Uniform stabilization of a shallow shell model with nonlinear boundary feedbacks

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Abstract

We consider a dynamic linear shallow shell model, subject to nonlinear dissipation active on a portion of its boundary in physical boundary conditions. Our main result is a uniform stabilization theorem which states a uniform decay rate of the resulting solutions. Mathematically, the motion of a shell is described by a system of two coupled partial differential equations, both of hyperbolic type: (i) an elastic wave in the 2-d in-plane displacement, and (ii) a Kirchhoff plate in the scalar normal displacement. These PDEs are defined on a 2-d Riemann manifold. Solution of the uniform stabilization problem for the shell model combines a Riemann geometric approach with microlocal analysis techniques. The former provides an intrinsic, coordinate-free model, as well as a preliminary observability-type inequality. The latter yield sharp trace estimates for the elastic wave—critical for the very solution of the stabilization problem—as well as sharp trace estimates for the Kirchhoff plate—which permit the elimination of geometrical conditions on the controlled portion of the boundary. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction and statement of main results

1.1. Boundary stabilization. Dynamic shallow shell

The goal of this paper is to provide a uniform stabilization result for a dynamic shallow shell model with suitable, natural, nonlinear dissipative boundary feedback terms in the form of moments and shears applied to an edge of the shell. More explicitly, what this means is the following. First, with homogeneous boundary conditions, the (linear) shell model is conservative (energy preserving). Next, we impose suitable nonlinear dissipative terms (tractions/shears/moments) in physical boundary conditions exercised only on a portion $\Gamma_1$ of the boundary $\Gamma$ of the shell and then seek to force the energy of the new corresponding closed loop, well-posed (Theorem 1.1) dissipative problem to decay to zero at a certain rate. The rate depends explicitly on pre-assigned growth properties of the dissipative terms. This is the content of our main Theorem 1.2.

Boundary stabilization of conservative PDE problems has received considerable attention and there is now a vast literature on this subject. However, most of the existing results suffer from two limitations: (i) they refer to single, scalar equations (wave, plates, Schrödinger equations, etc.), and, above all, (ii) the coefficients of the principal part operators are assumed constant. As has been widely recognized, removal of either one of the above restrictions introduces serious technical challenges. To exemplify: (i) coupling of two PDEs introduces a new array of difficulties even at the less demanding level of unique continuation, let alone at the seriously more demanding level of observability/stabilization a priori inequalities. Moreover, (ii) in the case of scalar second-order hyperbolic equations with variable coefficients, microlocal analysis and geometric optics method have been employed [1] to obtain sharp a priori observability/stabilization inequalities.

The shell model of the present paper encompasses both of these new features: the coefficients of the principal part operators are variable coefficients (due to the curved nature of the shell), and the model couples two hyperbolic-like equations: a 2-d system of elasticity and a plate-like Kirchhoff equation. In addition, uniform stabilization for (a system of elasticity, hence for) a shell requires the preliminary solution, via microlocal analysis, of the problem regarding sharp trace regularity of its solutions. More on this below. For all these reasons, stabilization of a shell has been a sought-after objective and open problem for some time now, given the curved nature of a shell as a geometric object, which translates, analytically, into a highly complicated mathematical model. As already noted, this consists of two coupled variable coefficient partial differential equations (PDEs), both of hyperbolic type, of which we will have to say more below. Classically, the topic of static shells is covered by many books. They all assume the middle surface of a shell to be described by one coordinate patch (the image in $\mathbb{R}^3$ of a smooth function defined on a connected domain of $\mathbb{R}^2$). In addition to the resulting
geometrical limitations of this approach (which forces the exclusion of, say, a sphere), the classical models use traditional geometry and end up with highly complicated analytical models. In these, the explicit presence of the Christoffel symbols $\Gamma^k_{ij}$ make them unsuitable for energy method computations of the type needed for continuous observability/stabilization estimates in cases such as the one of a shell, where the coefficients of the principal operator and of the energy level terms are variable in space. A recent advance in this area is an intrinsic model of a shallow shell as a 2-d Riemann manifold, within the intrinsic, coordinate free setting of differential geometry, as proposed in [2,3]. This approach allows for the use of a computational method, initiated by Bochner, for overcoming the complexity of the computations in proving identities of geometric/analytic interest. Accordingly, throughout this paper, the shallow shell is viewed as a 2-dimensional Riemann manifold in $\mathbb{R}^3$ as in [2,3], with an intrinsic model that features an array of differential geometric notions (for which we provide a concise, didactic appendix for a reader not accustomed to this machinery).

To solve the outstanding (nonlinear) boundary feedback stabilization problem of a dynamic shallow shell, we combine the differential geometric description of the shell—in particular, the continuous observability estimate in [3]—with a delicate PDE-microlocal analysis yielding sharp trace regularity of the solutions of elastic waves and of Kirchhoff plates, the two components of a shell. This way, we first of all solve the problem, and, in the process, we achieve two main benefits: (i) we dispense altogether with restrictive geometrical conditions on the controlled part of the boundary of the shell, of the type used in wave and plate literature [4]; (ii) we avoid unnatural and mathematically undesirable terms in the boundary feedbacks of the elastic wave [5] even in the flat case, whose purpose was to cancel out boundary traces, which one could not control without sharp trace theory, at the price of injecting boundary terms which are not in $L^2$. More explicitly, the sharp, microlocal trace theory of the plate component ($w$ below) is not strictly critical for achieving some solution of the present uniform stabilization problem: in fact, one could get a solution at the price of assuming, instead, restrictive and unnecessary geometrical conditions on the controlled part of the boundary $\Gamma_1$ as in prior literature [4]. By contrast, the contribution of a sharp, microlocal trace theory of the elastic wave component is indispensable for the very solution of the present uniform stabilization problem.

A more detailed description of the contributions of this paper over the literature is given below.

In the flat case, the results of this paper reduce to (a subset of) the nonlinear boundary stabilization for the full von Karman model, as solved in [6]; see also [7]. Indeed, we shall use the same strategy as the one employed in the flat (nonlinear) full von Karman model [6] in the Euclidean case, with the additional technical difficulties stemming from the curved nature of the shell (hence variable coefficients of the principal parts of the two components). This overall strategy critically relies, as mentioned above, on sharp trace regularity
results of (scalar wave equations [8], hence of) elastic wave equations [9] as in the Lame’s system of elasticity, and finally of Kirchhoff’s plate equations [10]. However, our problem is not Lame-type and a generalization of [8,9] is required (Section 3.2). An additional new component needed in the present curved shell’s problem over [6] is an observability inequality from [3], as already mentioned above. Describing a dynamic shell as a 2-dimensional Riemann manifold requires a suitable mathematical apparatus, which we relegate to the Appendix. Instead, in the present introductory section, we wish to arrive at the main content of the paper in a shortest possible way through a minimum amount of differential geometric notation. Thus, the aim of the present section is twofold.

First, we introduce a nonlinear, boundary feedback closed loop dynamic model of a shallow shell, based on the open loop differential geometric model in [3], in the form of a mixed, coupled system of two linear hyperbolic partial differential equations on a 2-dimensional orientable manifold: one linear “elastic wave-type” equation in the in-plane, 2-dimensional displacement $W$; and one linear “Kirchhoff plate-like” equation in the scalar normal displacement $w$. Here, the displacement vector is $\zeta(x) = W(x) + w(x)N(x)$ at the point $x$ of the middle surface of the shell, where $N(x)$ is the unit normal field. This model is reminiscent of the full nonlinear von Karman model in the flat case: in the von Karman case, the two equations are coupled by nonlinear terms which are unbounded on the energy space; in the curved shell’s case, the linear versions of these same equations are coupled via the curvature instead. Thus, when specialized to the flat case, the two equations in $W$ and $w$ reduce precisely to the elastic system of elasticity in $W$ and to the Kirchhoff plate equation in $w$, respectively. The accompanying four boundary conditions associated with these two linear hyperbolic equations—two boundary conditions in $W$ and two boundary conditions in $w$ of physical significance (moments and shears)—are, instead, nonlinear, and of a special choice. They are selected here in a suitable dissipative feedback form, which involves tangential and normal components of the velocity components $W_t$ and $w_t$, in a natural way. Handling these with no geometrical conditions imposed on the controlled part of the boundary of a shell is a major challenge of the present paper, for which we employ sharp trace regularity results of elastic waves by extending the microlocal arguments of [8] (scalar waves), [9] (system of elasticity in Lame form) and Kirchhoff plates [10], to our present problem which is not Lame-type.

After stating a preliminary well-posedness result (Theorem 1.1) on the overall mixed coupled PDE system in the variables $[W, w]$ for the displacement, we next provide the main result of the present paper, Theorem 1.2. This is a uniform stabilization result which asserts that, under suitable and physically natural assumptions, the solutions of the $[W, w]$-mixed problem decay to zero at a uniform rate with no geometrical conditions imposed on the controlled part of the shell’s boundary.
1.2. Dynamic shallow shell’s model in nonlinear, dissipative, feedback form

We make reference to Fig. 1. Throughout this paper, we shall use the notation of the literature [3] when applicable. Accordingly, the middle surface of the shell is a bounded region $\Omega$, which lies on a smooth orientable two-dimensional surface $M$ of $\mathbb{R}^3$. The regular boundary (on $M$) of $\Omega$ is denoted by $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and consists of two disjoint portions: $\Gamma_0$ which will be the “uncontrolled” part of the boundary; and $\Gamma_1$ which will be the “controlled” part of the boundary; that is, the one where the dissipative feedback is active. We write below the coupled system of two hyperbolic partial differential equations in $[W, w]$ which represent the dynamic model of a shallow shell in feedback form, to be considered throughout this paper. It has homogeneous Dirichlet boundary conditions on $\Gamma_0$, and suitable nonlinear dissipative feedback terms on $\Gamma_1$ involving $W_t$ and $w_t$. It is given by

\begin{align}
W_{tt} - [\Delta_\mu W + (1 - \mu)kW + \mathcal{F}(w)] &= 0 \\
in (0, \infty) \times \Omega &\equiv Q_\infty, \quad (1.1a) \\
[I - \gamma \Delta]w_{tt} + \gamma[\Delta^2 w - (1 - \mu)\delta(kdw)] \\
+ (H^2 - 2(1 - \mu)k)w + \mathcal{G}(W) &= 0 \\
in Q_\infty, &\quad (1.1b) \\
W \equiv 0, \quad w \equiv 0, \quad \frac{\partial w}{\partial n} \equiv 0 \\
in (0, \infty) \times \Gamma_0 &\equiv \Sigma_{0, \infty}, \quad (1.1c) \\
B_1(W, w) = g_1(\langle W_t, n \rangle), \quad B_2(W, w) = g_2(\langle W_t, \tau \rangle) \\
in (0, \infty) \times \Gamma_1 &\equiv \Sigma_{1, \infty}, \quad (1.1d) \\
\Delta w + (1 - \mu)B_3(w) &= -h_1(\frac{\partial w}{\partial n}) \\
in \Sigma_{1, \infty}, &\quad (1.1e) \\
\frac{\partial \Delta w}{\partial n} + (1 - \mu)B_4w - \gamma \frac{\partial w_t}{\partial n} &= -\frac{\partial}{\partial \tau} h_2(\frac{\partial w_t}{\partial \tau}) \\
in \Sigma_{1, \infty}, &\quad (1.1f) \\
\zeta(0, \cdot) \equiv [W(0, \cdot), w(0, \cdot)] &\equiv \zeta_0 = [W_0, w_0], \\
\zeta_t(0, \cdot) \equiv [W_t(0, \cdot), w_t(0, \cdot)] &\equiv \zeta_1 = [W_1, w_1]. \quad (1.1g)
\end{align}

In the boundary homogeneous case, where the boundary functions are all zero: $g_1 \equiv g_2 \equiv h_1 \equiv h_2 \equiv 0$, the mixed problem (1.1) specializes to the one considered in [2,3]. The choice of the boundary functions is a distinctive feature of the present paper. It will be shown to be the correct choice for the purpose of forcing the solutions of (1.1) decay to zero with a uniform rate. In the flat case, the feedback problem (1.1) reduces to (a special case of) the fully nonlinear von Karman system considered in [6] (see also [7]) here, the coupling between the $W$- and the $w$-equation is via nonlinear unbounded terms in the energy space; instead, problem (1.1) in the flat case yields no coupling terms: $\mathcal{F}(w) = 0, \mathcal{G}(W) = 0$; see below (1.2).
1.3. Essential glossary of notation

The proper definition of all symbols entering the shell’s model (1.1) requires a suitable notational and conceptual apparatus, to be given in the Appendix below. Here we merely introduce them and identify them. First, the smooth surface $M$ containing the shell’s middle surface $\Omega$ is viewed as a two-dimensional Riemann manifold with metric induced from $\mathbb{R}^3$. This induced metric on $M$ is denoted by $\langle \cdot, \cdot \rangle$, the dot product on $\mathbb{R}^3$. Next, $\Delta_\mu$ in (1.1a) is a Hodge–Laplace type operator [(A.26)] applied to 1-forms (equivalently, vector fields) on $M$, defined by

$$-\Delta_\mu \equiv \frac{1-\mu}{2} \delta d + d\delta,$$

(1.2)

where $d$ is the exterior differential and $\delta$ its formal adjoint [(A.12), (A.20) and (A.21)]. The constant $\mu$, $0 < \mu < 1$ (physically $0 < \mu < 1/2$), is the Poisson’s coefficient of the material of the shell. Moreover, $k$ and $H$ are, respectively, the Gaussian curvature and the mean curvature [(A.31) below] of the shell’s middle surface $\Omega$. Furthermore, $\Delta$ in (1.1b) is the Laplacian on the manifold $M$ (Hodge–Laplace operator [(A.26)] on 0-forms, that is on functions); the coupling terms $\mathcal{F}(w)$ and $\mathcal{G}(W)$ are first-order differential operators on $w$ and $W$, respectively, whose structure is not essential in the present paper (see [2, p. 1733, (1.33)]).

However, in the flat case: $\Pi = 0$ ($\Pi$ is the second fundamental form [(A.30)] of $M$) and $H = k = 0$, these coupling terms vanish: $\mathcal{F}(w) = 0$, $\mathcal{G}(W) = 0$. Also, $\gamma = h/12$, where $h$ is the thickness of the shell [(A.29)]. Next, regarding the boundary terms, we preliminarily let $n$ and $\tau$ be the unit normal and unit tangential vectors along the boundary curve $\Gamma$ of the middle surface $\Omega$, with $n$ pointing toward the exterior of $\Gamma$ and $\tau$ oriented counterclockwise with respect to $n$. Thus, $\partial/\partial n$ and $\partial/\partial \tau$ are the corresponding normal derivative and tangential derivative

$$\frac{\partial}{\partial n} = \langle D, n \rangle, \quad \frac{\partial}{\partial \tau} = \langle D, \tau \rangle,$$

(1.3)
where $D$ denotes the Levi-Civita connection on $M$ [(A.4)] in the induced metric. Finally, the boundary operators $B_1$, $B_2$, $B_3$, $B_4$ in (1.1d)–(1.1f) are defined by

$$
B_1(W, w) = (1 - \mu)\Upsilon(W, w)(n, n) + \mu(wH - \delta W), \quad (1.4)
$$
$$
B_2(W, w) = (1 - \mu)\Upsilon(W, w)(n, \tau), \quad (1.5)
$$
$$
B_3 \equiv -D^2w(\tau, \tau), \quad (1.6)
$$
$$
B_4 \equiv \frac{\partial}{\partial \tau}[D^2w(\tau, n)] + k(x)\frac{\partial w}{\partial n} + \ell w, \quad \ell \geq 0. \quad (1.7)
$$

In (1.4)–(1.7), $\Upsilon(\zeta) = \Upsilon(W, w)$ is the linearized 2-covariant strain tensor defined by [2, (1.22)]

$$
\Upsilon(W, w) = \frac{1}{2}(DW + D^*W) + w\Pi \quad (1.8)
$$
in terms of the covariant differential $DW$ of $W$ [(A.4)] and its transpose $D^*W$ [(A.5)], as well as of the second fundamental form $\Pi$ [(A.30)] of the surface $M$. Moreover, $D^2w$ is the Hessian of $w$ [(A.6)] ($-D^2w$ denotes the change of curvature tensor of the middle surface $\Omega$). We note that the operator $B_4$ associated with the plate component of the shell is given in terms of normal and tangential coordinates, precisely as in [11, Chapter 3, Appendix D], which is a more convenient geometric and analytic representation of that arising in the variation model [4,12, 13]. In (1.7), $\ell$ is a nonnegative constant, whose role is seen in hypothesis (H.4) below (1.26).

