Abstract

Different partial hypergroupoids are associated with binary relations defined on a set $H$. In this paper we find sufficient and necessary conditions for these hypergroupoids in order to be reduced hypergroups. Given two binary relations $\rho$ and $\sigma$ on $H$ we investigate when the hypergroups associated with the relations $\rho \cap \sigma$, $\rho \cup \sigma$ and $\rho \sigma$ are reduced. We also determine when the cartesian product of two hypergroupoids associated with a binary relation is a reduced hypergroup.

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1. Introduction and preliminaries

The first step in the history of the development of Hyperstructures Theory was the 8th Congress of Scandinavian Mathematicians from 1934, when Marty [12] introduced the notion of hypergroup, analyzed its properties and applied them to non-commutative groups, algebraic functions, rational fractions. Nowadays the hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, ethnology, etc. (see [6]).

Till now, several connections between hyperstructures and binary relations are established and studied by many researchers: Rosenberg [13], Corsini [3,4], Corsini and Leoreanu [5], Chvalina [1], Konstantinidou and Serafimidis [11], Spartalis [14–16], De Salvo and Lo Faro [8] and so on. In this paper we deal with the hypergroupoids associated with binary relations introduced by Rosenberg [13] and studied then by Corsini and Leoreanu [3–5].

In the following we present some results obtained on this argument.

For a non-empty set $H$, we denote by $\mathcal{P}(H)$ the set of all non-empty subsets of $H$.

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Definition 1.1. A non-empty set $H$, endowed with a mapping, called hyperoperation, $\circ : H^2 \rightarrow \mathcal{P}^*(H)$ is named hypergroupoid. A hypergroupoid which verifies the following conditions:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$,
(ii) $x \circ H = H = H \circ x$, for all $x \in H$

is called hypergroup.

If $A$ and $B$ are non-empty subsets of $H$, then $A \circ B = \bigcup_{a \in A, \ b \in B} a \circ b$.

Rosenberg [13] has associated a partial hypergroupoid $I H = (H, \circ)$ with a binary relation $\rho$ defined on a set $H$, where, for any $x, y \in H$,

$x \circ x = L_x = \{z \in H \mid (x, z) \in \rho\}$ and $x \circ y = L_x \cup L_y$.

Definition 1.2. An element $x \in H$ is called outer element of $\rho$ if there exists $h \in H$ such that $(h, x) \notin \rho^2$.

We need some of Rosenberg results that we recall in the next theorem.

Theorem 1.3 (Rosenberg [13, Proposition 2]). $I H$ is a hypergroup if and only if

(i) $\rho$ has full domain;
(ii) $\rho$ has full range;
(iii) $\rho \subseteq \rho^2$;
(iv) if $(a, x) \in \rho^2$, then $(a, x) \in \rho$, whenever $x$ is an outer element of $\rho$.

Remark. If $\rho$ is a quasiorder relation, then the hypergroupoid $I H$ associated with $H$ is a hypergroup.

Theorem 1.4 (Corsini and Leoreanu [5, Proposition 1.1, Corollary 1.2, Remark 1.3]). Let $\rho$ be a relation defined on a set $H$ and $a, x \in H$. Let “$\circ$” be the partial hyperoperation defined above.

(i) If $\rho \subseteq \rho^2$, then $(a, x) \in \rho^2$ if and only if $x \in a \circ a \circ a$.
(ii) If $\rho \subseteq \rho^2$, then $x$ is an outer element for $\rho$ if and only if there exists $a \in H$ such that $x \notin a \circ a \circ a$.
(iii) If $\rho \subseteq \rho^2$, then there are no outer elements for $\rho$ if and only if for any $a \in H$, we have $a \circ a \circ a = H$.

Theorem 1.5 (Corsini [3, Theorem 1.3]). If $I H$ is a hypergroup, then the following statements hold:

(i) $\rho^2$ is a transitive relation,
(ii) If $\rho$ is symmetric, then $\rho^2$ is an equivalence relation on $H$.
(iii) If $\rho$ is symmetric and $\mid H/\rho^2 \mid > 1$, then $\rho$ is an equivalence relation on $H$.

Corollary 1.6. Let $\rho$ be a reflexive, symmetric and non-transitive relation on $H$. The following assertions are equivalent:

(i) $I H$ is a hypergroup.
(ii) For any $x \in H$ we have $x \circ x \circ x = H$.
(iii) There are no outer elements for $\rho$.
(iv) $\rho^2 = H \times H$.

Proposition 1.7 (Corsini [3, Theorem 2.5]). Let $\rho$ and $\sigma$ be two binary relations on $H$ with full domain and full range such that $\rho^2 = \rho$, $\sigma^2 = \sigma$ and $\rho \sigma = \sigma \rho$. Then $I H_{\rho \sigma}$ is a hypergroup.

