The next definition formalizes the process of using the predecessors to construct an alternating trail:

**Definition 2.10 (Path Segment in a Snapshot).** Let \( A \) be a snapshot, and let \( Y = Y_A \). For a vertex \( u \in Y \), an *alternating path segment* \( Q_1(u) \) of \( u \) is defined as follows:

1. If \( p(u) = u \) then \( Q_1(u) \) is the trivial alternating path \((u)\).
2. If \( u \neq p(u) \) then \( Q_1(u) \) is the alternating trail (in fact, a path) \((p(u), u^+, u)\) with one forward edge (the edge \((p(u), u^+) \in A\)) and one backward edge (the edge \((u, u^+) \in E(P)\)).

We write \( Q_j(u) = Q_1(p^{j-1}(u)) \ast Q_1(p^{j-2}(u)) \ast \cdots \ast Q_1(p(u)) \ast Q_1(u) \). Note that \( |B_{Q_1(u)}| = |F_{Q_1(u)}| \) and hence \( |B_{Q_j(u)}| = |F_{Q_j(u)}| \). It is easy to check the following:

**Claim 2.11.** \( Q_1(u) \) is a subgraph of \( A \cup E(P) \) and is an alternating trail relative to \( P \), as defined in Definition 2.2.

**Lemma 2.12.** Let \( P \) be a path partition and \( A \) a 1-snapshot in \( P \). If there exists an edge \( e = (u, v) \) with \( u, v \in \text{first}_P(Y) \) and \( v \in \text{in}[P] \) then \( P \) is not optimal.

**Proof.** Let \( j \) be such that \( p^j(u) \in \text{ter}[P] \). Such an index exists by the finiteness of the graph and Lemma 2.9. Consider the proper alternating trail \( Q = Q_j(u) \ast e \). Let \( P' = P \Delta Q \). Since \( P' \) is a subgraph of \( A \cup E[P] \cup \{e\} \) which is acyclic (as was proved in Lemma 2.7), it follows that \( P' \) is a path partition. By Claim 2.3,

\[
|P'| - |P| = |B_Q| - |F_Q| = |B_{Q_j(u)}| - |F_{Q_j(u)}| - 1 = -1. \]

**Theorem 2.13 (Gallai–Milgram [6]).** Let \( P \) be a path partition in a graph \( G \). Then either there exists an independent transversal of \( P \), or there exists a path partition \( P' \) satisfying \( |P'| < |P| \).

**Proof.** We give an algorithmic proof based on the lemmas above.

**2.6. Algorithm**

0. **Input:** \( G, P \)
1. **Initialize snapshot:** as in Section 2.4. Let \( X = \text{first}_P(Y) \).
2. **While** \((\exists e = (u, v), u, v \in X, v \not\in \text{in}[P])\) **do**
   3. **Update snapshot** as in Section 2.4
   4. **If** \((\exists e = (u, v), u, v, e \in X, v \in \text{in}[P])\)
     5. find \( Q \leftarrow Q_j(u) \ast e \)
   6. **\( P' \leftarrow P \Delta Q \)**
   7. **stop**
3. **Else** \( X \) is an independent transversal

In line 6, \( |P'| < |P| \) by Lemma 2.12, and in line 8, \( X \) is independent since any edge \( e = (u, v), u, v \in X, v \not\in \text{in}[P] \) is used to update and increase the snapshot. Once the snapshot is maximal with respect to inclusion, and no edge \( e = (u, v), u, v \in X, v \in \text{in}[P] \) exists, \( X \) is an independent transversal. ■
3. Proof for $k = 2$

3.1. Preliminary definitions

Recall that for a path partition $\mathcal{P}$ we denote the set of non-trivial paths as $\mathcal{P}^{> 1}$ and the set of trivial paths as $\mathcal{P}^1$. We extend Definition 2.2 in order to take into account the two different types of paths.

**Definition 3.1** (Prim and Proper Alternating Trail). An alternating trail $Q = \{v_0, \ldots, v_l\}$ is prim if $v_l \in \text{in}[\mathcal{P}]$. A proper alternating trail is of type (a) if $v_0 \in \text{ter}[\mathcal{P}^{> 1}]$ and of type (b) otherwise (i.e., $v_0 \in V[\mathcal{P}^1]$). A prim alternating path is of type (1) if $v_l \in \text{in}[\mathcal{P}^{> 1}]$ and of type (2), otherwise (i.e., $v_l \in V[\mathcal{P}^1]$).