### 1.4. Well-posedness of feedback problem (1.1)

The following well-posedness/regularity results are known [33] for the feedback problem (1.1).

**Theorem 1.1.** (a) [Generalized (weak) solutions] Assume that the nonlinear functions $h_i$, $g_i$ in (1.1d)–(1.1f) are possibly multivalued, monotone (in the sense of [14]), and that $0 \in h_i(0), 0 \in g_i(0)$. Then there exists a unique, global solution of finite energy of problem (1.1). This is to say that for any initial data (see Appendix, (4) and (8), for these spaces)

$$
W_0, W_1 \in H^1_0(\Omega, \Lambda) \times L^2(\Omega, \Lambda),
$$
$$
w_0, w_1 \in H^2_0(\Omega) \times H^1_0(\Omega), \quad (1.9)
$$

that is, subject to the boundary conditions (B.C.) $W_0 = w_0 = \partial w_0/\partial n = w_1 = 0$ on $\Gamma_0$, there exists a unique solution

$$
\{W, w\} \in C([0, T]; H^1_0(\Omega, \Lambda) \times H^2_0(\Omega)),
$$
$$
\{W_t, w_t\} \in C([0, T]; L^2(\Omega, \Lambda) \times H^1_0(\Omega)), \quad (1.10)
$$

where $T > 0$ is arbitrary. This solution is described by a nonlinear semigroup acting on the finite energy space. The form of the generator is given in (2.3) below.
(b) [Regular solutions] Assume that the boundary functions $h_i, g_i$ satisfy, in addition to the above hypotheses of part (a), the following more specific hypotheses: $h_i, g_i$ are single valued, and moreover $h_i, g_i \in C(\mathbb{R}), h_i', g_i' \in L_\infty(\mathbb{R})$. Then for any initial data (see Appendix for these spaces)

$$W_0, W_1 \in H^2(\Omega, \Lambda) \times H^1(\Omega, \Lambda),$$
$$w_0, w_1 \in H^3(\Omega) \times H^2(\Omega),$$

subject to the B.C. below (1.9), there exists a unique solution of (1.1):

$$[W, w] \in C([0, T]; H^2(\Omega, \Lambda) \times H^3(\Omega)), \quad (1.12)$$
$$[W_t, w_t] \in C([0, T]; H^1(\Omega, \Lambda) \times H^2(\Omega)). \quad (1.13)$$

Henceforth, the boundary functions $h_i, g_i$ are assumed to satisfy the condition of Theorem 1.1 unless otherwise stated.

Comments on Theorem 1.1. (a) The well-posedness of generalized solutions stated in part (a) of Theorem 1.1 follows from standard techniques applicable to second-order hyperbolic equation with boundary feedback: see [14] where abstract models of this type are considered and, more specifically, [6,15] where the well-posedness of the related nonlinear model is dealt with. This is done by applying the nonlinear semigroup generation theorem due to Crandall–Ligett (see also [14]).

(b) As to part (b) of Theorem 1.1—dealing with regular solutions—the proof proceeds by analyzing the domain of the nonlinear generator and showing that it contains $H^2(\Omega, \Lambda) \times H^3(\Omega)$ elements corresponding to the 2-d vector field (1-form) $W$ and scalar $w$. Then, a solution initiating in the domain of the generator remains there continuously in time. This is a standard approach in nonlinear monotone problems, and details are given in [15].

We remark explicitly that the case considered in Theorem 1.1 above is actually a particular case of a more general theory for monotone operators as considered in [6,15] (the geometry of the shell plays no role in the arguments) and fits particularly well into the general theory developed over the years. See also [16] where the well-posedness of a nonlinear shell is addressed directly.

1.5. Uniform stabilization

The main goal of the present paper is to show that the solutions of problem (1.1), asserted by Theorem 1.1, decay to zero at $t \to \infty$ at a uniform rate. Since the dissipative feedback terms are located on the portion $\Gamma_1$ of the boundary $\Gamma$ of the mid-surface $\Omega$ on the surface $M$, and the dissipation needs to be propagated from the boundary onto the interior of the shell, then we surmise that the geometry of the shell is bound to play a critical role in the stabilization arguments. Indeed, we shall require geometric assumptions (H.1) and (H.2) below.
Preliminaries. Let $\zeta = [W, w]$ be the displacement field of the middle surface $\Omega$ of the shell and denote $\hat{\zeta} = [\hat{W}, \hat{w}]$. Introduce the bilinear form \[ B(\zeta, \hat{\zeta}) = a(\Upsilon(\zeta), \Upsilon(\hat{\zeta})) + \gamma a(D^2w, D^2\hat{w}), \quad \gamma = \frac{h^2}{12}. \] (1.14)

See [17, p. 15], for (1.14). In (1.14), the 2-covariant tensor $\Upsilon(\cdot, \cdot)$ was defined by (1.8), while the 2-covariant tensor $D^2w$ is the Hessian of $w$, which is defined in (A.6) below. Moreover, in (1.14), $a(\cdot, \cdot)$ is a bilinear form \[ a(T_1, T_1) = (1 - \mu)(T_1, T_1)_{T^2(\Omega)} + \mu(\text{tr} T_1)^2, \quad x \in \Omega, \ T_1 \in T^2(\Omega), \] (1.15)
defined on second-order tensors $T^2(\Omega)$ of $\Omega$; see (A.2), (A.7) below for the inner product and the trace $\text{tr}$. Finally, with (1.14) we can associate the following symmetric bilinear form, defined directly on the middle surface $\Omega$:

\[ B(\zeta, \hat{\zeta}) = \int_{\Omega} B(\zeta, \hat{\zeta}) \, dx, \quad \zeta(x) = W(x) + w(x)N(x), \]

$W(x) \in M_x, \ x \in \Omega, \ (1.16)$

$M_x$ being the tangent space at $x \in M$. $N(x)$ is the unit normal field.

After these preliminaries, to state our stabilization result, we recall that the energy functional associated with model (1.1) is given by

\[ E(t) = E_k(t) + E_p(t), \] (1.17)

where $E_k$ is the kinetic energy

\[ E_k(t) \equiv \int_{\Omega} \left\{ |W_t|^2_{T_x} + w_t^2 + \gamma |Dw_t|^2_{T_x} \right\} \, dx \] (1.18)

\[ = \|W_t\|_{L^2(\Omega, \Lambda)}^2 + \|w_t\|_{L^2(\Omega)}^2 + \gamma \|Dw_t\|_{L^2(\Omega, \Lambda)}^2 \] (1.19)

(see Appendix for these spaces), and $E_p$ is the potential energy (see (1.14)--(1.16))

\[ E_p(t) \equiv B(\zeta, \zeta) = \int_{\Omega} B(\zeta, \zeta) \, dx \] (1.20)

\[ = \int_{\Omega} \left[ a(\Upsilon(W, w), \Upsilon(W, w)) + \gamma a(D^2w, D^2w) \right] \, dx, \]

$\zeta = [W, w]$. (1.21)

Next, in line with the statement above (1.14), we need to impose some geometric conditions on the shell.
1.6. Geometric assumptions

We shall assume the following hypotheses that were needed in [3] to prove the observability estimate which we shall invoke in Section 2.

(H.1) Ellipticity of the shell strain energy: there exists a constant $\lambda_0 \geq 1$ such that

$$\lambda_0 B(\zeta, \zeta) \geq \|DW\|_{L^2(\Omega, \mathbb{R}^2)}^2 + \gamma \|D^2w\|_{L^2(\Omega, \mathbb{R}^2)}^2$$

for $\zeta = [W, w] \in H^1(\Omega, \Lambda) \times H^2(\Omega)$,

(1.22)

where the function spaces are defined in (A.2), (A.9) below.

In particular, a sufficient condition for (H.1) to hold true is that both $\Pi$ and $D\Pi$ are small enough [18], where $\Pi$ is the second fundamental form (A.30) of $M$. A much weaker condition where (H.1) holds true is given in [2, Theorem 3.2]: it basically says that the shell is “sufficiently shallow.”

Main assumption (H.2). We assume that there exists a vector field $V \in \mathcal{X}(M)$ such that [(A.4)] the covariant differential

$$DV(X, X) = b(x)|X|^2, \quad X \in M_x, \quad x \in \partial \mathcal{O},$$

(1.23)

where $b$ is a function on $\mathcal{O}$. Set [(A.1)]

$$a(x) = \frac{1}{2} \langle DV, \mathcal{E} \rangle_{T^2}, \quad x \in \partial \mathcal{O},$$

(1.24)

where $\mathcal{E}$ is the volume element of $M$. Moreover, suppose that $b$ and $a$ satisfy the following inequality:

$$2 \min_{x \in \partial \mathcal{O}} b(x) > \lambda_0 (1 + \mu) \max_{x \in \partial \mathcal{O}} |a(x)|.$$  

(1.25)

Assumption (H.2) consists of (1.23) and (1.25).

Illustrations where assumption (H.2) holds true are given in [2]. The include shells whose mid-surface lies on a surface of constant curvature or a surface of revolution.

Theorem 1.2. Assume (H.1) in (1.22) and (H.2) in (1.23), (1.25) above. In addition to the well-posedness assumptions on $h_i$, $g_i$ in Theorem 1.1(a), assume further that

(H.3) there exist positive constants $0 < m < M$ and a sufficiently large constant $R > 0$, such that for all $s \in \mathbb{R}$ with $|s| > R$, we have

$$m|s|^2 \leq g_i(s)s \leq M|s|^2, \quad m|s|^2 \leq h_i(s)s \leq M|s|^2,$$

for $i = 1, 2$.

(1.26)
Next, assume that

\[(H.4)\] or the coefficient \(\ell\) in \((1.1)\) is positive: \(\ell > 0\), or else \(\Gamma_0 \neq \emptyset\).

Let \([W, w]\) be a weak solution of the feedback problem \((1.1)\), as asserted by Theorem 1.1(a). Then there exists a constant \(T_0 > 0\) such that, with reference to the energy \(E(t)\) in \((1.17)\), the following estimate holds true:

\[
E(t) \leq C(E(0))s\left(\frac{t}{T_0} - 1\right), \quad \forall t \geq T_0,
\]

where \(C(E(0))\) denotes a constant depending on the initial energy \(E(0)\), and where \(s(t)\) is a real-valued function converging to zero: \(s(t) \to 0\) as \(t \to \infty\), which is constructed as a solution of the following Cauchy problem

\[
s_t(t) + q(s(t)) = 0, \quad s(0) = E(0),
\]

involving a nonlinear ordinary differential equation where the function \(q(\cdot)\) is, in turn, constructed from the data of problem \((1.1)\). More precisely, the nonlinear monotone increasing function \(q(\cdot)\) is determined entirely from the behavior at the origin of the nonlinear boundary functions \(g_i, h_i\), according to the following algorithm [19].

**Step 1.** Due to the assumed monotonicity of the nonlinear boundary functions \(h_i, g_i\) one can readily construct [19] functions \(\tilde{g}_i, \tilde{h}_i\), concave and strictly increasing; vanishing at the origin: \(\tilde{g}_i(0) = \tilde{h}_i(0) = 0\), such that the following inequalities are satisfied for \(|s| \leq 1\):

\[
\tilde{g}_i(sg_i(s)) \geq |s|^2 + |g_i(s)|^2, \quad \tilde{h}_i(sh_i(s)) \geq |s|^2 + |h_i(s)|^2,
\]

\(\forall |s| \leq 1\).

We then define first the functions \(r_0(\cdot)\) and its rescaled version \(r(\cdot)\) by

\[
r_0(s) = 2 \sum_{i=1}^2 \tilde{g}_i(s) + \tilde{h}_i(s), \quad r(\cdot) = r_0\left(\frac{\cdot}{\text{meas } \Sigma_1}\right),
\]

and next the function \(p\),

\[
p = (cI + r)^{-1},
\]

where \(c\) is a constant dependent on \((1/\text{meas } \Sigma_1)(1/m + M)\), where \(\Sigma_1 = (0, T] \times \Gamma_1\).

**Step 2.** Having constructed the function \(p(\cdot)\) in \((1.31)\) from the given boundary feedback functions \(h_i, g_i\) (data of the problem) via \((1.29)–(1.31)\), we next introduce the function \(q(\cdot)\) by [19]

\[
q = I - (I + p)^{-1},
\]
so that $q$ is monotone increasing and $q(0) = 0$. It is such function $q$ that defines the nonlinear ordinary differential equation in the Cauchy problem (1.28), whose solution $s(\cdot)$ determines the decay rate of the energy $E(t)$ in (1.27) as $t \to \infty$.

**Remark 1.1.** (i) Assume, in particular, that the nonlinear functions $g_i$, $h_i$ are bounded from below by a linear function; that is, that (reinforcing (1.26) valid for $|s| > R$)

$$|h_i(s)| \geq c|s|, \quad |g_i(s)| \geq c|s|, \quad \forall s \in \mathbb{R},$$

(1.33)

for some $c > 0$. Then, it can be shown that the decay rates predicted by Theorem 1.2 are exponential. That is, there exist positive constant $C$, $\omega$—possibly depending on $E(0)$—such that

$$E(t) \leq C e^{-\omega t}, \quad \forall t \geq T_0.$$  

(1.34)

(ii) Assume, instead, that the functions $h_i$, $g_i$ have polynomial growth at the origin; that is,

$$h_i(s)s \geq a_i|s|^{p+1}, \quad g_i(s)s \geq b_i|s|^{p+1}$$

for $|s| \leq 1$, $a_i, b_i$ positive constants, $p > 1$.

(1.35)

Then, the decay rates predicted by Theorem 1.2 are algebraic:

$$E(t) \leq C t^{2/(1-p)}, \quad p > 1,$$

(1.36)

where $C = C(E(0)) = a$ constant depending on $E(0)$.

1.7. Contribution of the present paper and literature

The one stated above amounts to a stabilization problem for a coupled system of two PDEs, which consists of a wave-like equation and a plate-like equation, both defined on a 2-d manifold.

In the flat case, a prototype of this model is the full von Karman system, where the coupling is, in fact, nonlinear and unbounded with respect to the finite energy space. The problem of boundary stabilization of the full von Karman model was solved in [6]. However, there are major differences between the aforementioned two stabilization problems: the one for the (curved) shell and the one for the full von Karman model (in Euclidean space). These differences may be summarized as follows.

In the case of the full von Karman system [6], the main mathematical difficulties stemmed from the following sources:

(1) the fact that the problem is nonlinear, with a strong nonlinear coupling;
(2) the fact that the (critical) elastodynamic component introduces boundary traces which are neither bounded by the energy terms nor bounded by the feedback terms.
By contrast, in the present paper, the shallow shell problem (1.1) offers the following features:

(i) On the good side, the dynamical equations (1.1a), (1.1b) are linear and display “lower-order, weak coupling terms” \( \mathcal{F}(w) \) and \( \mathcal{G}(W) \), which therefore are not a source of serious additional technical difficulties over the analysis of the linear uncoupled equations.