It may happen that the hyperoperation “$\circ$” does not discriminate between a pair of elements of $H$, when two elements play interchangeable roles with respect to the hyperoperation. On a hypergroupoid $(H, \circ)$, the following three
equivalence relations, called the operational equivalence, the inseparability and the essential indistinguishability, respectively, may be defined (see [9,10,7]):

- \( x \sim_o y \iff x \circ a = y \circ a \) and \( a \circ x = a \circ y \), for any \( a \in H \);
- \( x \sim_i y \iff \) for \( a, b \in H \), we have \( x \in a \circ b \iff y \in a \circ b \);
- \( x \sim_e y \iff x \sim_o y \) and \( x \sim_i y \).

For any \( x \in H \), let \( \hat{x}_o \), \( \hat{x}_i \) and \( \hat{x}_e \), respectively, denote the equivalence classes of \( x \) with respect to the relations \( \sim_o \), \( \sim_i \) and \( \sim_e \).

We say that a hypergroup \( \langle H, \circ \rangle \) is reduced if and only if, for any \( x \in H \), \( \hat{x}_e = \{x\} \).

**Proposition 1.8 (Jantosciak [10, Proposition 3]).** For any hypergroup \( \langle H, \circ \rangle \), the quotient hypergroup \( \langle H/\sim_e, \star \rangle \) is a reduced hypergroup, where the hyperoperation \( \star \) on \( H/\sim_e \) is defined by

\[
\hat{x}_e \star \hat{y}_e = \{ z \mid z \in x \circ y \}. 
\]

The quotient hypergroup \( \langle H/\sim_e, \star \rangle \) is called the reduced form of the hypergroup \( \langle H, \circ \rangle \).

It is known that the study of hypergroups falls into two parts: the study of reduced hypergroups and the study of all hypergroups having the same reduced form.

Our goal is to determine necessary and sufficient conditions such that the hypergroup \( \mathbb{H}_\rho \), associated with a binary relation \( \rho \), is reduced. Moreover, given two binary relations \( \rho \) and \( \sigma \) defined on \( H \), we investigate when the hypergroups \( \mathbb{H}_{\rho \cap \sigma}, \mathbb{H}_{\rho \cup \sigma}, \mathbb{H}_{\rho \sigma} \) are reduced. In the last part of the paper we talk about the cartesian product of the reduced hypergroups.

2. **Basic properties**

Let \( \rho \) be a binary relation defined on a non-empty set \( H \).

For any \( x \in H \), we denote \( L_x^\rho = \{ z \in H \mid (x, z) \in \rho \} \) and \( R_x^\rho = \{ z \in H \mid (z, x) \in \rho \} \).

If it is clear what is the relation we talk about, then we use the notations \( L_x \) and \( R_x \) instead of \( L_x^\rho \) and \( R_x^\rho \).

If \( \rho \) is a relation such that the associated hypergroupoid \( \mathbb{H}_\rho \) is a hypergroup, then, for any \( x \in H \), we have \( L_x = \emptyset \) and \( R_x = \emptyset \).

It is easy to see that

1. \( \rho \) is reflexive if and only if, for any \( x \in H \), \( x \in L_x \);
2. \( \rho \) is symmetric if and only if, for any \( x \in H \), \( L_x = R_x \);
3. \( \rho \) is transitive if and only if, for any \( x, y \in H \) with \( L_x \cap R_y \neq \emptyset \) it results \( y \in L_x \).

Let \( \rho \) and \( \sigma \) be two distinct binary relations defined on \( H \). One verifies that:

(i) \( L_x^{\rho \cap \sigma} = \{ z \in H \mid (x, z) \in \rho \cap \sigma \} = L_x^\rho \cap L_x^\sigma \),
\( R_x^{\rho \cap \sigma} = \{ z \in H \mid (z, x) \in \rho \cap \sigma \} = R_x^\rho \cap R_x^\sigma \).

(ii) \( L_x^{\rho \cup \sigma} = \{ z \in H \mid (x, z) \in \rho \cup \sigma \} = L_x^\rho \cup L_x^\sigma \),
\( R_x^{\rho \cup \sigma} = \{ z \in H \mid (z, x) \in \rho \cup \sigma \} = R_x^\rho \cup R_x^\sigma \).

(iii) \( L_x^{\rho \sigma} = \{ z \in H \mid (x, z) \in \rho \sigma \} = \{ z \in H \mid \exists t \in H : (x, t) \in \rho, (t, z) \in \sigma \}
= \{ z \in L_t^\rho \mid t \in L_x^\sigma \},
R_x^{\rho \sigma} = \{ z \in H \mid (z, x) \in \rho \sigma \} = \{ z \in H \mid \exists t \in H : (z, t) \in \rho, (t, x) \in \sigma \}
= \{ z \in R_t^\rho \mid t \in R_x^\sigma \}.