We denote a prim and proper trail of type (1) and (a) as an (a-1) type. Types (a-2), (b-1) and (b-2) are defined similarly. An alternating cycle is a closed prim and proper alternating trail of type (b-2) $Q = (v_0, \ldots, v_l)$ where $v_0 = v_l \in V[\mathcal{P}^1]$.

Claim 2.3 holds here as well, and is extended as follows:

**Lemma 3.2.** If $Q$ is a prim and proper alternating trail relative to $\mathcal{P}$, and $\mathcal{P}' = \mathcal{P} \Delta Q$ is acyclic, then the number of non-trivial paths in $\mathcal{P}'$ and $\mathcal{P}$ can differ by at most one, i.e., $-1 \leq |\mathcal{P}'^{> 1}| - |\mathcal{P}^{> 1}| \leq 1$.

More specifically, if $Q$ is of type (a-1) then $|\mathcal{P}'^{> 1}| = |\mathcal{P}^{> 1}| - 1$. If $Q$ is of type (a-2), (b-1) or a cycle then $|\mathcal{P}'^{> 1}| = |\mathcal{P}^{> 1}|$. Finally, if $Q$ is of type (b-2) and not an alternating cycle then $|\mathcal{P}'^{> 1}| = |\mathcal{P}^{> 1}| + 1$.

**Proof.** If $Q$ is of type (a-1) then $v_l$ and $v_0$ are initial and terminal vertices of non-trivial paths in $\mathcal{P}$, but not in $\mathcal{P}'$, $\text{in}[\mathcal{P}'^{> 1}] = \text{in}[\mathcal{P}^{> 1}] \setminus \{v_l\}$ and $\text{ter}[\mathcal{P}'^{> 1}] = \text{ter}[\mathcal{P}^{> 1}] \setminus \{v_0\}$. Therefore the number of non-trivial paths decreases by 1.

Similarly, if $Q$ is of type (a-2), (b-1) or a cycle then $|\mathcal{P}'^{> 1}| = |\mathcal{P}^{> 1}|$ and $|\text{ter}[\mathcal{P}'^{> 1}]| = |\text{ter}[\mathcal{P}^{> 1}]|$, and if it is of type (b-2) and not a cycle then $|\mathcal{P}'^{> 1}| = |\mathcal{P}^{> 1}| \cup \{v_0\}$ and $|\text{ter}[\mathcal{P}'^{> 1}]| = |\text{ter}[\mathcal{P}^{> 1}]| \cup \{v_l\}$, implying that the number of non-trivial paths increases by 1. ■

We now extend the notion of 1-transversal:

**Definition 3.3** (2-Transversal). Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a path partition in $G$. The sets $Y_1, Y_2 \subseteq V$, $Y_1 \cap Y_2 = \emptyset$ form a 2-transversal of $\mathcal{P}$ if for each $1 \leq i \leq 2$, and for each $P \in \mathcal{P}^{> 1}$, $|Y_i \cap V(P)| \geq 1$, and $V[\mathcal{P}^1] \subseteq Y_1 \cup Y_2$.

Denote by $X_i = \text{first}_\mathcal{P}(Y_i)$, $i = 1, 2$, the sets of vertices in $Y_i$ which appear first on each path in $\mathcal{P}$.

Note that since every path in $\mathcal{P}^1$ is trivial, then, naturally, $V[\mathcal{P}^1] \subseteq X_1 \cup X_2$. Therefore $X_1, X_2$ is also a 2-transversal.

To prove Conjecture 1.1 for $k = 2$ we extend the proof for the case $k = 1$: Given a path partition $\mathcal{P}$, either we find a 2-transversal $Y_1 \cup Y_2$ such that $X_1 \cup X_2$ is a 2-colouring orthogonal to $\mathcal{P}$, or we find an alternating trail $Q$ such that $\mathcal{P}' = \mathcal{P} \Delta Q$ is a path partition with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.

The algorithm is more complicated than the algorithm for $k = 1$ since the definition of a 2-snapshot and its update are more involved than those of a 1-snapshot.
3.2. Definition of a 2-snapshot

We begin with a definition of a quasi-2-snapshot, and then proceed with the definition of a 2-snapshot.