(ii) On the bad side, however, the present case of a curved geometry of the shell yields coefficients of the principal part which are nonconstants, a challenging difficulty for inverse-type inequalities even for single hyperbolic [1] or Petrowski-type inequalities. Variable coefficient principal part coefficients for equations on a manifold represent the main new difficulty of the present paper. Indeed, in the case of constant coefficients in the principal part, the results of the present paper are strictly included in [6].

(iii) As in the case of the von Karman system [6], boundary traces of both the wave and the plate components persist in the estimates, as terms which can neither be bounded by the energy terms, nor by the feedback terms. The wave boundary traces are particularly critical (see below). Accordingly, we need to absorb these wave-boundary traces, by posing further the microlocal arguments of [8] (for waves), hence of [9] for elastic waves (Lame systems), since our present case is of different form than the Lame system treated in [9]. Thus, [9] cannot be quoted directly, and the necessary additional analysis is given in Proposition 3.2.2 below.

As already noted in the Introduction, sharp, microlocal trace theory of the plate \( w \)-component given here in Section 3.3 below—while certainly useful in eliminating restrictive and unnecessary geometrical conditions—is not, however, critical for the very solution of the present uniform stabilization problem. By contrast, the present uniform stabilization problem (even in the linear feedback case) would not be solvable without the use of a sharp, microlocal trace theory of the elastic wave \( W \)-component, which we provide here in Section 3.2 below. The same holds true for the uniform stabilization of the elastic wave, with no coupling, even in the linear case. In short: sharp, microlocal trace theory of an elastic wave—either uncoupled, or else coupled as in the shell model; whether with linear or with nonlinear feedback; and, finally, whether in the flat or in the curved case—is needed to solve the corresponding uniform stabilization problem (but is not needed just to get some solution of the corresponding continuous observability/exact controllability problem, under geometrical conditions, as in [3]). This claim on the \( W \)-traces appears to be at variance with some work, such as the almost contemporaneous [20] (which follows closely [3] in seeking to extend [3] to a stabilization result), where the uniform boundary stabilization of the present shell model is claimed, albeit with a linear feedback, and under geometrical conditions, without, however, any use of a sharp trace theory of the elastic \( W \)-component, as in our Section 3.2 be-
Close inspection reveals, however, that such works on stabilization attempt to compensate for a lack of sharp trace theory on the elastic $W$-component, by employing the Korn inequality on the boundary, while the Korn inequality is valid only in the interior.

Finally, we note that uniform stabilization of spherical shells—with a higher-order coupling than in the present model—were solved in [21] (linear case) and [22] (nonlinear case): in the model of spherical shells, due to the intrinsic symmetry of the model, no differential geometry is needed in the analysis. Moreover, optimal regularity and exact controllability are proved in [34].

2. Preliminary results

In this section we shall formulate and prove several preliminary estimates which deal with the trace regularity of solutions to the nonlinear problem given by (1.1). These results, while important in proving the main theorem, are also of independent interest in their own right.

2.1. Dissipativity equality

A starting point is, as usual, the dissipativity equality which states that the energy $E(t)$ in (1.17) of the entire system is nonincreasing. This fact alone does not prove, of course, that the energy is decaying, but it is a necessary preliminary step of the stability analysis.

**Lemma 2.1.** Let $[W, w]$ be a finite energy solution of system (1.1), as guaranteed by Theorem 1.1. Then, for any $s \leq t$, the following identity holds true for the energy $E(t)$ defined by (1.17):

$$E(t) + 2 \int_s^t \int_{\Gamma_1} [g_1(\langle W_t, n \rangle \langle W_t, n \rangle) + g_2(\langle W_t, \tau \rangle \langle W_t, \tau \rangle) + h_1 \left( \frac{\partial w_t}{\partial n} \frac{\partial w_t}{\partial n} + h_2 \left( \frac{\partial w_t}{\partial \tau} \frac{\partial w_t}{\partial \tau} \right) \right] d\Gamma_1 dt = E(s).$$

**Proof.** The proof is standard and follows by a classical energy-type argument. We multiply Eqs. (1.1a), (1.1b) by $W_t, w_t$, respectively, integrate over $\Omega \times (s, t)$ and apply the divergence theorem (as in [2,3]) first to smooth solutions, and then we extend by density to all weak solutions. The main tool of our computations is the following Green’s formula [3, (3.1.35)] in the notation of [3]:

$$(A\eta, \hat{\eta})_{L^2(\Omega, A) \times L^2(\Omega)} = \int_{\Omega} (A\eta, \hat{\eta}) dx$$
\[\int_{\Omega} B(\eta, \hat{\eta}) \, dx - \int_{\Gamma} \left\{ B_1(W, w) \langle \hat{W}, n \rangle + B_2(W, w) \langle \hat{W}, \tau \rangle + \gamma \left[ (\Delta w + (1 - \mu)B_3 w) \frac{\partial \hat{w}}{\partial n} - \left( \frac{\partial \Delta w}{\partial n} + (1 - \mu)B_4 w \right) \hat{w} \right] \right\} \, d\Gamma,\]

(2.2)

where \(\eta = [W, w], \hat{\eta} = [\hat{W}, \hat{w}],\)

\[A\eta \equiv \left[ -\Delta_{\mu} W - (1 - \mu)kW - \mathcal{F}(w) - \gamma \left[ \Delta^2 w - (1 - \mu)\delta(kdw) \right] + \left( H^2 - 2(1 - \mu)k \right) w + \mathcal{G}(W) \right],\]

(2.3)

and where \(\Delta_{\mu}\) and \(B(\cdot, \cdot)\) are defined by (1.2) and (1.15), while \(B_1, B_2, B_3, B_4\) are defined by (1.4)–(1.7). Finally, the coupling \(\mathcal{F}(w)\) and \(\mathcal{G}(W)\) are first-order operators, defined in [2, p. 1733, (1.33)], whose precise structure is not essential in the present paper. However, \(\mathcal{F}(w) = 0, \mathcal{G}(W) = 0\) in the flat case. □

In our next step we apply multipliers to (1.1). These are the same as those used in the flat nonlinear case in [6], except that now they are in differential geometric form. The corresponding calculations in the curved shell case are the Riemann metric counterpart of those in the flat case, and thus follow the same philosophy as those in [6] where they were used for the full nonlinear von Karman system. More precisely, we apply:

(i) the multipliers \([DVW, V(w)]\), in the notation of (A.4) below, in order to handle the potential energy \(E_p(t)\) in (1.21), where \(V\) is the vector field on \(M\) assumed in (H.2);

(ii) the multipliers \([W, w]\), in order to obtain an estimate for the difference between kinetic energy \(E_k(t)\) and potential energy \(E_p(t)\).

The actual computations are performed in [2, particularly formulas (2.122), (2.123)] and lead to the following inequality, which is the counterpart, in the curved case, of a special case of the inequality in the flat case given in [6, Lemma 3.2], when specialized to the linear model (1.1a), (1.1b).

**Proposition 2.2.** Assume (H.1), (H.2). With reference to strong solutions of the original problem (1.1) as guaranteed by Theorem 1.1, the following inequality holds true for the energy \(E(t)\) defined in (1.17): for \(T > 0\) given, there exist constants \(C > 0, C_T > 0\), such that

\[\int_{0}^{T} E(t) \, dt \leq C[E(0) + E(T)] + C_T(\text{BT}^{\text{good}}) + C_T(\text{BT}^{\text{bad}}) + LOT(W, w),\]

(2.4)
where

(i) \( \text{LOT}(W, w) \) are lower terms with respect to the energy \( E(t) \) in (1.17) where \( E(t) \) is topologically equivalent to

\[
H^1(\Omega, \Lambda) \times L_2(\Omega, \Lambda) \times H^2(\Omega) \times H^1(\Omega)
\]

(2.5)

for \([W, W_t, w, w_t]\); see Appendix for these spaces;

(ii) if \( BT = BT^{\text{good}} + BT^{\text{bad}} \) are the boundary terms, divided into “good” and “bad” terms, these are defined by

\[
BT^{\text{good}} = \int_0^T [\|W_t\|_{L^2(\Gamma_1, \Lambda)}^2 + \|Dw_t\|_{L^2(\Gamma_1, \Lambda)}^2] \, dt,
\]

(2.6a)

\[
BT^{\text{bad}} = \int_0^T \|DW\|_{L^2(\Gamma_1, \Gamma_2)}^2 \, dt + \int_{\Sigma_1} B(\zeta, \zeta) \, d\Sigma_1,
\]

(2.6b)

where we recall \( \zeta = [W, w] \) and \( B(\zeta, \zeta) \equiv a(\Upsilon(\eta), \Upsilon(\eta)) + a(D^2 w, D^2 w) \) as in (1.14).

**Remark 2.1.** While all boundary terms \( B^{\text{good}} \) involving time derivatives in (2.6a) will be determined by the dissipation, the boundary integral in the term \( BT^{\text{bad}} \) in (2.6b) contains traces of the first order for \( W \) and of the second order for \( w \); see (1.14), (1.8). These traces are *not* determined either by the energy or by the boundary conditions. In fact, the main challenge to, and contribution by, this paper is to provide an estimate for the traces in the \( BT \) term. This will be done by extending microlocal estimates [8] for scalar waves, hence [9] for elastic waves in Lame form, to the present \( W \)-component which is now in Lame form, and [10] for the plate \( w \)-component and applying arguments as in [6] for the case of full von Karman model.

Henceforth, to streamline the notation in the estimates below, we shall generally adopt the following notation for functions, respectively \( k \)-tensors:

\[
|u|_{\alpha, \Omega}: \quad \text{for the } H^\alpha(\Omega) \text{- or } H^\alpha(\Omega, T^k) \text{-norm},
\]

(2.7a)

\[
|u|_{\alpha, \Gamma}: \quad \text{for the } H^\alpha(\Gamma) \text{- or } H^\alpha(\Gamma, T^k) \text{-norm},
\]

(2.7b)

\[
(u, v)_{\Omega}: \quad \text{for the } L^2(\Omega) \text{- or } L^2(\Omega, T^k) \text{-inner product}.
\]

(2.7c)

In other words, our notation will identify only the number of derivatives on \( \Omega \) or \( \Gamma \), and leave unspecified whether the argument of that norm is a function, or a \( k \)-tensor (\( k \)-form) on \( \Omega \) or \( \Gamma \). This should not generate any confusion and should make the paper easier to follow by readers not familiar with this full notation.
3. Main trace estimate for problem (1.1). Statement and proof

Throughout this section assumptions (H.1) and (H.2) are in force.

3.1. Main statement. First step of the proof: Local reduction to a Euclidean (flat) coordinate system

The following estimate is critical for the proof of Theorem 1.2.

**Theorem 3.1.1.** Let \(0 < \alpha < T/2\). Then, the following trace estimate holds true for any regular solution of problem (1.1), as guaranteed by Theorem 1.1: there exists a constant \(C_{\alpha T} > 0\) such that

\[
BT_{\text{bad}}^{\alpha, T - \alpha} \equiv \int_\alpha^{T - \alpha} \int_{\Gamma_1} \left[ B(\zeta, \zeta) + |DW|_{T_x}^2 \right] dx \, dt \\
\leq C_{\alpha T} \int_{\Sigma_1} \left[ |Dw_1|_{T_x}^2 + |W_1|_{T_x}^2 + |g_1((W_t, n))|^2 + |g_2((W_t, \tau))|^2 \right. \\
+ \left. \left| h_1 \left( \frac{\partial w_t}{\partial n} \right) \right|^2 + \left| h_2 \left( \frac{\partial w_t}{\partial \tau} \right) \right|^2 \right] d\Sigma_1 + LOT(W, w),
\]

where the lower-order terms \(LOT\) are below energy level and satisfy

\[
LOT(W, w) \leq C_\epsilon \sup_{t \in [0, T]} \left[ |W(t)|_{1-\epsilon, \Omega} + |W_1(t)|_{-\epsilon, \Omega} + |w(t)|_{2-\epsilon, \Omega} + |w_t(t)|_{1-\epsilon, \Omega} \right] + \text{LOT}_2(W, w).
\]

for any \(\epsilon > 0\); see notation adopted in (2.7) in 1-forms \(W, W_t\) or functions \(w, w_t\).

**Remark 3.1.1.** The traces \(B(\zeta, \zeta), DW\) in (3.1.1) are not bounded by the energy.

**Proof of Theorem 3.1.1.** By recalling the definition of \(B(\zeta, \zeta)\) in (1.15) we see that it suffices to prove the following estimate for regular solutions of problem (1.1): there exists a constant \(C_{\alpha T} > 0\), such that

\[
\int_\alpha^{T - \alpha} \int_{\Gamma_1} \left[ |D^2 w|_{T_x}^2 + |DW|_{T_x}^2 \right] dx \, dt \\
\leq C_{\alpha T} \int_{\Sigma_1} \left[ |Dw_1|_{T_x}^2 + |W_1|_{T_x}^2 + |g_1((W_t, n))|^2 + |g_2((W_t, \tau))|^2 \right.
\]
Orientation. The proof of estimate (3.1.3) requires the insertion of microlocal estimates. To carry this out, we shall apply a basic strategy similar to that already employed in [6] in the flat, nonlinear case of the full von Karman system. Indeed, the proof of estimate (3.1.3) will comprise three main steps. In Step 1—given in the present section—we shall introduce a coordinate cover of a boundary layer of and a subordinate partition of unity, and reduce our task to prove estimate (3.1.3) for just one coordinate system. Next, in Step 2, to be carried out in Section 3.2, we shall provide a sharp trace estimate for the linear model of dynamic elasticity, to be used for the in-plane \( W \)-components of the displacement vector \( \zeta \). Finally, in Step 3, to be carried out in Section 3.3, we shall provide a sharp trace estimate for the linear Kirchhoff plate model, to be used for the normal \( w \)-component of the displacement \( \zeta \).

Steps 2 and 3 are critical for the proof of the stabilizability estimates without assuming geometric conditions on the controlled portion \( \Gamma_1 \) of the boundary, as done in wave-plate literature [4,12,13] and without considering additional tangential components of the horizontal displacement \( W \) in the structure of the stabilizing feedback as done in [5] in the flat case.

**Step 1.** With reference to Fig. 1, let \( \mathcal{U} = \{ \mathcal{U}_1, \ldots, \mathcal{U}_n \} \) be a coordinate cover of a boundary layer of \( \overline{\Omega} \) on the surface \( M \), and \( \{ \phi_1, \ldots, \phi_n \} \) be a partition of unity subordinate to \( \mathcal{U} \), \( 0 \leq \phi_i \leq 1 \). Then we can write \( W \) and \( w \) as

\[
W = \sum_{i=1}^{n} \phi_i W, \quad w = \sum_{i=1}^{n} \phi_i w.
\] (3.1.4)

It suffices to establish estimate (3.1.3) with respect to one generic coordinate patch in the atlas, denoted by \( \{ U, \psi \} \), whose coordinate functions are denoted by \( x, y \). Invoking [23, p. 183], we have that there exists a positive smooth function \( \rho \) on \( U \) such that the Riemann metric \( g = \langle \cdot, \cdot \rangle \) of the manifold \( M \) is expressed by

\[
g = \langle \cdot, \cdot \rangle = \rho^{-1} (dx^2 + dy^2) = \rho^{-1} \bar{g}, \quad \text{or} \quad g_{ij} = \rho^{-1} \delta_{ij},
\] (3.1.5)

where \( \bar{g} \) is the Euclidean metric in \( \mathbb{R}^2 \), so that \( \bar{g}_{ij} = \delta_{ij} \). Accordingly we set

\[
\begin{cases}
W = w_1 \, dx + w_2 \, dy & \text{on } U, \\
n = n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y}, \\
\tau = \tau_1 \frac{\partial}{\partial x} + \tau_2 \frac{\partial}{\partial y}, & \tau_1 = -n_2, \quad \tau_2 = n_1,
\end{cases}
\] (3.1.6)

for the local representation of the 1-form \( W \) and vector fields \( n \) and \( \tau \), for the normal and tangential unit vectors \( n, \tau \) along the boundary \( \Gamma \) of \( \Omega \), whose normal and tangential derivatives are given by (1.3). Moreover, we shall set the following notation:
$\partial^\ell$ = general (unstructured) differential operator of order $\ell$, $\ell = 1, 2, \ldots$, in the space variables $x$ and $y$.