(iv) If, for any \( x \in H \), \( L_x^\rho = L_x^\sigma \), then \( \rho = \sigma \).
Proposition 2.1. Let $H_{\rho}$ be the hypergroup associated with the binary relation $\rho$ defined on $H$. For any $x, y \in H$, the following implications hold:

1. $x \sim_{o} y \iff L_x = L_y$,
2. $x \sim_{i} y \iff R_x = R_y$.

Proof. (1) By the definition of the relation “$\sim_{o}$”, we have that $x \sim_{o} y$ is equivalent to $x \circ a = y \circ a$, for any $a \in H$, which means $L_x \cup L_a = L_y \cup L_a$. If $L_x = L_y$, it is clear that $x \sim_{o} y$.

Now we suppose $x \sim_{o} y$, hence, for any $a \in H$, $L_x \cup L_a = L_y \cup L_a$.

- For $a = x$ it results $L_x = L_x \cup L_y$, so $L_y \subseteq L_x$.
- For $a = y$ it results $L_x \cup L_y = L_y$, so $L_x \subseteq L_y$.

We conclude that $L_x = L_y$.

(2) Take $x, y \in H, x \sim_{i} y$. This means that $x \in a \circ b \iff y \in a \circ b$, for $a, b \in H$, that is $x \in L_a \cup L_b \iff y \in L_a \cup L_b$. But, for any $x \in H$, $R_x \neq \emptyset$, therefore there exists $a \in H$ such that $a \in R_x$, that is $x \in L_a$; it follows $x \in L_a = a \circ a$ and since $x \sim_{i} y$, we obtain $y \in L_a$, that is $a \in R_y$. Similarly we obtain $R_y \subseteq R_x$ and then $R_x = R_y$. □

Now, if $R_x = R_y$ we have $x \in L_z \iff y \in L_z$, for $z \in H$, therefore $x \in z \circ t \iff y \in z \circ t$, for $z, t \in H$, which means $x \sim_{i} y$.

In the following, we investigate when two different elements $x, y \in H$ are in the relation $x \sim_{e} y$ in the hypergroups $H_{\rho \cap \sigma}$ and $H_{\rho \sigma}$.

Proposition 2.2. Let $\rho$ and $\sigma$ be two quasiorder relations on a non-empty set $H$. For any $x, y \in H, x \sim_{e} y$ in $H_{\rho \cap \sigma}$ if and only if $x \sim_{e} y$ in $H_{\rho}$ and $x \sim_{e} y$ in $H_{\sigma}$.

Proof. Since $\rho$ and $\sigma$ are two quasiorder relations, the hypergroupoids associated with $\rho, \sigma$ and $\rho \cap \sigma$ are hypergroups.

First, we suppose $x \sim_{e} y$ in $H_{\rho}$ and $x \sim_{e} y$ in $H_{\sigma}$; by the previous proposition we have $L_x^{\rho} = L_y^{\rho}, R_x^{\rho} = R_y^{\rho}, L_x^{\sigma} = L_y^{\sigma}$ and $R_x^{\sigma} = R_y^{\sigma}$, so $L_x^{\rho \cap \sigma} = L_y^{\rho \cap \sigma}$ and $R_x^{\rho \cap \sigma} = R_y^{\rho \cap \sigma}$, that is $x \sim_{e} y$ in $H_{\rho \cap \sigma}$.

Conversely, suppose $x \sim_{e} y$ in $H_{\rho \cap \sigma}$, that is $x \sim_{o} y$ and $x \sim_{i} y$ in $H_{\rho \cap \sigma}$. It is enough to show the implications:

1. $L_x^{\rho} \cap L_y^{\sigma} = L_y^{\rho} \cap L_y^{\sigma} \implies L_x^{\rho} = L_y^{\rho} \text{ and } L_x^{\sigma} = L_y^{\sigma}$;
2. $R_x^{\rho} \cap R_y^{\sigma} = R_y^{\rho} \cap R_y^{\sigma} \implies R_x^{\rho} = R_y^{\rho} \text{ and } R_x^{\sigma} = R_y^{\sigma}$.

We will prove the first one, the second one has a similar proof.

Since $\rho$ and $\sigma$ are reflexive relations, we write $x \in L_x^{\rho} \cap L_x^{\sigma}$, so $x \in L_x^{\rho} \cap L_y^{\sigma}$, that is $(y, x) \in \rho \cap \sigma$ and similarly, $(x, y) \in \rho \cap \sigma$.