**Definition 3.4 (Quasi-2-Snapshot).** Let \( \mathcal{P} \) be path partition in \( G \). A quasi-2-snapshot in \( \mathcal{P} \) is a sequence of edges \( \mathcal{A} = (a_1, a_2, \ldots) \) where \( a_i \in E(G) \setminus E[\mathcal{P}] \) for \( i = 1, 2, \ldots \) which satisfies the following axioms A1a and A2.

A1a. If \( a_i = (u, v) \) then \( v \not\in \text{in}[\mathcal{P}^+] \). In addition, either \( u \in \text{ter}[\mathcal{P}] \) or \( u^+ \in \text{ter}[\mathcal{P}] \) or there exists a \( j < i \) such that \( u^+ = \text{head}(a_j) \) or \( u^{++} = \text{head}(a_j) \).

A2. For each \( v \in V(G) \) there exists at most one edge \( a_i \) with \( \text{head}(a_i) = v \).

Given a quasi-2-snapshot \( \mathcal{A} \), we define the set \( Y_\mathcal{A} = \{u; u \in \text{ter}[\mathcal{P}] \text{ or } u^+ \in \text{ter}[\mathcal{P}] \text{ or } u^+ \in \text{head}(\mathcal{A}) \text{ or } u^{++} \in \text{head}(\mathcal{A})\} \). By A1a and A2, for each \( u \in Y \), there exists a predecessor \( p(u) \in Y \) defined as follows:

\[
p(u) = \begin{cases} u' & \text{if there exists no edge } (v, u') \in \mathcal{A} \\ v & \text{if there exists an edge } (v, u') \in \mathcal{A} \end{cases}
\]

where

\[
u' = \begin{cases} u & \text{if } u \in \text{ter}[\mathcal{P}] \\ u^+ & \text{otherwise.} \end{cases}
\]

Before we proceed to the definition of a 2-snapshot, we introduce three more axioms. The first one is parallel to Lemma 2.9.

A1b. For each \( u \in Y \), if \( p^j(u) = u \) then \( p(u) = u \).

As in the case \( k = 1 \), by the finiteness of the graph and axiom A1b, for any \( u \in Y \) there must be an index \( j \) such that \( p^j(u) = p^{j+1}(u) \in \text{ter}[\mathcal{P}] \). On the basis of the definition of \( p(u) \), we can now define an alternating path segment as the segment that connects \( p(u) \) with \( u \).

**Definition 3.5 (Path Segment in a 2-Snapshot).** Let \( \mathcal{A} \) be a quasi-2-snapshot and let \( Y = Y_\mathcal{A} \). For a vertex \( u \in Y \), an alternating path segment \( Q_1(u) \) of \( u \) is defined as follows:

\[
Q_1(u) = \begin{cases} (u) & \text{if there exists no edge } (v, u) \in \mathcal{A} \text{ and } u \in \text{ter}[\mathcal{P}] \\ (u^+, u) & \text{if there exists no edge } (v, u^+) \in \mathcal{A} \text{ and } u \in \text{ter}[\mathcal{P}] \\ (p(u), u^+) & \text{if there exists an edge } (v, u) \in \mathcal{A} \text{ and } u \in \text{ter}[\mathcal{P}] \\ (p(u), u^{++}) & \text{if there exists an edge } (v, u^+) \in \mathcal{A} \text{ and } u \in \text{ter}[\mathcal{P}] \end{cases}
\]

Note that in the first case, \( Q_1(u) = (u) \) is the trivial path consisting of no edges; in the second, \( Q_1(u) \) consists of one backward edge, in the third case it consists of one forward edge, and in the fourth, it consists of one forward edge and one backward edge.

We write \( Q_j(u) = Q_1(p^{j-1}(u)) * Q_1(p^{j-2}(u)) * \cdots * Q_1(p(u)) * Q_1(u) \).

We assume now that \( \mathcal{A} \) satisfies axiom A1b, and let \( j \) be the smallest index such that \( p^j(u) = p^{j+1}(u) \in \text{ter}[\mathcal{P}] \).

Define

\[
c(u) = \begin{cases} |B_{Q_j(u)}| - |F_{Q_j(u)}| + 1 & \text{if } p^j(u) \in V[\mathcal{P}^+] \\ |B_{Q_j(u)}| - |F_{Q_j(u)}| + 2 & \text{if } p^j(u) \in V[\mathcal{P}] \end{cases}
\]

**Remark 3.6.** By the definition above, in all cases, a path segment satisfies \( |B_{Q_1(u)}| - |F_{Q_1(u)}| = \)
c(u) - c(p(u)) and thus |B_{Q_j(u)}| - |F_{Q_j(u)}| = c(u) - c(p_j(u)).