Lemma 3.1.2. In the coordinate system $\{U, \psi\}$ in the atlas and the setting (3.1.4)–(3.1.7) introduced above, the corresponding version of problem (1.1) is given by

$$
\begin{align*}
(\phi W)_{tt} - \rho \Delta_{xy}(\phi W) &= F(\partial^1(\varphi_1 W), \partial^1(\varphi_1 w)) \\
\text{on} \ (0, \infty) \times \tilde{\Omega}, & \quad (3.1.8a)
\end{align*}
$$

$$
\begin{align*}
(\phi w)_{tt} - \gamma \rho \Delta_0(\phi w)_{tt} + \gamma \rho^2 \Delta_0^2(\phi w) &= G(\partial^1(\varphi_1 W), \partial^3(\varphi_1 w)) + \partial^1(\varphi_1 w_{tt}) \\
\text{on} \ (0, \infty) \times \tilde{\Omega}, & \quad (3.1.8b)
\end{align*}
$$

with the following boundary conditions on $\tilde{T}_0$:

$$
W = 0, \quad w = 0, \quad \frac{\partial}{\partial n} w = 0 \quad \text{on} \ (0, \infty) \times \tilde{T}_0, & \quad (3.1.8c)
$$

and the following boundary conditions on $(0, \infty) \times \tilde{T}_1$:

$$
\begin{align*}
B_{1xy}(\phi W) &= f_1(\varphi_1 W, \varphi_1 W_{n}), \quad (\varphi_1 w), \quad (3.1.8d) \\
B_{2xy}(\phi W) &= f_2(\varphi_1 W, \varphi_1 W_{\tau}), \quad (\varphi_1 w), \quad (3.1.8e) \\
\frac{\partial^2}{\partial n^2}(\phi w) + \mu \frac{\partial^2}{\partial \tau^2}(\phi w) &= f_3(\partial^1(\varphi_1 w), \delta_1(\varphi_1 w)), \quad (3.1.8f) \\
\frac{\partial^3}{\partial n^3}(\phi w) + \frac{\partial}{\partial n} \frac{\partial^2}{\partial \tau^2}(\phi w) &= f_4(\partial^2(\varphi_1 w), w_{tt}, \frac{\partial}{\partial \tau} w_t), \quad (3.1.8g)
\end{align*}
$$

$$
\tilde{\Omega} = \psi(U \cap \Omega \cap \text{supp} \phi), \\
\tilde{T}_0 = \psi(T_0 \cap \text{supp} \phi), \quad \tilde{T}_1 = \psi(T_1 \cap \text{supp} \phi), \quad (3.1.9)
$$

where

(i) $\varphi_1$ is a smooth function (arising from commutators) such that $\text{supp} \varphi_1 \subset \text{supp} \phi$; more precisely, $\text{supp} \varphi_1 = \text{supp} \phi \setminus \{p \in M: \phi(p) \equiv 1\}$;

(ii) $\Delta_{xy}$ and $\Delta_0$ are defined by

$$
\Delta_{xy} W = \left[ \frac{\partial^2}{\partial x^2} w_1 + \frac{1-\mu}{2} \frac{\partial^2}{\partial y^2} w_1 + \frac{1+\mu}{2} \frac{\partial^2}{\partial x \partial y} w_2 \right],
$$

$$
\Delta_0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad (3.1.10)
$$

[warning: $\Delta_0$ is not $\Delta_{\mu=0}$; see (1.2)];

(iii) the functions $F, G, f_1, \ldots, f_4$ are linear combinations of their arguments by means of smooth functions;
(iv) the boundary operators $B_{1_{xy}}$ and $B_{2_{xy}}$ are defined by

$$
B_{1_{xy}} W \equiv (1 - \mu) \rho^{-1} \left[ n_1 \frac{\partial}{\partial n} w_1 + n_2 \frac{\partial}{\partial n} w_2 \right] + \mu \, \text{div}_0 W + \text{lot},
$$

(3.1.11)

$$
B_{2_{xy}} W \equiv \frac{\rho^{-1}}{2} \left[ n_1 \frac{\partial}{\partial \tau} w_1 + n_2 \frac{\partial}{\partial \tau} w_2 + \tau_1 \frac{\partial}{\partial n} w_1 + \tau_2 \frac{\partial}{\partial n} w_2 \right].
$$

(3.1.12)

A subscript 0 refers to the Euclidean metric: $\text{div}_0 W = \partial w_1 / \partial x + \partial w_2 / \partial y$.

**Proof of Lemma 3.1.2.** First, we handle the $W$- and $w$-equations.

(i) We multiply each equation (1.1a) by $\phi$ and use commutators, as usual.

(ii) We next verify that, in light of the metric relation in (3.1.5), the Laplace–Beltrami operator $\Delta$ on function on $M$ and the Laplace operator $\Delta_0$ in $(x, y)$ coordinates (see (3.1.10)) are related locally by

$$
\Delta = \rho \Delta_0, \quad \Delta^2 = \rho^2 \Delta_0^2 + \text{lot},
$$

(3.1.13)

with appropriate lower-order terms. In fact, (3.1.5) yields $g_{ij} = \rho^{-1} \delta_{ij}$, with $\delta_{ij}$ the Kronecker symbol. Thus, for the $2 \times 2$ matrix $g_{ij}$ we have

$$
\det\{g_{ij}\} = \rho^{-2}.
$$

(3.1.14)

The inverse matrix is

$$
\{g^{ij}\} = (g_{ij})^{-1} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}.
$$

(3.1.15)

Next from the well-known expression ([24, p. 137], [25, p. 153]) of the Laplace–Beltrami operator $\Delta$ in local coordinates $x = x_1$, $y = x_2$, we compute from (3.1.14), (3.1.15)

$$
\Delta = \frac{1}{\sqrt{\det\{g_{ij}\}}} \sum_{i,j}^n \frac{\partial}{\partial x_i} \left( \sqrt{\det\{g_{ij}\}} g_{ij} \frac{\partial}{\partial x_j} \right) = \rho \Delta_0,
$$

(3.1.16)

and (3.1.13) for $\Delta$ is proved. Then (3.1.13) for $\Delta^2$ follows at once.

(iii) Given a vector field $X = \sum_{i=1}^2 \alpha_i (\partial / \partial x_i) \in \mathcal{X}(M)$; let $x = x_1$, $y = x_2$, and let [25, p. 153]

$$
\text{div} X = \sum_{i=1}^2 (D_{\partial / \partial x_i} X)_i, \quad \text{div}_0 X = \frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_2}{\partial y}.
$$

(3.1.17)

Then we have (with appropriate lower-order terms)

$$
\text{div} X = \text{div}_0 X + \text{lot}.
$$

(3.1.18)

In fact, in local coordinates, $\text{div} X$ is given by ([24, p. 127], [25, p. 153])
\[
\text{div } X = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g_{ij})} \alpha_i \right) \quad (3.1.19)
\]

\[
= \rho \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\frac{1}{\rho} \alpha_i \right) = \text{div}_0 X + \text{lot}, \quad (3.1.20)
\]

by using (3.1.14), and (3.1.18) is verified.

(iv) Consider the gradients \( Df = \nabla_g f \) and \( \nabla_0 f \) of a smooth (scalar) function \( f \) in the metric \( g \) in local coordinates and in the usual Euclidean sense

\[
Df(x) = \nabla_g f(x) = \sum_{i,j=1}^{2} g^{ij}(x) \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}, \quad (3.1.21)
\]

\[
\nabla_0 f(x) = \sum_{i=1}^{2} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.
\]

Then, by (3.1.15) used in the expression of \( \nabla_g f \) above, we obtain via (3.1.5)

\[
\begin{align*}
\{ Df = \nabla_g f &= \rho \nabla_0 f, \quad \text{hence} \\
\langle \nabla_g f, X \rangle_g &= \rho^{-1} \langle \nabla_g f, X \rangle_0 = \langle \nabla_0 f, X \rangle_0
\end{align*}
\quad (3.1.22a, b)
\]

for any vector field \( X \in \mathcal{X}(M) \). Taking \( X = n \), or \( X = \tau \), we have that the normal and tangential derivatives \( \frac{\partial}{\partial n}, \frac{\partial}{\partial \tau} \) are the same under \( g \) and \( \bar{g} \) in local coordinates, and we need not distinguish between them. Accordingly, we preserve the same notation in either case.

(v) Let \( \omega \in \Lambda^1(M) \) be a 1-form on \( M \). There exists a unique \( X \in \mathcal{X}(M) \) by the identification \( \Lambda^1(M) = \mathcal{X}(M) \) in (A.0): \( \langle \omega, Y \rangle = g(Y, X) = \langle Y, X \rangle, \forall Y \in \mathcal{X}(M) \). Then, recalling that the exterior derivative \( d \) does not depend on the metric, so \( d = d_0 \) (see (A.17)), we have

\[
d \delta \omega = -\rho \nabla_0 (\text{div}_0 X) + \text{lot} = \rho d \delta_0 \omega + \text{lot} \in \Lambda^1(M), \quad (3.1.23)
\]

with appropriate lower-order terms. In fact, we first invoke the well-known result [24, p. 164, Eq. (10.25)] \( \delta \omega = -\text{div } X \) in (A.24). Next, recalling (3.1.18), (A.24) and (A.14),

\[
\delta \omega = -\text{div } X = -\text{div}_0 X + \text{lot} = \delta_0 \omega + \text{lot} \in \Lambda^0(M). \quad (3.1.24)
\]

Further, from (3.1.24), recalling (A.14) and (3.1.22a),

\[
d \delta \omega = -d(\text{div}_0 X) + \text{lot} = -\nabla_g (\text{div}_0 X) + \text{lot} \\
= -\rho \nabla_0 (\text{div}_0 X) + \text{lot} = \rho d \delta_0 \omega + \text{lot} \in \Lambda^1(M), \quad (3.1.25)
\]

and (3.1.23) is verified, again with appropriate lower-order terms.
(vi) Let $\omega$ be a 1-form: $\omega = \sum_{k=1}^{2} \omega_k dx_k \in \Lambda^1(M)$. Then the Hodge–Laplacian operator $\Delta$ in (A.26) [24, p. 162] on 1-forms yields

$$-(d\delta + \delta d)\omega \equiv \Delta \omega = \rho \sum_{k=1}^{2} \Delta_0 \omega_k dx_k + lot = \rho \Delta_0 \omega + lot,$$

(3.1.26)

with $\Delta_0$ being $\Delta$ in the metric $\bar{g}$ (flat case). Indeed, recalling [26, p. 226], we have

$$\Delta \omega = \sum_{k=1}^{2} (\Delta \omega)_k dx_k, \quad (\Delta \omega)_k = \sum_{i,j} g^{ij} \frac{\partial^2 \omega_k}{\partial x_i \partial x_j} + lot = \rho \Delta_0 w_k + lot.$$

(3.1.27)

Recalling (3.1.15) and (3.1.10) for $\Delta_0$ on 0-forms, i.e., functions, in the last equality, (3.1.27) yields (3.1.26).

(vii) We return to the Hodge–Laplace-type operator $\Delta_\mu$ applied on 1-forms defined by (1.2). Denote by $(\Delta_\mu)_0$ its version in the metric $\bar{g}$ in (3.1.5) (flat case). Then we have

$$\Delta_\mu = \rho (\Delta_\mu)_0 + lot.$$

(3.1.28)

In fact, using (3.1.25) for $d\delta$ and (3.1.26) we obtain

$$\delta d\omega = -\Delta \omega - d\delta \omega = -\rho \Delta_0 \omega - \rho d\delta_0 \omega + lot$$

$$= \rho (-\Delta_0 - d\delta_0) \omega + lot = \rho \delta_0 d\omega + lot.$$

(3.1.29)

Hence recalling $\Delta_\mu$ in (1.2), (3.1.25) and (3.1.29),

$$\Delta_\mu = -\left[ \frac{1}{2} \delta d + d\delta \right] = -\rho \left[ \left( \frac{1}{2} \mu \right) \delta_0 d + d\delta_0 \right] + lot$$

$$= \rho (\Delta_\mu)_0 + lot,$$

(3.1.30)

and (3.1.28) is established.

(viii) Identity (3.1.28) reduces $\Delta_\mu$ on 1-forms to the flat case version $(\Delta_\mu)_0$. But in the flat case, we have the following results ([35, p. 279, Eq. (7.8)], [26, p. 228, with $p = 1, n = 2$]) for the 1-form $W = w_1 dx + w_2 dy$ in (3.1.6):

$$-\delta_0 W = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = \text{div} W \quad (\text{see (A.24)}),$$

(3.1.31)

$$-d\delta_0 W = \left( \frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_2}{\partial x \partial y} \right) dx + \left( \frac{\partial^2 w_1}{\partial y \partial x} + \frac{\partial^2 w_2}{\partial y^2} \right) dy,$$

(3.1.32)

$$dW = dw_1 \wedge dx + dw_2 \wedge dy \left( \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} \right) dx \wedge dy,$$

(3.1.33)

$$-d\delta_0 W = \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx$$

$$= \left( \frac{\partial^2 w_1}{\partial y^2} - \frac{\partial^2 w_2}{\partial x \partial y} \right) dx + \left( \frac{\partial^2 w_2}{\partial x \partial y} - \frac{\partial^2 w_1}{\partial x^2} \right) dy = \Delta_0 W + d\delta_0 W,$$

(3.1.34)
with \( f = (\partial w_2/\partial x - \partial w_1/\partial y) \); see [35, p. 279, Eq. (7.8)] and (A.17) for (3.1.33) and (A.25) for (3.1.34). These results (3.1.32), (3.1.34) are also noted, via different computations, in [3, (3.3.42), (3.3.43)]. Thus, by (3.1.32), (3.1.34), we readily conclude, via (1.2), that

\[
(\Delta \mu)W = -\left[ \frac{1 - \mu}{2} \delta_0 d + d \delta_0 \right] W \equiv \Delta_{xy} W + \text{lot},
\]

with \( \Delta_{xy} \) defined in (3.1.10).

**Remark 3.1.1.** We note explicitly that, in the flat case, with \( \Delta_0 \) given by (3.1.10), if we set on 1-forms (vector fields) \( W = w_1 \, dx + w_2 \, dy \):

\[
A_a W = a \left[ \frac{\Delta_0 w_1}{\Delta_0 w_2} \right] + \nabla_0 \text{div}_0 W, \quad \Delta(v) = v \delta_0 d + d \delta_0,
\]

we then have via (3.1.32), (3.1.34):

\[
(1 - v) A_{v/(1-v)} = \Delta(v).
\]

(ix) Using the results in (ii)–(viii), and multiplying each equation (1.1a), (1.1b) by \( \phi \) and using commutators, as stated in (i), we then obtain (3.1.8a) and (3.1.8b).

Next, we handle the boundary conditions.