Let us consider $z \in L_x^{\rho}$, that is $(x, z) \in \rho$ and since $(y, x) \in \rho$, by the transitivity of $\rho$, it results $(y, z) \in \rho, z \in L_y^{\rho}$. We have $L_x^{\rho} \subseteq L_y^{\rho}$ and similarly $L_y^{\rho} \subseteq L_x^{\rho}$. We obtain $L_x^{\rho} = L_y^{\rho}$ and, in the same way, $L_x^{\sigma} = L_y^{\sigma}$. □

Proposition 2.3. Let $\rho$ and $\sigma$ be two binary relations on $H$ with full domain and full range such that $\rho^2 = \rho, \sigma^2 = \sigma$ and $\rho \sigma = \sigma \rho$. If, for $x, y \in H$, $x \sim_{o} y$ in $H_{\rho}$ and $x \sim_{i} y$ in $H_{\sigma}$, then $x \sim_{e} y$ in $H_{\rho \sigma}$.

Moreover, $x \sim_{e} y$ in $H_{\rho}$ and $x \sim_{e} y$ in $H_{\sigma}$ lead to $x \sim_{e} y$ in $H_{\rho \sigma}$.

Proof. In this hypothesis, the hypergroupoids $H_{\rho}, H_{\sigma}$ and $H_{\rho \sigma}$ are hypergroups.

Let us consider $x, y \in H$ such that $x \sim_{o} y$ in $H_{\rho}$ and $x \sim_{i} y$ in $H_{\sigma}$, so we have $L_x^{\rho} = L_y^{\rho}$ and $R_x^{\sigma} = R_y^{\sigma}$. It is enough to prove the implications:

1. $L_x^{\rho} = L_y^{\rho} \implies L_x^{\rho \sigma} = L_y^{\rho \sigma}$;
2. $R_x^{\sigma} = R_y^{\sigma} \implies R_x^{\rho \sigma} = R_y^{\rho \sigma}$.
Let \( z \in L_x^{\rho \sigma} \); there exists \( t \in L_x^\rho \) such that \( z \in L_t^\rho \), so there exists \( t \in L_y^\rho \) such that \( z \in L_t^\rho \); therefore \( z \in L_y^{\rho \sigma} \). Similarly \( L_y^{\rho \sigma} \subseteq L_y^\rho \).

In the same way we can show the second implication.

Thus, if \( x \sim_{\rho, \sigma} y \) in \( \mathbb{H}_\sigma \) and \( x \sim_{1, \rho} y \) in \( \mathbb{H}_\rho \), it results \( x \sim_{\rho \sigma} y \) in \( \mathbb{H}_{\rho \sigma} \) and since \( \rho \sigma = \sigma \rho \) we obtain the last assertion of the proposition. \( \square \)

### 3. Reduced hypergroups associated with binary relations

In this section, first, we determine a necessary and sufficient condition for the hypergroup \( \mathbb{H}_\rho \) in order to be reduced; then we analyze this condition for different types of relations. Secondly, we prove that the hypergroupoid \( \mathcal{H}_\rho \) associated with a binary relation defined by Corsini [4] is not a reduced hypergroup.

#### Theorem 3.1

The hypergroup \( \mathbb{H}_\rho \) is reduced if and only if, for any \( x, y \in H \), \( x \) different from \( y \), either \( L_x \neq L_y \) or \( R_x \neq R_y \).

**Proof.** By the definition, the hypergroup \( \mathbb{H}_\rho \) is reduced if and only if, for any \( x \neq y \), it is true \( x \gamma_{\rho, \sigma} y \) or \( x \gamma_{1, \rho} y \) and by the Proposition 2.1 this is equivalent with \( L_x \neq L_y \) or \( R_x \neq R_y \). \( \square \)

For some particular relations, the condition expressed in the previous theorem is simpler, as we see in the following results.

#### Proposition 3.2

If \( \rho \) is an equivalence on \( H \), then the hypergroupoid \( \mathbb{H}_\rho \) is a reduced hypergroup if and only if \( \rho = \Delta_H = \{(x, x) \mid x \in H\} \).

**Proof.** If \( \rho \) is an equivalence on \( H \), then \( \langle \mathbb{H}_\rho, \circ \rangle \) is a hypergroup.

Since \( \rho \) is symmetric, we have, for any \( x \in H \), \( L_x = R_x \) and then, \( \mathbb{H}_\rho \) is reduced if and only if, for any \( x \neq y \), \( L_x \neq L_y \). We show that this condition is equivalent with the following one: for any \( x \in H \), \( L_x = \{x\} \) and then, it is clear \( \rho = \Delta_H \).

If, for any \( x \in H \), \( L_x = \{x\} \), it results for all \( x \neq y \) that \( L_x \neq L_y \).