Define

Y_1 = \{u; c(u) = 1\}
Y_2 = \{u; c(u) = 2\} \text{ and there exists no } v \text{ with } c(v) = 1, \text{ and } v > P u \}.

By this definition we have:

**Claim 3.7.** For each \( P \in \mathcal{P}^>1 \), all the vertices in \( V(P) \cap Y_2 \) appear before the vertices in \( V(P) \cap Y_1 \).

We are now ready to define axioms A3a and A4. Loosely speaking, A3a is similar to A3, except that we apply it to each colour class \( c(u) \) separately. Axiom A4 asserts that we are interested only in two colour classes, since we are dealing with the case \( k = 2 \).

**A3a.** If \( a_i = (u, v), a_j = (w, x), v > P w \) and \( c(v) = c(w) \) then \( i > j \).

**A4.** If \( a = (u, v) \in \mathcal{A} \) then \( u \in Y_1 \cup Y_2 \).

**Definition 3.8** (2-Snapshot). A 2-snapshot in \( \mathcal{P} \) is a sequence of edges \( \mathcal{A} = \{a_1, a_2, \ldots\} \) where \( a_i \in E(G) \setminus E[\mathcal{P}] \) for \( i = 1, 2, \ldots \) which satisfies axioms A1a, A1b, A2, A3a, and A4 defined above.

**Example 3.9.** Let \( \mathcal{A} = \emptyset \). Then \( Y_1 = \text{ter}[\mathcal{P}^>1], Y_2 = Y_1^\cup \cup V[\mathcal{P}^1], p(u) = u \) for \( u \in Y_1 \cup V[\mathcal{P}^1] \) and \( p(u) = u^+ \) for \( u \in Y_2 \cap V[\mathcal{P}^>1] \).

### 3.3. Acyclicity of the auxiliary digraph

**Observation 3.10.** Let \( \mathcal{A} \) be a 2-snapshot on \( \mathcal{P} \) and let \( u, v \in \text{first}_{\mathcal{P}}(Y_j) \) for some \( j = 1 \) or \( 2 \). Then the graph induced by \( E[\mathcal{P}] \cup \{e = (x, y) \in \mathcal{A}; x \in Y_j \} \cup \{(u, v)\} \) is acyclic.

**Proof.** The proof is similar to the proof of **Lemma 2.7** using axiom A3a.

**Lemma 3.11.** Let \( \mathcal{A} \) be a 2-snapshot on \( \mathcal{P} \). Then the graph \( H \) induced by \( \mathcal{A} \cup E[\mathcal{P}] \) is acyclic. Furthermore, if \( u, v \in \text{first}_{\mathcal{P}}(Y_j) \) for some \( j = 1, \) or \( 2, \) then the graph induced by \( H \cup \{(u, v)\} \) is acyclic.

**Proof.** Assume, by contradiction, that \( H \cup \{(u, v)\} \) contains a cycle \( C \). Assume first that \( C \) intersects both \( Y_1 \) and \( Y_2 \). Let \( (x_1, x_2, \ldots, x_r) \) be the shortest path contained in \( C \) such that \( x_1 \in Y_1 \) and \( x_r \in Y_2 \). This implies that \( x_i \not\in V[\mathcal{P}^1] \) for \( 2 \leq i \leq r - 1 \) since, otherwise, \( x_i \not\in Y_1 \cup Y_2 \), and by A4 no edges of \( H \) can leave \( x_i \). Hence \( x_2, \ldots, x_r \) are contained in some path \( P \in \mathcal{P}^>1 \). But now \( x_2^- \in Y_1 \), and \( x_2^- > x_r \in Y_2 \), contradicting **Claim 3.7**. Therefore, all the edges in \( \mathcal{C} \) in \( \mathcal{A} \) are in \( \{e = (x, y) \in \mathcal{A}; x \in Y_j \} \cup \{(u, v)\} \), for \( j = 1 \) or \( 2 \) which is impossible by the previous observation.