(x) We first prove (3.1.11). By (1.4) we have in the flat case:

\[
B_1(W, w) = (1 - \mu) \Upsilon(W, w)(n, n) - \mu \delta_0 W + \mu w H.
\]

Next, for \( W = [w_1, w_2], n = [n_1, n_2] \) as in (3.1.6), by using (1.8), (A.4), (A.5) and (3.1.5), (3.1.6), we compute

\[
\Upsilon(W, w)(n, n) = \frac{1}{2} \left[ DW(n, n) + D^* W(n, n) \right] + \omega \Pi = DW(n, n) + \omega \Pi
\]

\[
= \langle D_n W, n \rangle + \omega \Pi = \rho^{-1} \left[ \begin{array}{c} \frac{\partial w_1}{\partial n} \\ \frac{\partial w_2}{\partial n} \end{array} \right] \cdot \left[ \begin{array}{c} n_1 \\ n_2 \end{array} \right]_{\mathbb{R}^2} + \omega \Pi
\]

\[
= \rho^{-1} \left[ n_1 \frac{\partial w_1}{\partial n} + n_2 \frac{\partial w_2}{\partial n} \right] + \omega \Pi,
\]

recalling the conclusion of point (iv) above on \( \partial/\partial n, \partial/\partial \tau \). Next, we recall that \( \delta_0 W = -\text{div}_0 W \) by (3.1.31), or (A.24). Using this and (3.1.41) in (3.1.38) yields

\[
B_1(W, w) = (1 - \mu) \rho^{-1} \left[ n_1 \frac{\partial w_1}{\partial n} + n_2 \frac{\partial w_2}{\partial n} \right] + \mu \left( \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} \right) + \mu w H + \omega \Pi,
\]

and (3.1.11) is established.
(xi) Next, we prove (3.1.12). By (1.5),

\[ B_2(W,w) = (1 - \mu) \mathcal{Y}(W, w)(n, \tau), \]

where by (1.8), (A.4), (A.5) we compute via (3.1.5), (3.1.6),

\[ \mathcal{Y}(W, w)(n, \tau) = \frac{1}{2} \left[ DW(n, \tau) + D^* W(n, \tau) \right] + \omega \Pi \]

(by (3.1.5))

\[ = \frac{\rho^{-1}}{2} \left\{ \left[ \begin{array}{c} \frac{\partial w_1}{\partial \tau} \\ \frac{\partial w_2}{\partial \tau} \end{array} \right] \cdot \left[ \begin{array}{c} n_1 \\ n_2 \end{array} \right] \right\} + \left\{ \left[ \begin{array}{c} \frac{\partial w_1}{\partial n} \\ \frac{\partial w_2}{\partial n} \end{array} \right] \cdot \left[ \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right] \right\} \]

(by (3.1.6))

\[ = \frac{\rho^{-1}}{2} \left\{ n_1 \frac{\partial w_1}{\partial \tau} + n_2 \frac{\partial w_2}{\partial \tau} + n_1 \frac{\partial w_1}{\partial n} + n_2 \frac{\partial w_2}{\partial n} \right\} \]

and (3.1.12) is established. Lemma 3.1.2 is proved.

\[ \square \]

3.2. Trace regularity for elastic waves (W-component)

**Orientation.** This section and the next provide sharp trace regularity results which are critical for the proof of stability estimates without assuming geometric conditions on \( \Gamma_1 \) and without considering artificial tangential components of the in-plane displacement \( W \) in the structure of the stabilizing feedback (as was done in [5] in the study of the von Karman problem) which are not in \( L^2 \). These estimates are based on the corresponding trace estimates valid for (i) linear model of dynamic elasticity and (ii) linear Kirchhoff model. They are obtained by methods of microlocal analysis.

As to (i), we need to extend to the present non-Lame elastic \( W \)-component the analysis begun in [8] for second-order hyperbolic equations, which was then the basis for the analysis in [9] of Lame-type elastic systems. We cannot merely quote [9] as our \( W \)-system is not of Lame-type. As to (ii) instead, we shall invoke [10] for the sharp trace regularity of second-order derivatives for plates. The sharp trace regularity of first-order traces for \( W \)—to be given in Proposition 3.2.1 below—are critical for solving the stabilization problem in the first place. Instead, the sharp trace regularity of second-order traces for \( w \)—to be given in Proposition 3.3.1—merely avoid unnecessary and restrictive geometrical conditions on the controlled portion of the boundary \( \Gamma_1 \). The main idea is to obtain the estimates for the tangential derivatives on the boundary in terms of the velocity traces and lower-order terms: see Proposition 3.2.1 for \( W \) and Proposition 3.3.1 for \( w \).
To formulate these results we introduce some notation. Let $T > 0$ be fixed. In fact, from now on we shall assume that $T$ is sufficiently large depending on the finite speed of propagation corresponding to Eq. (1.1). We denote $Q \equiv [0, T] \times \Omega$, $\Sigma_{1,\alpha} = [\alpha, T - \alpha] \times \Gamma_1$ where $\alpha < T/2$, $\Sigma_1 \equiv [0, T] \times \Gamma_1$, $\Sigma_0 \equiv [0, T] \times \Gamma_0$, $\Sigma \equiv [0, T] \times \Gamma$ and similarly with $\tilde{\Gamma}$, $\tilde{\Gamma}_1$, etc. We shall follow the notation set up in (2.7) to denote Sobolev norms.

The constant $C$ is a generic constant, different in various occurrences.

**Proposition 3.2.1.** Let $W, w$ be a finite energy solution corresponding to the system (1.1) as guaranteed by Theorem 1.1(a). Then, for any $0 < \epsilon < 1/4$ and $0 < \alpha < T/2$, there exist constants $C > 0$ such that the following trace regularity takes place:

$$\int_0^T \int_0^{\alpha} \int_{\Gamma_1} |DW|_T^2 d\Sigma_1 \alpha \leq C_{\epsilon T} \int_{\Sigma_1} |W_t|^2 + |g_1(\langle W_t, n \rangle)|^2 + |g_2(\langle W_t, \tau \rangle)|^2 d\Sigma_1 + C_{\alpha T} \int_0^T [\|w\|_{2, \Omega}^2 + \|W\|_{1, -\epsilon, \Omega}^2] dt. \quad (3.2.1)$$

**Remark 3.2.1.** Proposition 3.2.1 is a counterpart of [6, Lemma 2.2], proved for the “flat” but nonlinear case. Notice that the regularity of the trace of $DW$, claimed by Proposition 3.2.1 (see also Proposition 3.3.1 for the Kirchhoff plate below), does not follow from the standard interior regularity of finite energy solutions via trace theory. These are independent regularity results which rely heavily on microlocal arguments applied to both the dynamic system of elasticity and the dynamic Kirchhoff plate.

**Proof of Proposition 3.2.1.** Step 1. It suffices to prove Proposition 3.2.1 for each function $(\phi W)$. Then, by using partition of unity property (3.1.4), summing up all local estimates provides the global estimate (3.2.1) stated above. To accomplish this, we recall the function $F$ in (3.1.8a) as a linear combination of its arguments, by means of smooth functions (statement (iii) below (3.1.10)) and define

$$F(x, y, t) \equiv F(\partial_1(x, y) W(x, y)), \quad (3.2.2)$$

where $(\phi W), (\phi w)$ is a finite energy solution corresponding to system (3.1.8), as guaranteed by Theorem 1.1. Then, in particular, $\phi W$ satisfies the system of dynamic elasticity (3.1.8a) which we rewrite here now in terms of the $(x, y)$-coordinates:

$$(\phi W)_{tt} - \rho \Delta_{xy}(\phi W) = F \quad \text{in} \quad (0, \infty) \times \widehat{\Omega}. \quad (3.2.3)$$
We shall be extending the trace regularity result developed in [9] for the classical Lame model of dynamic elasticity with tractions prescribed on the boundary (based on the analysis of [8] where an analogous result was first proved for the wave equation). In our case, the system considered in (3.2.3) is of a different form than the Lame system treated in [9]. Thus, the regularity stated in [9] cannot be quoted directly. Nevertheless, we will be able to adopt the techniques of [8,9] in order to cope with the present new situation, dealing with a system which is not of Lame type. In fact, we shall first prove

**Proposition 3.2.2.** Let \((\phi, W)\) be a smooth solution of Eq. (3.2.3). For all \(\epsilon < 1/2\) and all \(0 < \alpha < T/2\), there exists \(C = C_{\alpha \epsilon} > 0\) such that we have the estimate

\[
\int_0^T \int_{\bar{\Gamma}_1} \left| \nabla_0 (\phi W) \cdot \tau \right|^2 d\bar{\Sigma}_{1,\alpha} \leq C \int_0^T \left[ |W|^2_{0,\bar{\Gamma}_1} + \sum_{i=1}^2 |B_{i,xy}(\phi W)|^2_{0,\bar{\Gamma}_1} + |F|^2_{-1/2,\bar{\Omega}} + |W|^2_{1-\epsilon,\bar{\Omega}} \right] dt,
\]

(3.2.4)

where \(F\) is given by (3.2.2), and the boundary operators \(B_{i,xy}\) are defined in (3.1.11), (3.1.12).

**Proof.** Estimate (3.2.4) above would follow from Theorem 1.2 in [9], if the system (3.2.3) considered and the boundary operators (3.1.11), (3.1.12) were the same as those treated in [9]. However, this is not the case, and we need to adjust and extend the arguments. We assume that the reader has [9] in hand. We shall use the same notation. First of all, we notice that the microlocal argument used to prove Theorem 1.2 in [9] is fully applicable to our situation in a hyperbolic microlocal sector \(\mathcal{R}_2 \cup \mathcal{R}_{tr}\); see Fig. 2. Indeed, in this sector, the a priori information on time derivatives of the boundary traces is sufficient for the estimates. In \(\mathcal{R}_1\) instead, the structure of the equations and boundary conditions is critical. However, by rescaling appropriately the dual variables \(\sigma, \eta\), we can claim that \(\mathcal{R}_1\) corresponds to an elliptic sector. Thus, elliptic theory for pseudodifferential elliptic systems [24,27] will yield appropriate estimates provided that we verify the following two properties:

(P1) With reference to the time localization of problem (3.1.8), in the neighborhood of the boundary, we can express the second normal derivatives in terms of, at most, second-order tangential (time and space) derivatives in \(W\) and first-order normal derivative of an operator of order at most 1 in the tangential variable on \(\bar{\Gamma}\), along with other quantities appearing on the RHS of inequality (3.2.4).
Fig. 2. $\sigma$ and $\eta$, the dual variables of time and tangential space variable, respectively.

(P2) The transformed boundary conditions are noncharacteristic. This is to say, we can express the normal derivatives on the boundary in terms of tangential derivatives and pre-assigned values on the boundary determined by the system of boundary operators $B_{i,x,y}$, $i = 1, 2$. In other words, the boundary operators $B_{i,x,y}$, $i = 1, 2$, in (3.1.11), (3.1.12), along with only tangential derivatives suffice to determine normal derivatives on the boundary.

To accomplish these two tasks, we shall use local representations of the operators involved.

Step 1. Proof of property (P1). The first main step in establishing property (P1) is the following lemma, which refers to Eq. (3.1.8a) = (3.2.3) for now.

Lemma 3.2.3. Let $0 < \mu < 1$ as in (1.2). Let $(\phi W)$ be a smooth solution of Eq. (3.2.3). Then the following identity holds true on a boundary layer $\tilde{\Gamma} = \partial \tilde{\Omega}$ of the boundary:

$$\frac{\partial^2 (\phi W)}{\partial n^2} = \rho^{-1} M^{-1} \left\{-F + (\phi W)_{tt} + PO_2(\phi W) \right\} + \text{lot},$$

where

(i) $M$ is the following nonsingular $2 \times 2$ matrix

$$M = \begin{bmatrix} n_2^2 + \frac{1-\mu}{2}n_2^2 & \frac{1+\mu}{2}n_1n_2 \\ n_1n_2 & \frac{1+\mu}{2}n_1^2 + \frac{1-\mu}{2}n_2^2 \end{bmatrix},$$

$$\det M = n_1^2n_2^2(1-\mu) + \left(n_1^2 + n_2^4\right)\frac{1-\mu}{2} > 0, \quad 0 < \mu < 1. \quad (3.2.6)$$

(ii) $F$ is given by (3.2.2) (as a linear combination of its arguments).
(iii) With \( W = w_1 \, dx + w_2 \, dy \), see (3.1.6), we have

\[
PO_2(\phi W) = \left[ c_{11} \frac{\partial^2 (\phi w_1)}{\partial \tau^2} + c_{12} \frac{\partial^2 (\phi w_2)}{\partial \tau^2} \right],
\]

\[
PO_1(\phi W) = \left[ b_{11} \frac{\partial (\phi w_1)}{\partial \tau} + b_{12} \frac{\partial (\phi w_2)}{\partial \tau} \right],
\]

with coefficients \( c_{11}, \ldots, b_{11}, \ldots, b_{22} \) which are given explicitly in terms of \( \{n_1, n_2, \mu\} \) (in the proof below), but whose exact expression is irrelevant in our present argument.

**Proof.** To prove (3.2.5), we return to the operator \( \Delta_{x,y} \) given by (3.1.10), and express it exclusively in terms of normal and tangential derivatives, \( \partial/\partial n \) and \( \partial/\partial \tau \), rather than derivatives in \( x \) and \( y \). We shall use the following identities:

\[
\frac{\partial}{\partial x} = n_1 \frac{\partial}{\partial n} - n_2 \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial y} = n_2 \frac{\partial}{\partial n} + n_1 \frac{\partial}{\partial \tau},
\]

\[
\frac{\partial^2}{\partial x^2} = n_1^2 \frac{\partial^2}{\partial n^2} + n_2^2 \frac{\partial^2}{\partial \tau^2} - 2n_1 n_2 \frac{\partial^2}{\partial n \partial \tau} + \text{lot},
\]

\[
\frac{\partial^2}{\partial y^2} = n_2^2 \frac{\partial^2}{\partial n^2} + n_1^2 \frac{\partial^2}{\partial \tau^2} + 2n_1 n_2 \frac{\partial^2}{\partial n \partial \tau} + \text{lot},
\]

\[
\frac{\partial^2}{\partial y \partial x} = n_1 n_2 \frac{\partial^2}{\partial n^2} - n_1 n_2 \frac{\partial^2}{\partial \tau^2} + (n_1^2 - n_2^2) \frac{\partial^2}{\partial n \partial \tau} + \text{lot}.
\]

Identities (3.2.9) are given, e.g., in [11, p. 299]. From these, one readily obtains (3.2.10)--(3.2.12). Next, we return to the definition of \( \Delta_{x,y} \) in (3.1.10), as applied, say, to \( W \) given by (3.1.6) in terms of its coordinates \( w_1 \) and \( w_2 \), substitute (3.2.10)--(3.2.12), and obtain

\[
\text{top row of } [\Delta_{x,y} W] = \frac{\partial^2 w_1}{\partial x^2} + \frac{1 - \mu}{2} \frac{\partial^2 w_1}{\partial y^2} + \frac{1 + \mu}{2} \frac{\partial^2 w_2}{\partial y \partial x}
\]

\[
= \left[ n_1^2 + \left( \frac{1 - \mu}{2} \right) n_2^2 \right] \frac{\partial^2 w_1}{\partial n^2} + \left[ n_1 n_2 \left( \frac{1 + \mu}{2} \right) \right] \frac{\partial^2 w_2}{\partial n^2}
\]

\[
+ \left[ n_2^2 + \left( \frac{1 - \mu}{2} \right) n_1^2 \right] \frac{\partial^2 w_1}{\partial \tau^2} + \left[ -n_1 n_2 \left( \frac{1 + \mu}{2} \right) \right] \frac{\partial^2 w_2}{\partial \tau^2}
\]

\[
+ \left[ -n_1 n_2 (1 + \mu) \right] \frac{\partial^2 w_1}{\partial n \partial \tau} + \left[ \frac{n_1^2 - n_2^2}{2} (1 + \mu) \right] \frac{\partial^2 w_2}{\partial n \partial \tau} + \text{lot},
\]

\[
\text{bottom row of } [\Delta_{x,y} W] = \frac{1 - \mu}{2} \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} + \frac{1 + \mu}{2} \frac{\partial^2 w_1}{\partial y \partial x}
\]

\[
= \left[ n_1 n_2 \left( \frac{1 + \mu}{2} \right) \right] \frac{\partial^2 w_1}{\partial n^2} + \left[ \frac{1 - \mu}{2} \right] n_1^2 + n_2^2 \right] \frac{\partial^2 w_2}{\partial n^2}.
\[
+ \left[ -n_1 n_2 \left( \frac{1 + \mu}{2} \right) \right] \frac{\partial^2 w_1}{\partial \tau^2} + \left[ \frac{1 - \mu}{2} n_2^2 + n_1^2 \right] \frac{\partial^2 w_2}{\partial \tau^2} \\
+ \left[ n_1^2 - n_2^2 \left( \frac{1 + \mu}{2} \right) \right] \frac{\partial^2 w_1}{\partial n \partial \tau} + \left[ n_1 n_2 (1 + \mu) \right] \frac{\partial^2 w_2}{\partial n \partial \tau} + \text{lot.} \quad (3.2.14)
\]