Conversely, let \( y \neq x \), \( y \in L_x \); we obtain \( \{x, y\} \subseteq L_y \). For any \( z \in L_y \setminus \{x, y\} \) we have \( (y, z) \in \rho \), \( (x, y) \in \rho \) and by transitivity it follows \( (x, z) \in \rho \), so \( z \in L_x \). Similarly, it results \( L_x \subseteq L_y \), thus \( L_x = L_y \) which is in contradiction with the hypothesis. \( \square \)

#### Proposition 3.3

If \( \rho \) is a non-symmetric quasiorder on \( H \), then the hypergroup \( \langle \mathbb{H}_\rho, \circ \rangle \) is reduced if and only if, for any \( x \neq y \), \( L_x \neq L_y \).

**Proof.** If \( \rho \) is a quasiorder on \( H \) then, for any \( x \neq y \in H \), we have the implication \( x \sim_{\rho, \sigma} y \Rightarrow x \sim_{1, \rho} y \).

Indeed, if we suppose \( L_x = L_y \) and \( R_x \neq R_y \), there exists \( z \in R_x, z \notin R_y \); then \( (z, x) \in \rho \) and \( (z, y) \notin \rho \). But \( \rho \) is reflexive and then \( y \in L_y = L_x \); thus \( (x, y) \in \rho \) and by transitivity we obtain \( (z, y) \in \rho \), which is false.

So, for any \( x \neq y \), the condition \( "L_x \neq L_y \) or \( R_x \neq R_y" \) is equivalent with \( "L_x \neq L_y". \( \square \)

#### Proposition 3.4

If \( \rho \) is a reflexive symmetric non-transitive relation on \( H \), such that \( \rho^2 = H \times H \), then the hypergroup \( \langle \mathbb{H}_\rho, \circ \rangle \) is reduced if and only if \( L_x \neq L_y \), for all \( x, y \in H \), \( x \) different from \( y \).

**Proof.** As in the previous proposition it is enough to prove that, for any \( x \neq y \), \( x \sim_{\rho, \sigma} y \Rightarrow x \sim_{1, \rho} y \).

If we suppose there exists \( a \in H \) such that \( x \in L_a \) and \( y \notin L_a \), then, by the symmetry, we have \( a \in L_x = L_y \) and thus \( a \in L_y \), so \( y \in L_a \), contradiction.

Given a binary relation \( \rho \) on \( H \), Corsini [4] has defined another hyperoperation: for any \( x, y \in H \),

\[
x \otimes_{\rho} y = L_x \cap R_y,
\]

and he has proved that \( \mathcal{H}_\rho = \langle H, \otimes_{\rho} \rangle \) is a hypergroupoid if and only if \( \rho^2 = H \times H \).
4. The hypergroups $\mathbb{H}_{\rho \cap \sigma}$, $\mathbb{H}_{\rho \cup \sigma}$, $\mathbb{H}_{\rho \sigma}$ as reduced hypergroups

Let $\rho$ and $\sigma$ be two binary relations defined on a non-empty set $H$. The hypergroups $\mathbb{H}_{\rho \cap \sigma}$, $\mathbb{H}_{\rho \sigma}$ and $\mathbb{H}_{\rho \cup \sigma}$ are reduced independently if $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ are or are not reduced hypergroups, as we will see in the following results.

**Proposition 4.1.** Let $\rho$ and $\sigma$ be two quasiorder relations on $H$. If the hypergroups $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ are reduced, then the hypergroup $\mathbb{H}_{\rho \cap \sigma}$ is reduced too.

**Proof.** If we suppose that the hypergroup $\mathbb{H}_{\rho \cap \sigma}$ is not reduced, then it results there exist $x \neq y$ in $H$ such that $x \sim_{\rho \cap \sigma} y$ in $\mathbb{H}_{\rho \cap \sigma}$ and therefore $x \sim_{\rho} y$ in $\mathbb{H}_{\rho}$, $x \sim_{\sigma} y$ in $\mathbb{H}_{\sigma}$, which is impossible because the hypergroups $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ are reduced. □

**Remark.** If the hypergroup $\mathbb{H}_{\rho \cap \sigma}$ is reduced, then the hypergroups $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ can be reduced or not, as one sees from the following examples.

**Example 4.2.** Let $H = \{1, 2, 3, 4\}$.

(i) If $\rho \cap \sigma = \Delta_H = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and $\rho$, $\sigma$ are equivalences on $H$ different from the diagonal relation $\Delta_H$, then the hypergroup $\mathbb{H}_{\rho \cap \sigma}$ is reduced, but neither $\mathbb{H}_{\rho}$ nor $\mathbb{H}_{\sigma}$ is a reduced hypergroup (see Proposition 3.2).

(ii) Set $\rho = \Delta_H \cup \{(1, 2)\}$ and $\sigma = \Delta_H \cup \{(1, 3)\}$. Then $\rho \cap \sigma = \Delta_H$, so $\mathbb{H}_{\rho \cap \sigma}$ is a reduced hypergroup and also $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$.