### 3.4. Update of a 2-snapshot

**Example 3.9** above describes an initial 2-snapshot \( \mathcal{A} \). If \( X_i = \text{first}_{\mathcal{P}}(Y_i) \) for \( i = 1, 2 \) is an independent set, then it is a 2-colouring orthogonal to \( \mathcal{P} \), and we are done. Otherwise, we shall update \( \mathcal{A} \). In the following we show how \( \mathcal{A} \) can be updated in an ‘easy’ way.
3.4.1. Direct update

Let \( \mathcal{A} \) be a 2-snapshot. Let \( X_i = \text{first}_p(Y_i) \) for \( i = 1, 2 \). In the following cases \( \mathcal{A} \) can be updated, remaining a 2-snapshot, where either \( Y_1 \) is increased, or \( Y_1 \) remains the same and \( Y_2 \) is increased.

1. If there exists an edge \( e = (u, v), u, v \in X_2, v \in V[\mathcal{P}^{>1}], v \notin \text{in}[\mathcal{P}^{>1}] \) then update as follows:
   \[ \mathcal{A} \leftarrow \mathcal{A} + e. \]
   Note that here \( Y_1 \) is unchanged, \( Y_2 \leftarrow Y_2 + v^-; p(v^-) \leftarrow u. \)

2. If there exists an edge \( e = (u, v), u, v \in X_1, v, v^- \in V[\mathcal{P}^{>1}] \setminus \text{in}[\mathcal{P}^{>1}], v^- \in Y_2 \) and no edges of \( \mathcal{A} \) leave \( v^- \) then update as follows:
   \[ \mathcal{A} \leftarrow \mathcal{A} + e - (x, v) \quad \text{for any} \ (x, v) \in \mathcal{A}, \ x \neq u, \text{if exists}. \]
   Note that here \( Y_1 \) is increased: \( Y_1 \leftarrow Y_1 + v^-; Y_2 \leftarrow Y_2 - v^- + v^-; p(v^-) \leftarrow u; p(v^-) \leftarrow v^- \).

3. If there exists an edge \( e = (u, v), u, v \in X_2, v \in V[\mathcal{P}^{1}] \) and no edges of \( \mathcal{A} \) leave \( v \) then update as follows:
   \[ \mathcal{A} \leftarrow \mathcal{A} + e. \]
   Note that here \( Y_1 \) is increased: \( Y_1 \leftarrow Y_1 + v; Y_2 \leftarrow Y_2 - v; p(v) \leftarrow u. \)

In each of the cases above, \( e \) is labeled in \( \mathcal{A} \) as \( a_{i+1} \), where \( i = \max\{j; a_j \in \mathcal{A}\} \). It is routine to check that axioms A1a, A1b, A2, A3a, and A4 are maintained after the update.

3.4.2. Update with regret

Recall that in cases 2 and 3 above we needed the condition that no edges of \( \mathcal{A} \) leave a certain vertex \( x \) (where \( x = v^- \) in case 2, and \( x = v \) in case 3). If this condition is violated, i.e., there are edges in \( \mathcal{A} \) leaving \( x \), then after updating \( \mathcal{A} \) as in Section 3.4.1 we may violate axioms A3a and A4. (We remark that the violation of A4 is more severe than the violation of A3a since the violation of A3a can be remedied by reordering the edges in \( \mathcal{A} \); however, a violation of A4 indicates an intrinsic problem which we need to understand better if we want to generalize the proof to \( k > 2 \).) The most serious problem that may arise is the existence of paths in \( \mathcal{A} \cup E[\mathcal{P}] \) from \( x \) to \( Y_2 \), which may lead, if we continue the algorithm, to a cycle in \( \mathcal{A} \cup E[\mathcal{P}] \).

**Definition 3.12** \( (p^\infty(v), p^{-\infty}(v)) \). Let \( p^\infty(v) = \{p(v), p(p(v)), p^3(v), \ldots\} = \bigcup_{i=1}^{\infty} \{p^i(v)\} \).

Define \( p^{-1}(v) = \{u; p(u) = v\} \). Note that \( p \) is a function (not necessarily a bijection) and \( p^{-1} \) is the inverse relation.

Let \( p^{-\infty}(v) = \bigcup_{i=1}^{\infty} p^{-i}(v) \).

We will suggest now a more elaborate update where all edges leaving \( x \) (and others) are erased, unless \( p^{-\infty}(x) \cap Y_1 \neq \emptyset \). Details will follow this definition:

**Definition 3.13** (Regret Function). Let \( \mathcal{A} \) be a 2-snapshot, \( v \in Y_2 \). Assume that \( p^{-\infty}(v) \subseteq Y_2 \). The function regret\( (v) \) updates the 2-snapshot \( \mathcal{A} \) as follows (see Fig. 2):

\[ \mathcal{A} \leftarrow \mathcal{A} \setminus \{(u, w) : u \in \{p^{-\infty}(v) \cup \{v\}\}\}. \]

As a result, \( Y_2 \) gets updated as follows: \( Y_2 \leftarrow Y_2 \setminus p^{-\infty}(v) \) and \( Y_1 \) is unchanged.
Fig. 2. Regret(v).