Thus, returning to (3.2.3), we rewrite it as

\[
\rho \Delta_y(\phi W) = -F + (\phi W)_{tt} \quad \text{in} \quad (0, \infty) \times \tilde{\Omega}. \quad (3.2.15)
\]

Hence, using identities (3.2.13), (3.2.14) on the right side of (3.2.15), with the argument \(W\) replaced by \((\phi W)\), we obtain

\[
\rho \left[ n_1^2 + \frac{1 - \mu}{2} n_2^2 \right] \frac{\partial^2 (\phi w_1)}{\partial n^2} - \left[ \frac{1 - \mu}{2} n_1^2 + n_2^2 \right] \frac{\partial^2 (\phi w_2)}{\partial n^2} = -F + (\phi W)_{tt} + P O_2(\phi W) + \frac{\partial}{\partial n} P O_1(\phi W) + \text{lot}, \quad (3.2.16)
\]

recalling (3.2.7), (3.2.8). Thus, (3.2.16) shows (3.2.5), by invoking the definition (3.2.6) of the nonsingular matrix \(M\) and the property that \(\rho > 0\). Lemma 3.2.3 is proved. \(\square\)

Next, we complete the proof of property (P1), which refers to the time localization of Eq. (3.1.8a) = (3.2.3), not to (3.1.8a) itself. Let \(\chi(t)\) be a \(C^\infty\) time-dependent function, which is identically 1 on \([\alpha, T - \alpha]\), and which vanishes outside \([\alpha/2, T - \alpha/2]\). We then set

\[
W_c = \chi W, \quad w_c = \chi w, \quad \chi = \begin{cases} 
1 & \alpha \leq t \leq T - \alpha, \\
0 & t < \alpha/2, \ t > T - \alpha/2. 
\end{cases} \quad (3.2.17)
\]

Instead of multiplying Eq. (3.1.8a) = (3.2.3) by \(\chi(t)\) and repeating the argument of Lemma 3.2.3, we prefer to multiply Eq. (3.2.5) by \(\chi(t)\). This way, by (3.2.17), since \(\partial^1\) in (3.1.7) is only an operator in the space variables, we obtain

\[
\frac{\partial^2 (\phi W_c)}{\partial n^2} = \rho^{-1} M^{-1} \left\{ -F(\partial^1(\phi_1 W_c), \partial^1(\phi_1 w_c)) + (\phi W_c)_{tt} \\
+ P O_2(\phi W_c) + \frac{\partial}{\partial n} P O_1(\phi W_c) \right\} + \text{lot}, \quad (3.2.18)
\]

where in \text{lot} in (3.2.18) we have now included also a new main term—over the \text{lot} in (3.2.5)—given by the time commutator \([\partial^2/\partial t^2, \chi](\phi W)\), which is first order in \(t\). Then (3.2.18) establishes property (P1), as desired.

\textbf{Step 2. Proof of property (P2).} Property (P2) is established by the following
Lemma 3.2.4. Let $0 < \mu < 1$ as in (1.2). Let $(\phi W)$ be a smooth solution of Eq. (3.2.3). Then the following identity holds true on the boundary $\partial \tilde{\Omega}$:
\[
\left[ \frac{\partial (\phi w_1)}{\partial n}, \frac{\partial (\phi w_2)}{\partial n} \right] = \frac{\partial (\phi W)}{\partial n} = N^{-1} \left\{ \begin{bmatrix} B_{1xy}(\phi W) \\ B_{2xy}(\phi W) \end{bmatrix} + \tilde{P}O_1(\phi W) \right\},
\] 
(3.2.19)
where
(i) $N$ is the following nonsingular $2 \times 2$ matrix:
\[
N = \begin{bmatrix} (1 - \mu)\rho^{-1} + \mu n_1 & (1 - \mu)\rho^{-1} + \mu n_2 \\ \tau_1 & \tau_2 \end{bmatrix},
\]
de $\det N = (1 - \mu)\rho^{-1} + \mu > 0$, for $0 < \mu < 1$, (3.2.20)
\[
\tau_1 = -n_2, \quad \tau_2 = n_1 \text{ as in (3.1.6)}, \quad \tau \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \tau_2^2 + \tau_1^2 = 1.
\]
(ii) The operators $B_{i,xy}$, $i = 1, 2$, are given in (3.1.11), (3.1.12).
(iii) $\tilde{P}O_1(\phi W) = \left[ \begin{array}{c} \mu \left[ -n_2 \frac{\partial (\phi w_1)}{\partial \tau} + n_1 \frac{\partial (\phi w_2)}{\partial \tau} \right] \\ n_1 \frac{\partial (\phi w_1)}{\partial \tau} + n_2 \frac{\partial (\phi w_2)}{\partial \tau} \end{array} \right]$.
(3.2.21)

Proof. We return to the definition of $B_{1,xy}$ in (3.1.11), and express $B_{1,xy}$ fully in terms of the normal and tangential derivatives $\partial/\partial n$ and $\partial/\partial \tau$: to this end, all we need is to express this way the last divergence term in (3.1.11):
\[
\text{div}_0 W = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = n_1 \frac{\partial w_1}{\partial n} - n_2 \frac{\partial w_1}{\partial \tau} + n_2 \frac{\partial w_2}{\partial n} + n_1 \frac{\partial w_2}{\partial \tau},
\] 
(3.2.22)
recalling identities (3.2.9). Substituting (3.2.22) in (3.1.11) and using $(\phi W)$ rather than $W$ as an argument yields the top line of the following two identities, while the bottom line is, instead, just a rewriting of $B_{2,xy}$ in (3.1.12):
\[
B_{1,xy}(\phi W) = \left[ (1 - \mu)\rho^{-1} + \mu \right] \left[ n_1 \frac{\partial (\phi w_1)}{\partial n} + n_2 \frac{\partial (\phi w_2)}{\partial n} \right]
\]
\[
\quad + \mu \left[ -n_2 \frac{\partial (\phi w_1)}{\partial \tau} + n_1 \frac{\partial (\phi w_2)}{\partial \tau} \right],
\] 
(3.2.23)
\[
B_{2,xy}(\phi W) = \frac{\rho^{-1}}{2} \left\{ \tau_1 \frac{\partial (\phi w_1)}{\partial n} + \tau_2 \frac{\partial (\phi w_2)}{\partial n} \right. 
\]
\[
\left. + \left[ n_1 \frac{\partial (\phi w_1)}{\partial \tau} + n_2 \frac{\partial (\phi w_2)}{\partial \tau} \right] \right\}.
\] 
(3.2.24)
Thus, (3.2.23), (3.2.24) can be rewritten as the system given by (3.2.19), with matrix $N$ as in (3.2.20) which is nonsingular for $0 < \mu < 1$, and term $\tilde{P}O_1(\phi W)$ given by (3.2.21). Thus, the transformed boundary conditions are noncharacteristic. Lemma 3.2.4 is proved. □
Completion of proof of Proposition 3.2.2. Having established properties (P1) and (P2) in the last two lemmas and (3.2.18), we can then assert that Proposition 3.2.2 is proved on the basis of [9].

\[ \square \]

Step 2. In this step we prove “half” of the desired estimate (3.2.1) of Proposition 3.2.1; namely, the half involving the tangential component of \( \nabla_0(\phi W) \). More precisely, we shall prove

**Proposition 3.2.5.** Let \( (\phi W) \) be a smooth solution of Eq. (3.2.3). For all \( 0 < \epsilon < 1/2 \) and \( 0 < \alpha < T/2 \), there exists \( C_{\epsilon \alpha} > 0 \) such that the following estimate holds true:

\[
T^{-\alpha} \int_{\tilde{\Gamma}_1} \int_{\alpha} |\nabla_0(\phi W) \cdot \tau|^2 d\tilde{\Sigma}_{1,\alpha} \\
\leq C_{\epsilon \alpha} \int_0^T \left[ |W_t|^2_{0,\tilde{\Gamma}_1} + |w|^2_{1,\tilde{\Omega}} + |W|^2_{1-\epsilon,\tilde{\Omega}} \right] dt \\
+ C_{\epsilon \alpha} \int_{\tilde{\Sigma}_1} \left[ |g_1((W_t, n))|^2 + |g_2((W_t, \tau))|^2 \right] d\tilde{\Sigma}_1. \tag{3.2.25}
\]

**Proof.** Our starting point is estimate (3.2.4) in Proposition 3.2.2. In its right side, we substitute the feedback expressions given by (3.1.8e), (3.1.8f) for the boundary operators \( B_{1xy} \), thus obtaining

\[
T^{-\alpha} \int_{\tilde{\Gamma}_1} \int_{\alpha} |\nabla_0(\phi W) \cdot \tau|^2 d\tilde{\Sigma}_{1,\alpha} \\
\leq C_{\epsilon \alpha} \int_0^T \left[ |W_t|^2_{0,\tilde{\Gamma}_1} + |f_1((\phi_1 W), g_1((W_t, n)), (\phi_1 w))|^2_{0,\tilde{\Gamma}_1} \\
+ |f_2((\phi_1 W), g_2((W_t, \tau)), (\phi_1 w))|^2_{0,\tilde{\Gamma}_1} \\
+ |F|^2_{-1/2,\tilde{\Omega}} + |W|^2_{1-\epsilon,\tilde{\Omega}} \right] dt. \tag{3.2.26}
\]

We now recall that \( f_1, f_2 \), as well as \( F \) given by (3.2.2), are all functions which are linear combinations of their arguments, by means of smooth functions. Thus, we can estimate \( F \) in (3.2.2) recalling (3.1.7) for \( \partial^1 \)-conservatively as

\[
|F(t)|_{-1/2,\tilde{\Omega}} \leq C[|w(t)|_{1,\tilde{\Omega}} + |W(t)|_{1-\epsilon,\tilde{\Omega}}], \quad 0 < \epsilon < \frac{1}{2}. \tag{3.2.27}
\]
and apply trace theory to the terms $|\langle \varphi_1 W \rangle|_0, \tilde{r}_1$ and $|\langle \varphi_1 w \rangle|_0, \tilde{r}_1$ coming from $f_i$, to obtain (3.2.25) from (3.2.26) and (3.2.27), as desired. □

**Remark 3.2.1.** Estimate (3.2.25), when applied to the *homogeneous system of dynamic elasticity*, states that the traces of the tangential derivatives of $W$ are bounded by the traces of velocity modulo lower-order terms. A result of similar nature was obtained first for the classical wave equation in [8].

**Step 3.** In this step, we prove the second “half” of the desired final estimate (3.2.1) of Proposition 3.2.1; namely, the half involving the normal component of $\nabla_0(\phi W)$. More precisely, we shall prove

**Proposition 3.2.6.** Let $(\phi W)$ be a smooth solution of Eq. (3.2.3). For all $0 < \epsilon < 1/2$ and $0 < \alpha < T/2$, there exists $C_{\epsilon\alpha} > 0$ such that the following estimates hold true:

(i)  
\[
\int_{\tilde{r}_1}^{T-\alpha} \int_{\tilde{\Sigma}_{1,\alpha}} |\nabla_0(\phi W) \cdot n|^2 \, d\tilde{\Sigma}_{1,\alpha} \leq C_{\epsilon\alpha} \int_{\tilde{r}_1}^{T-\alpha} \int_{\tilde{r}_1}^{T} |\nabla_0(\phi W) \cdot \tau|^2 \, d\tilde{\Sigma}_{1,\alpha} + C_{\epsilon\alpha} \int_{\tilde{\Sigma}_1} \left[ |g_1(\langle W_t, n \rangle)|^2 + |g_2(\langle W_t, \tau \rangle)|^2 \right] \, d\tilde{\Sigma}_1.
\]

(ii) From (3.2.25) and (3.2.28), we obtain

\[
\int_{\tilde{r}_1}^{T-\alpha} \int_{\tilde{\Sigma}_{1,\alpha}} |\nabla_0(\phi W) \cdot n|^2 \, d\tilde{\Sigma}_{1,\alpha} \leq C_{\epsilon\alpha} \int_{0}^{T} \left[ |W_t|^2_{0, \tilde{r}_1} + |w|^2_{1, \tilde{\Omega}} + |W|^2_{1-\epsilon, \tilde{\Omega}} \right] \, dt + C_{\epsilon\alpha} \int_{\tilde{\Sigma}_1} \left[ |g_1(\langle W_t, n \rangle)|^2 + |g_2(\langle W_t, \tau \rangle)|^2 \right] \, d\tilde{\Sigma}_1.
\]

**Proof.** (i) Recalling, again, the feedback expressions (3.1.8d), (3.1.8e) for the boundary operators $B_{i,xy}$, and the definition of $\widetilde{P} O_1(\phi W)$ in (3.2.21), we estimate
\[
\begin{align*}
\left[ B_{1xy}(\phi W) \right] + P_0(\phi W) & \bigg|_{0, \tilde{\Gamma}_1} \\
\left[ B_{2xy}(\phi W) \right] + \tilde{\Gamma}_1 & \bigg|_{0, \tilde{\Gamma}_1} \\
& = f_1((\varphi_1 W), g_1((W_t, n)), (\varphi_1 w)) \\
& + \left[ \begin{array}{c}
-n_2 \frac{\partial (\varphi w_1)}{\partial \tau} + n_1 \frac{\partial (\varphi w_2)}{\partial \tau}
\end{array} \right] \bigg|_{0, \tilde{\Gamma}_1} \\
& + f_2((\varphi_1 W), g_2((W_t, \tau)), (\varphi_1 w)) \\
& + \left[ \begin{array}{c}
n_1 \frac{\partial (\varphi w_1)}{\partial \tau} + n_2 \frac{\partial (\varphi w_2)}{\partial \tau}
\end{array} \right] \bigg|_{0, \tilde{\Gamma}_1} \\
& \leq \left\{ \begin{array}{c}
|\nabla_0(\phi W) \cdot \tau|_{0, \tilde{\Gamma}_1} + |w|_{1, \Omega} + |W|_{1-\epsilon, \Omega} + |g_1((W_t, n))|_{0, \tilde{\Gamma}_1} \\
+ |g_2((W_t, \tau))|_{0, \tilde{\Gamma}_1}
\end{array} \right\}. \\
\end{align*}
\] (3.2.30)

In going from (3.2.30) to (3.2.31), we have recalled once more that \( f_1 \) and \( f_2 \) are linear combinations of their arguments by means of smooth functions, and we have used trace theory on \( |(\varphi_1 w)|_{0, \tilde{\Gamma}_1} \) and \( |(\varphi_1 W)|_{0, \tilde{\Gamma}_1} \). Inserting estimate (3.2.30) into the right side of (3.2.19) and integrating over \((\alpha, T - \alpha) \times \tilde{\Gamma}_1\) yields estimate (3.2.28), as desired.