(iii) Set $\rho = \Delta_H \cup \{(1, 2), (2, 1), (1, 3), (2, 3)\}$, $\sigma = \Delta_H \cup \{(1, 2), (3, 4)\}$, $\rho \cap \sigma = \Delta_H \cup \{(1, 2)\}$. It results the hypergroups $\mathbb{H}_{\rho \cap \sigma}$ and $\mathbb{H}_{\sigma}$ are reduced, but the hypergroup $\mathbb{H}_{\rho}$ is not $(L^0_1 = L^0_2$, $R^0_1 = R^0_2)$.

**Proposition 4.3.** Let $\rho$ and $\sigma$ be two binary relations on $H$ with full domain and full range such that $\rho^2 = \rho$, $\sigma^2 = \sigma$ and $\rho \sigma = \sigma \rho$. If the hypergroup $\mathbb{H}_{\rho \sigma}$ is reduced, then both hypergroups $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ are reduced.

**Proof.** From the Proposition 2.3 we have the implications:

1. $L^\rho_x \neq L^\rho_y \Rightarrow L^\rho_x \neq L^\rho_y$ and $L^\sigma_x \neq L^\sigma_y$.
2. $R^\rho_x \neq R^\rho_y \Rightarrow R^\rho_x \neq R^\rho_y$ and $R^\sigma_x \neq R^\sigma_y$.

If $\mathbb{H}_{\rho \sigma}$ is a reduced hypergroup then, for any $x \neq y$, we have $x \not\sim_{\rho \sigma} y$, so, for any $x \neq y$, $L^\rho_x \neq L^\rho_y$ or $R^\rho_x \neq R^\rho_y$. It follows $(L^\rho_x \neq L^\rho_y$ and $L^\sigma_x \neq L^\sigma_y$) or $(R^\rho_x \neq R^\rho_y$ and $R^\sigma_x \neq R^\sigma_y$) and therefore the hypergroups $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ are reduced. □

**Remark.** In the same hypothesis as in the Proposition 4.3, if $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ are reduced hypergroups, then the hypergroup $\mathbb{H}_{\rho \sigma}$ is reduced or not, as the following examples show.

**Example 4.4.** We consider the following two situations.

1. Set $H = \{1, 2, 3, 4\}$, $\rho = \Delta_H \cup \{(1, 2)\} = \rho^2$ and $\sigma = \Delta_H \cup \{(1, 3)\} = \sigma^2$. Clearly, $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ are reduced hypergroups (see the Proposition 3.3); since $\rho \sigma = \Delta_H \cup \{(1, 2), (1, 3)\} = \rho \sigma$, it results that the hypergroup $\mathbb{H}_{\rho \sigma}$ is reduced.

2. Set $H = \{1, 2, 3\}$, $\rho = \Delta_H \cup \{(2, 1), (2, 3)\} = \rho^2$ and $\sigma = \Delta_H \cup \{(1, 3), (1, 2)\} = \sigma^2$. Again it results that $\mathbb{H}_{\rho}$ and $\mathbb{H}_{\sigma}$ are reduced hypergroups; we obtain $\rho \sigma = \Delta_H \cup \{(1, 2), (1, 3), (2, 1), (2, 3)\} = \rho \sigma$, and then $L^\rho_1 = H = L^\rho_2$, $R^\rho_1 = \{1, 2\} = R^\rho_2$, therefore the hypergroup $\mathbb{H}_{\rho \sigma}$ is not reduced.
Remark. Let $\rho$ and $\sigma$ be two binary relations defined on $H$ such that the hypergroupoids $H_\rho$, $H_\sigma$ and $H_{\rho \cup \sigma}$ are hypergroups. If $H_\rho$ and $H_\sigma$ are reduced hypergroups, then the hypergroup $H_{\rho \cup \sigma}$ can be reduced or not and conversely, if $H_{\rho \cup \sigma}$ is a reduced hypergroup, it does not result that the hypergroups $H_\rho$ and $H_\sigma$ are reduced, too, as it follows from the following examples.

Example 4.5. We present the following situations.

1. Set $H = \{1, 2, 3\}$, $\rho = A_H \cup \{(1, 2)\} = \rho^2$ and $\sigma = A_H \cup \{(2, 1)\} = \sigma^2$; we find $\rho \cup \sigma = A_H \cup \{(1, 2), (2, 1)\} = (\rho \cup \sigma)^2$. It is clear that $H_\rho$ and $H_\sigma$ are reduced hypergroups, but the hypergroup $H_{\rho \cup \sigma}$ is not reduced, since $L_{\rho \cup \sigma}^1 = \{1, 2\} \neq L_\rho^1 \cap L_\sigma^1$ (see the Proposition 3.3).