Note that if \( p^{-\infty}(v) = \emptyset \) then regret(v) does nothing to \( A \). In this case no regret is necessary (Piaf [11]).

**Lemma 3.14.** Let \( A \) be a 2-snapshot to \( \mathcal{P} \), and let \( v \in Y_2 \). If \( p^{-\infty}(v) \subseteq Y_2 \) then by applying regret(v) we obtain a 2-snapshot.

**Proof.** Let \( A' \) be the snapshot after regret(v) is applied. Axioms A1b, A2, A3a and A4 hold trivially for \( A' \) since \( A' \subseteq A \) and the order of the edges in \( A' \) is the same as in \( A \). To show that axiom A1a holds, note that if \( (u, w) \) is erased from \( A \), then all edges of type \( (p^{-i}(u), x) \), for \( i = 1, 2 \ldots \) are also erased. This implies that if \( a_i = (u, v) \in A' \) then either \( u^+ \in \text{ter}[\mathcal{P}^{>1}] \) or there exists an edge of the form \( a_j = (p(u), u^+) \in A' \) and \( j < i \), implying axiom A1a. Hence all axioms hold for \( A' \) and it is a 2-snapshot. ■

**Lemma 3.14** together with cases 2 and 3 of Section 3.4.1 imply

**Lemma 3.15** (Update 2-snapshot with Regret). Let \( A \) be a 2-snapshot to \( \mathcal{P} \), and \( X_i = \text{first}_{\mathcal{P}}(Y_i), i = 1, 2 \). In the following cases \( A \) can be updated, remaining a 2-snapshot, where \( Y_1 \) is increased.

1. If there exists an edge \( (u, v), u, v \in X_1, v, v^- \in V[\mathcal{P}^{>1}] \setminus \text{in}[\mathcal{P}^{>1}], v^- \in Y_2 \) and \( p^{-\infty}(v^-) \subseteq Y_2 \), then
   \( \text{regret}(v^-) \),
   \( A \leftarrow A + e - (x, v) \) for any \( (x, v) \in A, x \neq u \), if exists.
2. If there exists an edge \( (u, v), u, v \in X_2, v \in V[\mathcal{P}^1] \) and \( p^{-\infty}(v) \subseteq Y_2 \), then
   \( \text{regret}(v) \),
   \( A \leftarrow A + e \).

3.5. Backtracking and improving the path partition

The following lemma extends Lemma 2.12 to \( k = 2 \).

**Lemma 3.16.** Let \( \mathcal{P} \) be a path partition, and let \( A \) be a 2-snapshot. Let \( e = (u, v) \) be an edge in \( G \). Then \( \mathcal{P} \) is not 2-optimal in the following cases (see Fig. 3).
1. \( u, v \in X_2 \) and \( v \in \text{in}[\mathcal{P}^{>1}] \).
2. \( u, v \in X_1 \) and \( v \in V[\mathcal{P}^1] \).
3. \( u, v \in X_1 \) and \( v^- \in \text{in}[\mathcal{P}^{>1}] \).
4. \( u, v \in X_1 \), \( v^- \in Y_2 \) and \( p^{-\infty}(v^-) \cap Y_1 \neq \emptyset \).
5. \( u, v \in X_2 \), \( v \in V[\mathcal{P}^1] \) and \( p^{-\infty}(v) \cap Y_1 \neq \emptyset \).

**Proof.** In each of the cases mentioned, by Lemma 3.11, the graph \( H \) induced by \( A \cup E[\mathcal{P}] \cup \{e\} \) is acyclic. Thus, for any alternating trail \( Q \) in \( H \), \( \mathcal{P}' = \mathcal{P} \Delta Q \) is acyclic. We shall show in each case that \( Q \) can be found such that \( |\mathcal{P}'|_2 < |\mathcal{P}|_2 \). Note that
\[
|\mathcal{P}'|_2 - |\mathcal{P}|_2 = |\mathcal{P}^{>1}| - |\mathcal{P}^{>1}| + |\mathcal{P}'| - |\mathcal{P}| = |\mathcal{P}^{>1}| - |\mathcal{P}^{>1}| + B_Q - F_Q,
\]
and the value of \( |\mathcal{P}^{>1}| - |\mathcal{P}^{>1}| \) can be computed using Lemma 3.2.