(ii) We substitute the tangential component of \( \nabla_0(\phi W) \), given by (3.2.25), into the right side of inequality (3.2.28), and finally obtain estimate (3.2.29). Proposition 3.2.6 is proved. \( \square \)

**Step 4.** We now combine the last two propositions to obtain the desired final estimate of \( \nabla_0(\cdot) \) on the boundary, at least in the argument \((\phi W)\).

**Proposition 3.2.7.** Let \((\phi W)\) be a smooth solution of Eq. (3.2.3). For all \( 0 < \epsilon < 1/2 \) and \( 0 < \alpha < T/2 \), there exists \( C_{\epsilon \alpha} > 0 \) such that the following estimate holds true:

\[
\begin{align*}
\int_\alpha^{T - \alpha} \int_{\tilde{\Gamma}_1} |\nabla_0(\phi W)|^2 \, d\tilde{\Sigma}_{1\alpha} & \\
& \leq C_{\epsilon \alpha} \int_{\tilde{\Sigma}_1} \left[ |W_t|^2 + |g_1((W_t, n))|^2 + |g_2((W_t, \tau))|^2 \right] \, d\tilde{\Sigma}_1 \\
& + C_{\epsilon \alpha} \int_0^T \left[ |w(t)|_{1, \tilde{\Omega}}^2 + |W(t)|_{1-\epsilon, \tilde{\Omega}}^2 \right] \, dt. \\
& \end{align*}
\] (3.2.32)
Proof. We combine estimate (3.2.25) of Proposition 3.2.5 with estimate (3.2.29) of Proposition 3.2.6, to obtain estimate (3.2.32) at once.

Step 5. Conclusion of the proof of Theorem 3.2.1. Recalling the partition of unity (3.1.4), we add all the estimates (3.2.32) for all finitely many \( \phi \) and obtain the desired estimate (3.2.1) of Proposition 3.2.1.

3.3. Trace regularity for normal component \( w \)

Our next result deals with the improved trace regularity for the normal displacement \( w \).

Proposition 3.3.1. Let \( W,w \) be a finite energy solution to problem (1.1) as guaranteed by Theorem 1.1. Then, there is a constant \( C_{\alpha T\epsilon} > 0 \) such that

\[
T^{-\alpha} \int_0^T \int_{\Gamma_1} |D^2 w|^2_{T^2} d\Sigma_{1\alpha} \leq C \left\{ \int_{\Sigma_1} \left[ |Dw_1|^2 + \left| h_1 \left( \frac{\partial w_1}{\partial n} \right) \right|^2 + \left| h_2 \left( \frac{\partial w_1}{\partial \tau} \right) \right|^2 \right] d\Sigma_1 \right. \\
+ \int_0^T \left[ |w(t)|^2_{2-\epsilon,\Omega} + |w_1(t)|^2_{1-\epsilon,\Omega} + |W(t)|^2_{1-\epsilon,\Omega} \right] dt \right\}, 
\]  

(3.3.1)

Proof. As before, it suffices to prove this estimate locally in a layer of the boundary \( \Gamma \) of \( \Omega \) for solutions supported in supp of \( \phi \).

Step 1. Define

\[
\tilde{G}(x,y,t) \equiv \tilde{G}(\partial^1(\varphi_1 W), \partial^3(\varphi_1 w)),
\]

(3.3.2)

\[
f_3(x,y,t) \equiv f_3 \left( \partial^1(\varphi_1 w), h_1 \left( \frac{\partial w_1}{\partial n} \right) \right),
\]

(3.3.3)

\[
f_4(x,y,t) \equiv f_4 \left( \partial^2(\varphi_1 w), \frac{\partial w_1}{\partial \tau}, h_2 \left( \frac{\partial}{\partial \tau} w_1 \right) \right),
\]

(3.3.4)

in the notation (3.1.7) for \( \partial^\ell \) as a general differential operator of order \( \ell \) in the space variables \( x \) and \( y \), where \( \tilde{G} \), as well as \( f_3 \) and \( f_4 \), are linear combinations of their arguments, by means of smooth functions. As such, \( \tilde{G} \), \( f_3 \), \( f_4 \) are Nemytski operators. With the above notation, the variable \( (\phi w) \) satisfies Eq. (3.1.8b), along with the corresponding boundary conditions (3.1.8f) and (3.1.8g). We rewrite here the corresponding linear Kirchhoff plate problem in \( (\phi w) \), where the right side
operator \( G \) in (3.1.8b) is rewritten here in the form

\[
G = \tilde{G} + \mathcal{O}((\partial/\partial t)^1(\phi_1w_t)),
\]

— with \( \tilde{G} \) as in (3.3.2)—to distinguish it between space and time derivatives. Thus, \( (\phi w) \) satisfies

\[
\begin{cases}
(\phi w)_{tt} - \gamma \rho \Delta_0 (\phi w)_{tt} + \rho^2 \gamma \Delta_0^2 (\phi w) = G \\
= \tilde{G} + \mathcal{O}(\partial/\partial t)^1(\phi_1w_t), \\
\frac{\partial^2}{\partial n^2} (\phi w) + \mu \frac{\partial^2}{\partial \tau^2} (\phi w) = f_3, \\
\frac{\partial^3}{\partial n^3} (\phi w) + \frac{\partial}{\partial n} \frac{\partial^2}{\partial \tau^2} (\phi w) + (1 - \mu) \frac{\partial^2}{\partial \tau^2} \frac{\partial}{\partial n} (\phi w) \\
- \gamma \frac{\partial}{\partial n} (\phi w)_{tt} = f_4,
\end{cases}
\]

where \( f_3 \) and \( f_4 \) are given by (3.1.8f) and (3.1.8g). To problem (3.3.5a), (3.3.5b) we apply [10, p. 279, Theorem 2.1], which gives an improved trace regularity estimate. We obtain

\[
\begin{align*}
T^{-\alpha} &\int_{\tilde{\Sigma}_1} \int_{\tilde{\Sigma}_1} |D^2(\phi w)|^2 d\tilde{\Sigma}_1 \\
&\leq C \int_{\tilde{\Sigma}_1} \int_{\tilde{\Sigma}_1} \left[ \left| \frac{\partial^2 (\phi w)}{\partial \tau^2} \right|^2 + \left| \frac{\partial^2 (\phi w)}{\partial n^2} \right|^2 + \left| \frac{\partial^2 (\phi w)}{\partial \tau \partial n} \right|^2 \right] d\tilde{\Sigma}_1, \\
&\leq C \left\{ \int_{\tilde{\Sigma}_1} \left| \nabla_0 w_t \right|^2 d\tilde{\Sigma}_1 + \int_0^T \left[ |\tilde{G}(t)|^2_{(H^{3/2}(\Omega))} + |\partial^1(\phi(w_t))(t)|^2_{-1/2+\epsilon,\tilde{\Omega}} \right. \\
&\left. + |w(t)|^2_{2-\epsilon,\tilde{\Omega}} \right] dt + |f_3|^2_{0,\tilde{\Sigma}_1} + |f_4|^2_{-1,\tilde{\Sigma}_1} \right\},
\end{align*}
\]

recalling the boundary conditions (3.3.5b). The first estimate in (3.3.6) follows from the definition of norm of \( D^2 w(\cdot, \cdot) \) given in (A.1) and (A.2).

**Remark 3.3.1.** Estimate (3.3.7) for the Kirchhoff plate problem (3.3.5) is lifted from [11, p. 279, Theorem 2.1], except that we are giving in (3.3.7) a sharper result for the right-hand side \( G \) in (3.3.5a). This calls for an explanation. The result given in [11, Theorem 2.1] penalizes the “right-hand side” term \( G \) in the norm of \( H^{-s}, s < 1/2 \), in time and space. However, a sharper regularity result may be given on \( G \), if one distinguishes between tangential and normal directions. (This explains why we have rewritten \( G \) as \( G = \tilde{G} + \mathcal{O}((\partial/\partial t)^1(\phi_1w_t)) \), to distinguish between space and time derivatives.) More precisely, restrictions on the regularity of the forcing term \( G \) in (3.3.5) are dictated by the applicability of elliptic theory in an “elliptic sector” (see [10]). In our case, the Kirchhoff plate equation (3.3.5a) is supplemented with the free boundary conditions (3.3.5b) and (3.3.5c). If \( L \) denotes the realization in \( L^2(V) \) of \( \Delta_0^2 \) with the free B.C. (3.3.5b) and (3.3.5c),
then \( L \) is positive, self-adjoint, and \( \mathcal{D}(L^{s/4}) = H^s(V) \), \( s < 5/2 \) [11, Chapter 3, Appendix B]. Thus, we have been conservative in (3.3.7) on \( \tilde{G} \). Moreover, the microlocal argument in [10] is applied to functions with compact support in \([0, T]\) in time: thus, all boundary terms produced by integration by parts in time vanish. In conclusion, the term \( G \) in (3.3.5a), when viewed as forcing term on the right-hand side of the elliptic equation in an “elliptic sector,” may be allowed to lie in a space which is dual to \( H^s(\tilde{\Omega}) \). Thus, our estimate for \( G \) in (3.3.7) is still conservative, though explicitly not contained in [11, Theorem 2.1].

**Step 2.** We estimate next the contribution of the terms \( \tilde{G}, f_3, f_4 \) in (3.3.7).

**Lemma 3.3.2.** Let \( \{W, w\} \) be a regular solution of problem (1.1). Then, for all \( \epsilon < 1/2 \) the following estimates hold:

(i) \[
|\tilde{G}(t)|_{H^{3/2}(\tilde{\Omega})} \leq C |w(t)|_{2-\epsilon, \tilde{\Omega}} + |W(t)|_{1-\epsilon, \tilde{\Omega}} + |f_3(t)|_{0, \tilde{\Gamma}_1},
\]

(ii) \[
|f_3|_{0, \tilde{\Sigma}_1} \leq C \left[ h_1 \left( \frac{\partial}{\partial n} w_i \right) \right]_{0, \tilde{\Sigma}_1} + \left[ \int_0^T |w(t)|_{2-\epsilon, \tilde{\Omega}}^2 dt \right]^{1/2},
\]

(iii) \[
|f_4(t)|_{-1, \tilde{\Sigma}_1}^2 \leq C \int_0^T \left[ h_2 \left( \frac{\partial w_i(t)}{\partial \tau} \right) \right]_{0, \tilde{\Gamma}_1}^2 + \left| h_1 \left( \frac{\partial w_i(t)}{\partial n} \right) \right|^2
+ |w_i|_{0, \tilde{\Gamma}_1}^2 + |w|^2_{2-\epsilon, \tilde{\Omega}} dt.
\]

**Proof.** (i) For the proof of (3.3.8), we shall use duality. Let \( \psi \in H^{3/2}(\tilde{\Omega}) \). Then, using the definition (3.3.2) of \( \tilde{G} \) (as a linear combination of its arguments by means of smooth functions), we obtain

\[
\left| \left( \tilde{G}(t), \psi \right)_{\tilde{\Omega}} \right| \leq C \left| \left( \partial^1 (\varphi_1 W), \psi \right)_{\tilde{\Omega}} \right| + \left| \left( \partial^3 (\varphi_1 w), \psi \right)_{\tilde{\Omega}} \right|
\leq C |W|_{1-\epsilon, \tilde{\Omega}} |\psi|_{\epsilon, \tilde{\Omega}} + \left| \left( \partial^3 (\varphi_1 w), \psi \right)_{\tilde{\Omega}} \right|.
\]
Regarding the first term in (3.3.11), writing $\partial^1 = \text{div}_{0,x,y} + l0$, and applying the divergence theorem and accounting for the boundary conditions (3.3.5b), we obtain, recalling $\text{supp} \varphi_1 \subset \text{supp} \psi$ by Lemma 3.1.2 below (3.1.9),

$$
(\partial^3 (\varphi_1 w), \psi)_{\tilde{\Omega}} \\
\leq C \left| (\partial^2 (\varphi_1 w), \partial^1 \psi)_{\tilde{\Omega}} \right| + \left| \left( \frac{\partial^2}{\partial n^2} (\varphi_1 w), \psi \right)_{\tilde{\Gamma}_1} \right|
$$

(by (3.3.5b))

$$
\leq C \left\{ \left| (\partial^2 (\varphi_1 w), \partial^1 \psi)_{\tilde{\Omega}} \right| + \left| \left( \frac{\partial^2}{\partial \tau^2} (\varphi_1 w), \psi \right)_{\tilde{\Gamma}_1} \right| + \left| (f_3, \psi)_{\tilde{\Gamma}_1} \right| \right\}
$$

$$
\leq C |\varphi_1 w|_{2-\epsilon, \tilde{\Omega}} |\partial^1 \psi|_{\epsilon, \tilde{\Omega}} + C |\varphi_1 w|_{1, \tilde{\Gamma}_1} |\psi|_{1, \Gamma_1} + C |f_3|_{0, \tilde{\Gamma}_1} |\psi|_{0, \Gamma_1}
$$

$$
\leq C \left[ |w|_{2-\epsilon, \tilde{\Omega}} |\psi|_{3/2, \tilde{\Omega}} + |f_3|_{0, \Gamma_1} |\psi|_{0, \Gamma_1} \right].
$$

(3.3.12)

where in the last steps we have also used trace theory on $|(\varphi_1 w)|_{1, \tilde{\Gamma}_1}$ and $|\psi|_{0, \tilde{\Gamma}_1}$.

Combining (3.3.11) and (3.3.12) proves the first estimate (3.3.8) in part (i).

(ii) As to the proof of the second part, estimate (3.3.9), the argument is simpler and uses only trace theory: recalling $f_3$ in (3.3.3), we have

$$
|f_3|_{0, \tilde{\Sigma}_1} \leq C |\partial^1 (\varphi_1 w)|_{0, \tilde{\Sigma}_1} + C \left[ h_1 \left( \frac{\partial w_t}{\partial n} \right) \right]_{0, \tilde{\Sigma}_1}
$$

$$
\leq C \left[ h_1 \left( \frac{\partial w_t}{\partial n} \right) \right]_{0, \tilde{\Sigma}_1} + \left[ \int_0^T |w|^2_{2-\epsilon, \tilde{\Omega}} \, dt \right]^{1/2},
$$

(3.3.13)

and (3.3.9) in part (ii) is proved.