2. Set $H = \{1, 2, 3\}$, $\rho = A_H \cup \{(1, 2)\} = \rho^2$ and $\sigma = A_H \cup \{(1, 3)\} = \sigma^2$; we obtain $\rho \cup \sigma = A_H \cup \{(1, 2), (2, 1), (1, 3)\} = (\rho \cup \sigma)^2$. It follows that all the three hypergroups $H_\rho$, $H_\sigma$ and $H_{\rho \cup \sigma}$ are reduced.

3. Set again $H = \{1, 2, 3\}$ and the relations $\rho = A_H \cup \{(1, 2), (2, 1)\}$ and $\sigma = A_H \cup \{(1, 3)\} = \sigma^2$, therefore $\rho \cup \sigma = A_H \cup \{(1, 2), (2, 1), (1, 3)\}$, which is different from $(\rho \cup \sigma)^2 = A_H \cup \{(1, 2), (2, 1), (1, 3), (2, 3)\}$. In this case the hypergroup $H_\rho$ is not reduced, the hypergroup $H_\sigma$ is reduced and the hypergroup $H_{\rho \cup \sigma}$ is reduced, too. The hypergroupoid $H_{\rho \cup \sigma}$ is a hypergroup because $\rho \cup \sigma \subseteq (\rho \cup \sigma)^2$ and for the outer elements $1$ and $2$ of $\rho \cup \sigma$, condition (iv) of the Theorem 1.3 holds.

5. The cartesian product of the reduced hypergroups

Let $\langle H_1, o_1 \rangle$, $\langle H_2, o_2 \rangle$ be two hypergroups. On the cartesian product $H_1 \times H_2$ we define the hyperoperation

$$(x_1, x_2) \otimes (y_1, y_2) = (x_1 \circ_1 y_1, x_2 \circ_2 y_2)$$

and we obtain the hypergroup $\langle H_1 \times H_2, \otimes \rangle$ [2].

Proposition 5.1. In the hypergroup $\langle H_1 \times H_2, \otimes \rangle$, the following implications hold:

(i) $(x_1, x_2) \sim_o (y_1, y_2) \iff x_1 \sim_0 y_1$ in $H_1$ and $x_2 \sim_0 y_2$ in $H_2$;

(ii) $(x_1, x_2) \sim_i (y_1, y_2) \iff x_1 \sim_i y_1$ in $H_1$ and $x_2 \sim_i y_2$ in $H_2$.

Proof. (i) By the definition of the relation $\sim_o$ we have $(x_1, x_2) \sim_o (y_1, y_2)$ if and only if, for any $(a_1, a_2) \in H_1 \times H_2$, it is true: $(x_1, x_2) \otimes (a_1, a_2) = (y_1, y_2) \otimes (a_1, a_2)$ and $(a_1, a_2) \otimes (x_1, x_2) = (a_1, a_2) \otimes (y_1, y_2)$, which is equivalent with $x_1 \circ_1 a_1 = y_1 \circ_1 a_1$, $x_2 \circ_2 a_2 = y_2 \circ_2 a_2$ and $a_1 \circ_1 x_1 = a_1 \circ_1 y_1$, $a_2 \circ_2 x_2 = a_2 \circ_2 y_2$, that is, $x_1 \sim_o y_1$ and $x_2 \sim_o y_2$.

(ii) By the definition of the relation $\sim_i$ we get $(x_1, x_2) \sim_i (y_1, y_2)$ if and only if, for $(a_1, a_2), (b_1, b_2) \in H_1 \times H_2$, we have $(x_1, x_2) \in (a_1, a_2) \otimes (b_1, b_2)$ equivalently $(y_1, y_2) \in (a_1, a_2) \otimes (b_1, b_2)$, therefore $x_1 \in a_1 \circ_1 b_1$ and $x_2 \in a_2 \circ_2 b_2$ if and only if $y_1 \in a_1 \circ_1 b_1$ and $y_2 \in a_2 \circ_2 b_2$, that is, $x_1 \sim_i y_1$ and $x_2 \sim_i y_2$. □

Theorem 5.2. The hypergroup $\langle H_1 \times H_2, \otimes \rangle$ is reduced if and only if the hypergroups $\langle H_1, o_1 \rangle$ and $\langle H_2, o_2 \rangle$ are reduced.

Proof. First, we suppose that $\langle H_1 \times H_2, \otimes \rangle$ is a reduced hypergroup and that $H_1$ is not reduced. Then there exists $x_1 \neq y_1$ in $H_1$ such that $x_1 \sim_e y_1$, that is, $x_1 \sim_o y_1$ and $x_1 \sim_i y_1$. It follows that, for any $x_2 \in H_2$, we have $(x_1, x_2) \sim_o (y_1, x_2)$ and $(x_1, x_2) \sim_i (y_1, x_2)$ (by the previous proposition), that is $(x_1, x_2) \sim_e (y_1, y_2)$ with $(x_1, x_2) \neq (y_1, y_2)$; this means that $\langle H_1 \times H_2, \otimes \rangle$ is not reduced, which is in contradiction with the hypothesis.