1. Let \( j \) be the smallest integer such that \( p^j(u) = p^{j+1}(u) \). Consider the alternating trail \( Q = Q_j(u) \ast e \). Then, by Remark 3.6,
\[
|B_Q| - |F_Q| = |B_{Q_j(u)}| - |F_{Q_j(u)}| - 1 = c(u) - c(p^j(u)) - 1 = 1 - c(p^j(u)).
\]

Then
\[
|\mathcal{P}'|_2 - |\mathcal{P}|_2 = |\mathcal{P}^{>1}| - |\mathcal{P}^{>1}| + 1 - c(p^j(u)).
\]

If \( c(p^j(u)) = 1 \) then \( Q \) is an alternating trail of type (a-1) and thus \( |\mathcal{P}'^{>1}| < |\mathcal{P}^{>1}| \). If \( c(p^j(u)) = 2 \) then \( Q \) is of type (b-1), so we have \( |\mathcal{P}'^{>1}| = |\mathcal{P}^{>1}| \).

Note that the argument in this part is valid even if \( p^{j_0}(u) = v \) for some \( j_0 \leq j \). In the following parts, however, we have to be more careful.

2. Let \( j \) be minimal such that either \( p^j(u) = p^{j+1}(u) \) or \( p^j(u) = v \). Consider the alternating trail \( Q = Q_j(u) \ast e \). Then \( |\mathcal{P}'|_2 - |\mathcal{P}|_2 = |\mathcal{P}^{>1}| - |\mathcal{P}^{>1}| - c(p^j(u)) \).

If \( c(p^j(u)) = 1 \) and \( Q \) is not a cycle, then \( Q \) is an alternating trail of type (a-2) and thus \( |\mathcal{P}'^{>1}| = |\mathcal{P}^{>1}| \). If \( c(p^j(u)) = 2 \), then \( Q \) is of type (b-2) and \( |\mathcal{P}'^{>1}| = |\mathcal{P}^{>1}| + 1 \). Finally, if \( p^j(u) = v \), then \( Q \) is an alternating cycle with \( c(p^j(u)) = 1 \) and \( |\mathcal{P}'^{>1}| = |\mathcal{P}^{>1}| \). In all cases
\[
|\mathcal{P}'|_2 - |\mathcal{P}|_2 < 0.
\]

3. Note that in this case we must have \( c(v^-) = 2 \). Let \( j \) be minimal such that either \( p^j(u) = p^{j+1}(u) \) or \( p^j(u) = v^- \). Consider the alternating trail \( Q = Q_j(u) \ast (u, v, v^-) \). Then \( |\mathcal{P}'|_2 - |\mathcal{P}|_2 = |\mathcal{P}^{>1}| - |\mathcal{P}^{>1}| + 1 - c(p^j(u)) \).

![Fig. 3. Cases of a breakthrough.](image-url)
If $p^j(u) = v^-$ or $p^j(u) \in V[P^1]$ we have $|P^{>1}| = |P^{>1}| + c(p^j(u)) = 2$. If $p^j(u) \in \text{ter}[P^{>1}]$ we have $|P^{>1}| = |P^{>1}| + 1$ and $c(p^j(u)) = 1$. So in all cases, the inequality $|P^{>1}|-|P| < 0$ holds as required.

4. As in the previous case, let $j$ be minimal such that either $p^j(u) = p^j(u) + 1$ or $p^j(u) = v^-$. Assume first that $p^j(u) = v^-$. Consider the alternating cycle $Q = Q_j(u) * (u, v, v^-)$. Here $|P^{>1}|-|P| - |P^{>1}| + c(u) - c(v^-) = -1$. Thus $P$ is not optimal.

Assume now $p^j(u) = p^j(u) + 1$. Let $j$ be minimal such that $p^j(w) = v^-$ for some $w \in Y_1$. By the minimality of $j$ we have $w \in V[P^1]$. Let $Q = Q_j(u) * (u, v, v^-) * Q_j(w)$. Here we have

$$
|B_Q| - |F_Q| = |B_{Q_j(u)}| - |F_{Q_j(u)}| + |B_{Q_j(w)}| - |F_{Q_j(w)}| = c(u) - c(p^j(u)) + c(w) - c(v^-) = 1 - c(p^j(u)) + 1 - 2 = -c(p^j(u)).
$$

Thus,

$$|P^{>1}|-|P| = |P^{>1}| - |P^{>1}| + |B_Q| - |F_Q| = |P^{>1}| - |P^{>1}|-c(p^j(u)).$$

If $p^j(u) \in V[P^1]$ we have $|P^{>1}| = |P^{>1}| + 1$ and $c(p^j(u)) = 2$. If $p^j(u) \in \text{ter}[P^{>1}]$ we have $|P^{>1}| = |P^{>1}| + c(p^j(u)) = 1$. In both case we get $|P^{>1}|-|P| < 1$. So in all cases, the alternating cycle $Q = Q_j(u) * e$. Otherwise, like in Case 4, let $j$ be minimal such that $p^j(w) = v^-$ for some $w \in Y_1$, and let $Q = Q_j(u) * (u, v) * Q_j(w)$. As in previous parts, it is routine to check that in both cases $|P^{>1}|-|P| = 1$. Thus $P$ is not 2-optimal.

**Theorem 3.17.** Let $P$ be a path partition. Let $A$ be a 2-snapshot in $P$ in which $Y_1$ is maximal with respect to inclusion among all possible snapshots. Then either $X_1 = \text{first}_P(Y_1)$ is an independent set, or there exists a path partition $P'$ with $|P^{>1}| < |P|$.

**Proof.** Assume that $e = (u, v) \in E(G)$ and $u, v \in X_1$.

Case 1: $v \in V[P^1]$. By case 2 of Lemma 3.16 there exists a path partition $P'$ with $|P^{>1}| < |P|$.

Case 2: $v \in V[P^{>1}]$ and $v^- \in \text{in}[P^{>1}]$. By case 3 of Lemma 3.16 there exists a path partition $P'$ with $|P^{>1}| < |P|$.

Case 3: $v, v^- \in V[P^{>1}]$, $v^- \in Y_2$, $p^-\infty(v^-) \cap Y_1 \neq \emptyset$. By case 4 of Lemma 3.16 there exists a path partition $P'$ with $|P^{>1}|-|P| < |P|$. In both cases we get $|P^{>1}|-|P| < |P|$. By Lemma 3.15 case 1, $Y_1$ can be increased, contradicting the assumption of the theorem.

**Theorem 3.18.** Let $P$ be a path partition. Let $A$ be a 2-snapshot in $P$ in which $Y_2$ is maximal with respect to inclusion among all possible snapshots in which $Y_1$ is maximal with respect to inclusion. Then either $X_2 = \text{first}_P(Y_2)$ is an independent set or there exists a path partition $P'$ with $|P^{>1}| < |P|$.

**Proof.** Assume that $u, v \in X_2$ and $e = (u, v) \in E(G)$.

Case 1: $v \in \text{in}[P^{>1}]$. By case 1 of Lemma 3.16 there exists a path partition $P'$ with $|P^{>1}| < |P|$. In both cases we get $|P^{>1}| < |P|$. By case 1 of Section 3.4.1, $Y_2$ can be increased, contradicting the assumption of the theorem.

Case 3: $v \in V[P^1]$ and $p^-\infty(v^-) \subseteq Y_2$. By case 2 of Lemma 3.15, $Y_1$ can be increased, contradicting the assumption of the theorem.
Case 4: $v \in V[\mathcal{P}^1]$ and $p^{-\infty}(v) \cap Y_1 \neq \emptyset$. By case 5 of Lemma 3.16 there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.  ■

Theorems 3.17 and 3.18 easily imply:

**Corollary 3.19.** Let $\mathcal{P}$ be a 2-optimal path partition and let $\mathcal{A}$ be a 2-snapshot in $\mathcal{P}$ in which $Y_2$ is maximal with respect to inclusion among all possible snapshots in which $Y_1$ is maximal with respect to inclusion. Then the sets $X_i = \text{first}_{\mathcal{P}}(Y_i)$ for $i = 1, 2$ form a 2-colouring orthogonal to $\mathcal{P}$, proving Conjecture 1.1 for $k = 2$.

**Corollary 3.20.** Given a path partition $\mathcal{P}$. Then there exists a polynomial algorithm of complexity $O(|E|)$ that either finds a 2-colouring orthogonal to $\mathcal{P}$, or finds a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.

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**References**