Conversely, we suppose that $\langle H_1, o_1 \rangle$ and $\langle H_2, o_2 \rangle$ are reduced hypergroups, but $\langle H_1 \times H_2, \otimes \rangle$ is not. Then there exist $(x_1, x_2) \neq (y_1, y_2) \in H_1 \times H_2$ such that $(x_1, x_2) \sim_e (y_1, y_2)$. By the previous proposition we find $x_1 \sim_e y_1$ and $x_2 \sim_e y_2$. Since $\langle H_1, o_1 \rangle$ and $\langle H_2, o_2 \rangle$ are reduced, it follows that $x_1 = y_1$, $x_2 = y_2$, thus $(x_1, x_2) = (y_1, y_2)$ which is false. □
Proposition 5.3. Let \( \rho_1, \rho_2 \) be two binary relations defined on the non-empty sets \( H_1, H_2 \) such that the associated hypergroupoids \((H_1)_{\rho_1} \) and \((H_2)_{\rho_2} \) are hypergroups.

(i) If \((H_1)_{\rho_1} \) and \((H_2)_{\rho_2} \) are reduced hypergroups and, for \( j \in \{1, 2\} \), the implication \( \rho_j^2 \neq H_j^2 \implies \rho_{3-j}^2 = \rho_{3-j}^2 \) (that is \((H_1 \times H_2)_{\rho_1 \times \rho_2} \) is a hypergroup) \((4)\) holds, then \((H_1 \times H_2)_{\rho_1 \times \rho_2} \) is a reduced hypergroup.

(ii) If \((H_1 \times H_2)_{\rho_1 \times \rho_2} \) is a reduced hypergroup, then at least one of the hypergroups \((H_1)_{\rho_1} \) and \((H_2)_{\rho_2} \) is reduced.

Proof. (i) If we suppose that \((H_1 \times H_2)_{\rho_1 \times \rho_2} \) is not reduced, then there exist \((x_1, x_2) \neq (y_1, y_2) \in H_1 \times H_2 \) such that \( L(x_1, x_2) = L(y_1, y_2) \) and \( R(x_1, x_2) = R(y_1, y_2) \), that is \( L_{x_1} = L_{y_1}, L_{x_2} = L_{y_2}, R_{x_1} = R_{y_1}, R_{x_2} = R_{y_2} \). This implies that \( x_1 \sim y_1 \) in \((H_1)_{\rho_1} \) and \( x_2 \sim y_2 \) in \((H_2)_{\rho_2} \), but since \((H_1)_{\rho_1} \) and \((H_2)_{\rho_2} \) are reduced, it follows \( x_1 = y_1 \) and \( x_2 = y_2 \), therefore \((x_1, x_2) = (y_1, y_2) \), which is false.

(ii) Now, if \((H_1 \times H_2)_{\rho_1 \times \rho_2} \) is a reduced hypergroup and if we suppose that both hypergroups \((H_1)_{\rho_1} \) and \((H_2)_{\rho_2} \) are not reduced, it follows there exist \( x_1 \neq y_1 \in H_1 \) and \( x_2 \neq y_2 \in H_2 \) such that \( x_1 \sim y_1 \) in \((H_1)_{\rho_1} \) and \( x_2 \sim y_2 \) in \((H_2)_{\rho_2} \); we obtain \( L_{x_1} = L_{y_1}, R_{x_1} = R_{y_1} \) and \( L_{x_2} = L_{y_2}, R_{x_2} = R_{y_2} \), which lead to the relations \( L_{(x_1, x_2)} = L_{(y_1, y_2)} \) and \( R_{(x_1, x_2)} = R_{(y_1, y_2)} \). This is in contradiction with the hypothesis that \((H_1 \times H_2)_{\rho_1 \times \rho_2} \) is reduced. \( \square \)

6. Conclusions

The hypergroup associated with a binary relation \( \rho \) in the sense of Rosenberg is reduced if and only if, for any \( x, y \in H \), either \( L_x \neq L_y \) or \( R_x \neq R_y \). The unique equivalence relation \( \rho \) defined on \( H \) such that the hypergroup \( H_\rho \) is reduced is the diagonal relation \( \Delta_H \). Given two binary relations \( \rho \) and \( \sigma \) on \( H \), the property of being reduced of the associated hypergroups \( H_\rho \) and \( H_\sigma \) may or may not influence the same property of the hypergroups \( H_{\rho \cap \sigma} \) and \( H_{\rho \cup \sigma} \), respectively. Finally, we proved that the cartesian product of reduced hypergroups is a reduced hypergroup.

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