(iii) We now prove estimate (3.3.10). Recalling $f_4$ in (3.3.4), we estimate

$$
|f_4(t)|_{-1, \tilde{\Sigma}_1} \leq C \left[ |\partial^2 (\varphi_1 w)|_{-1, \tilde{\Sigma}_1} + |w_{tt}|_{-1, \tilde{\Sigma}_1} + \frac{\partial}{\partial \tau} h_2 \left( \frac{\partial w_t}{\partial \tau} \right) \right]_{-1, \tilde{\Sigma}_1}
$$

$$
\leq C \left[ |\partial^2 (\varphi_1 w)|_{-1, \tilde{\Sigma}_1} + |w_t|_{0, \tilde{\Sigma}_1} + h_2 \left( \frac{\partial w_t}{\partial \tau} \right) \right]_{0, \tilde{\Sigma}_1}
$$

$$
\leq C \left[ \frac{\partial^2}{\partial n^2} (\varphi_1 w) \right]_{-1, \tilde{\Sigma}_1} + |\partial^1 (\varphi_1 w)|_{0, \tilde{\Sigma}_1} + h_2 \left( \frac{\partial w_t}{\partial \tau} \right)_{0, \tilde{\Sigma}_1}
$$

$$
+ |w_t|_{0, \tilde{\Sigma}_1}
$$

$$
\leq C \left[ \frac{\partial^2}{\partial n^2} (\varphi_1 w) \right]_{-1, \tilde{\Sigma}_1} + \left[ \int_0^T |w|^2_{2-\epsilon, \tilde{\Omega}} \, dt \right]^{1/2}
$$
The second normal derivatives terms in (3.3.14) can be eliminated by using boundary conditions for the boundary moments: to this end, we invoke (3.3.5b) and recall that \( \text{supp } \varphi_1 \subset \text{supp } \varphi \) (see below (3.1.9)). We thus obtain via (3.3.3)

\[
\left| \frac{\partial^2}{\partial n^2} (\varphi_1 w) \right|_{-1, \tilde{\Gamma}_1} \leq C \left[ \left| \frac{\partial^2}{\partial \tau^2} \varphi_1 w \right|_{-1, \tilde{\Gamma}_1} + \left| f_3 \left( \partial^1 (\varphi_1 w), h_1 \left( \frac{\partial w_i}{\partial n} \right) \right) \right|_{-1, \tilde{\Gamma}_1} \right] \tag{3.3.15}
\]

\[
\leq C \left[ |\varphi_1 w|_{1, \tilde{\Gamma}_1} + |\partial^1 (\varphi_1 w)|_{0, \tilde{\Gamma}_1} + \left| h_1 \left( \frac{\partial w_i}{\partial n} \right) \right|_{0, \tilde{\Gamma}_1} \right] \tag{3.3.16}
\]

\[
\leq C \left[ |w|_{2-\epsilon, \tilde{\Omega}} + \left| h_1 \left( \frac{\partial w_i}{\partial n} \right) \right|_{0, \tilde{\Gamma}_1} \right], \quad 0 < \epsilon < \frac{1}{2}, \tag{3.3.17}
\]

where \( \varphi_2 \) in (3.3.15) is a smooth function with support larger than the support of \( \varphi_1 \): \( \text{supp } \varphi_1 \subset \text{supp } \varphi_2 \); the passage from (3.3.15) to (3.3.16) is based on crude estimate; finally, the last step from (3.3.16) to (3.3.17), we have used trace theory on \( |w|_{1, \tilde{\Gamma}_1} \) and \( |Dw|_{0, \tilde{\Gamma}_1} \). Substituting (3.3.17) into the right-hand side of (3.3.14) proves estimate (3.3.10) of part (iii). Lemma 3.3.2 is proved. \( \square \)

**Step 3.** We substitute inequalities (3.3.8)–(3.3.10) for \( \tilde{G}, f_3, f_4 \) into the right side of (3.3.7), and sum up all such resulting estimates with respect to the partition of unity as in (3.1.4) for \( w \), and obtain the sought-after estimate (3.3.1). Proposition 3.3.1 is thus proved. \( \square \)

### 3.4. Completeness of proof of Theorem 3.1.1

Finally, we combine estimate (3.2.1) on \( W \) in Proposition 3.2.1 with estimate (3.3.1) on \( w \) in Proposition 3.3.1. This way, we obtain the desired inequality (3.1.3) and hence the desired inequality (3.1.1). Theorem 3.1.1 is thus proved. \( \square \)

### 4. Stabilizability estimate and completion of the proof of Theorem 1.2

The remaining part of the proof of Theorem 1.2 is contained (as a strict subset) in the arguments presented in [19] (see also [32]). For the reader’s convenience, we shall outline the remaining steps.
Step 1. We proceed as in [8, Section 7.3] and [19]. We return to estimate (2.4) of Proposition 2.2, this time over the interval \([\alpha, T - \alpha]\) rather than over \([0, T]\). On the left-hand side, we obtain \((T - 2\alpha)E(T) \leq \int_{\alpha}^{T-\alpha} E(t) \, dt\) by the dissipativity property in (2.1). On the right-hand side, we use: (i) estimate (3.1.1) of Theorem 3.1.1 for the boundary terms \(BT_{bad}[\alpha, T - \alpha]\) over \([\alpha, T - \alpha]\); (ii) identity (2.1) again with \(s = 0, t = T\), to express \(E(0)\) in terms of \(E(T)\) plus boundary terms, where we use \(2g_i(s)s \leq |g_i(s)|^2 + s^2, 2h_i(s)s \leq |h_i(s)|^2 + s^2\) for these boundary terms, as well as (A.1), (A.2) to estimate them by the norms of \(Dw_t\) and \(W_t\). This way, using again (2.4), we readily obtain the counterpart of [19, Lemma 3.3].

**Proposition 4.1.** With reference to the energy in (1.17), the strong solutions of the original dynamics (1.1), guaranteed by Theorem 1.1, satisfy the following estimate for all \(T > 0\) sufficiently large: there is a constant \(C_T > 0\) such that

\[
E(T) + \int_0^T E(t) \, dt \leq C_T \int_{\Sigma_1} \left\{ \|W_t\|^2_{L^2(\Gamma, \Lambda)} + \|Dw_t\|^2_{L^2(\Gamma, \Lambda)} + |g_1((W_t, n))|^2 + |g_2((W_t, \tau))|^2 + \left| h_1\left( \frac{\partial w_t}{\partial n} \right) \right|^2 + \left| h_2\left( \frac{\partial w_t}{\partial \tau} \right) \right|^2 \right\} \, d\Sigma_1 + LOT(W, w),
\]

(4.1)

where, as before, \(LOT(W, w)\) are lower terms (with respect to the energy) as defined in (3.1.2c).

Step 2. Lower-order terms are absorbed, as usual, by a compactness/uniqueness argument. This requires an appropriate unique continuation result (from the boundary). Since the coefficients are time-independent, it is far convenient to require a uniqueness result for the corresponding static problem. The latter is established in [3, Proposition 2.3], by reducing the Cauchy problem to a system of three equations of the fourth order with the same principal part \(\Delta^2\), where \(\Delta\) is the Laplacian on the manifold \(M\). For this latter problem, the result in [28] is then invoked to obtain uniqueness. (One could also use results of [27]). Thus, a by-now standard compactness/uniqueness argument (see [19, Lemma 4.1]) leads to the absorption of lower-order terms. We thus obtain

**Lemma 4.2.** With reference to the \(LOT(W, w)\) in estimate (4.1) for the strong solutions of problem (1.1), there exists \(T > 0\) large enough, so that

\[ LOT(W, w) \equiv \int_0^T \left[ |W(t)|^2_{1-\epsilon,\Omega} + |w(t)|^2_{2-\epsilon,\Omega} \right] dt \]

\[ \leq C_T \int_{\Sigma_1} \left[ |W_t|^2 + |Dw_t|^2 + |g_1((W_t, n))|^2 + |g_2((W_t, \tau))|^2 \right. \]

\[ + \left. \left| h_1 \left( \frac{\partial w_t}{\partial n} \right) \right|^2 + \left| h_2 \left( \frac{\partial w_t}{\partial \tau} \right) \right|^2 \right] d\Sigma_1. \]  

(4.2)

**Step 3.** By combining the results of Proposition 4.1 and Lemma 4.2, we obtain the final stabilization estimate.

**Proposition 4.3.** Let \([W, w]\) be a regular solution to the original system (1.1), as guaranteed by Theorem 1.1. Then, there exists a constant \(T_0 > 0\) such that for any \(T > T_0\), there is a constant \(C_T > 0\) such that the following estimate holds true:

\[ E(0) + E(T) + \int_0^T E(t) dt \]

\[ \leq C_T \int_{\Sigma_1} \left[ |W_t|^2 + |Dw_t|^2 + |g_1((W_t, n))|^2 + |g_2((W_t, \tau))|^2 \right. \]

\[ + \left. \left| h_1 \left( \frac{\partial w_t}{\partial n} \right) \right|^2 + \left| h_2 \left( \frac{\partial w_t}{\partial \tau} \right) \right|^2 \right] d\Sigma_1. \]  

(4.3)

**Step 4.** Our next step is to express the boundary terms in terms of the feedbacks in (1.1d)–(1.1f). To accomplish this, we shall use the growth conditions (1.26) imposed on the nonlinear dissipation terms \(g_i, h_i\) at infinity, together with the consequent properties (1.29) of the “comparison” functions \(\tilde{g}_i, \tilde{h}_i\) which contain information on the growth at the origin. It is only at this point that we use the growth conditions (1.26) imposed on the nonlinear functions \(g_i, h_i\) given in assumption (H.3) = (1.26), together with the construction of the functions \(\tilde{g}_i, \tilde{h}_i\) which capture the behavior of the nonlinearity at the origin.

By splitting the integration on the boundary between “low” and “high” frequencies, and using Jensen’s inequality in the same manner as it is used on [19, pp. 1400–1401], we arrive at the following conclusion.

**Proposition 4.4.** Consider regular solutions of problem (1.1), as guaranteed by Theorem 1.1. Let \(E(t)\) be defined in (1.17). Then there exists \(T > 0\) such that

\[ p(E(T)) + E(T) \leq E(0), \]  

(4.4)

where the monotone function \(p\) is defined constructively in (1.31).
Step 5. The final conclusion of Theorem 1.2 now follows from (4.4) and [19, Lemma 3], which is based on a comparison theorem. □

Appendix A. Middle surface as a Riemann manifold with the induced metric of $\mathbb{R}^3$

That given in (1.1) of Section 1 is the classical shallow shell model, however, in a version produced in [2,3], in which the middle surface is viewed as a Riemann manifold with the induced metric of $\mathbb{R}^3$. This contrasts with the classical approach, where the middle surface is instead the image, under a smooth map, of a two-dimensional connected domain, and therefore described under just one coordinate patch. The Riemann geometric view of the middle surface, while much more flexible and general than the classical approach, requires, however, a heavy differential geometric setting and apparatus. This is given here for the purpose of explaining the quantities entering model (1.1) and providing the necessary background for the analysis of the shell.

A.1. Riemann geometric background and notation

We reproduce here from [2,3] the Riemann geometric background indispensable to obtain and understand model (1.1) of the shell, with middle surface viewed as a Riemann manifold. See [24–26,29,30].

(1) The usual inner product (dot product) in $\mathbb{R}^3$ is denoted by $\langle \cdot, \cdot \rangle$. Let $M$ be a smooth, orientable surface in $\mathbb{R}^3$ with unit normal field $N(x), x \in M$. We view $M$ as a 2-dimensional Riemann manifold with metric induced from $\mathbb{R}^3$, which will be denoted by $g(\cdot, \cdot)$ or $\langle \cdot, \cdot \rangle$, as convenient. For each $x \in M$, $M_x$ denotes the (2-dimensional) tangent space of $M$ at $x$. We denote: the set of all vector fields on $M$ by $X(M)$; these to fall $k$-order tensors on $M$ by $T^k(M)$; and the set of all $k$-forms on $M$ by $\Lambda^k(M)$, respectively, for any nonnegative integer $k$. Then $\Lambda^0(M) = T^0(M) = C^\infty(M) = \mathcal{X}(M)$, the set of all $C^\infty$-functions on $M$; moreover, \begin{equation}
T^1(M) = T(M) = \mathcal{X}(M) = \Lambda(M), \tag{A.0}
\end{equation}
where $\Lambda(M) = \mathcal{X}(M)$ has to be interpreted as the following isomorphism: given $X \in \mathcal{X}(M)$; then $U(Y) = \langle Y, X \rangle, \forall Y \in \mathcal{X}(M)$ determines a unique $U \in \Lambda(M)$.

(2) For each $x \in M$, the $k$-order tensor space $T^k_x$ on $M$ is an inner product space whose inner product is defined as follows. Let $[e_1, e_2]$ be an orthonormal basis of $M_x$. For any two $k$-order tensors $\alpha, \beta \in T^k_x$, $x \in M$, define the inner product of $T^k_x$ by \begin{equation}
\langle \alpha, \beta \rangle_{T^k_x} = \sum_{i_1, \ldots, i_k = 1}^2 \alpha(e_{i_1}, \ldots, e_{i_k})\beta(e_{i_1}, \ldots, e_{i_k}) \quad \text{at } x. \tag{A.1a}
\end{equation}
In particular, for $k = 1$, definition (A.1a) becomes
\[ \langle \alpha, \beta \rangle_{T^k_x} = \langle \alpha, \beta \rangle = g(\alpha, \beta), \quad \forall \alpha, \beta \in M_x, \]
(A.1b)
and we obtain the inner product of $M_x$ induced by $\mathbb{R}^3$.

(3a) Let $\Omega$ be a bounded region of the surface $M$ with boundary $\Gamma$ which is either regular or else empty. By (A.1), $T^k(\Omega)$ are inner product spaces in the following sense: the inner product is defined by
\[ (T_1, T_2)_{T^k(\Omega)} \equiv \int_\Omega \langle T_1, T_2 \rangle_{T^k_x} \, dx, \quad T_1, T_2 \in T^k(\Omega), \]
(A.2)
using (A.1), where $dx$ is the “volume element” of the surface $M$ with respect to its Riemann metric $g$ [30, p. 30].

(3b) We denote by $L^2(\Omega, T^k)$ the space which is the completion of $T^k(\Omega)$ with respect to the inner product (A.2). In particular, via (A.0), we have that $L^2(\Omega, \Lambda) = L^2(\Omega, T)$.

(4) The space $L^2(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the following inner product:
\[ (f, h)_{L^2(\Omega)} = \int_\Omega f(x)h(x) \, dx, \quad f, h \in C^\infty(\Omega). \]
(A.3)

(5) Let $D$ denote the Levi-Civita connection on $M$ in the metric $g$ of $M$ induced from $\mathbb{R}^3$ [24,29,30].

(5a) For a vector field $U \in X(M)$, $DU \in T^2(M)$ denotes the covariant differential of $U$: it is a second-order covariant tensor field (bilinear map) defined by
\[ DU(X, Y) = \langle DY U, X \rangle = DY U(X), \quad \forall X, Y \in M_x, \ x \in M. \]
(A.4)
One similarly defines the second-order covariant tensor field $D^*U \in T^2(M)$ by
\[ D^*U(X, Y) = DU(Y, X), \quad \forall X, Y \in M_x, \ x \in M, \]
(A.5)
so that $D^*U$ is the transpose of $DU$ in $T^2(M)$.

(5b) Let $f \in C^2(M)$. By definition, the Hessian $D^2 f$ of $f$ with respect to the metric $g$ is of second-order tensor defined by
\[ D^2 f(X, X) = \{DX(Df), X\}. \]
(A.6)

(6) For any second-order tensor $T \in T^2(M)$, the trace of $T$ at $x \in M$ is defined by
\[ \text{tr} T = \sum_{i=1}^2 T(e_i, e_i) \in C^\infty(M), \]
(A.7)
where $[e_1, e_1]$ is an orthonormal basis of $M_x$ with respect to $g$. 
(7) Let $D^k$ be the $k$th covariant differential of $f$ in the induced metric $g$ of $M$, where $f \in C^\infty(\Omega)$. Then $D^k f$ is a $k$-order tensor field on $\Omega$ and $\| \cdot \|_{L^2(\Omega,T^k)}$ and $\| \cdot \|_{L^2(\Omega)}$ are the norms induced by the inner products (A.2) and (A.3), respectively. Then, the Sobolev space $H^n(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the norm

$$\| f \|_{H^n(\Omega)} = \left\{ \sum_{k=1}^n \| D^k f \|_{L^2(\Omega,T^k)}^2 + \| f \|_{L^2(\Omega)}^2 \right\}^{1/2}. \quad (A.8)$$

For details on Sobolev spaces on Riemann manifolds, we refer to [31] or [24].

(8) The Sobolev space $H^k(\Omega,\Lambda)$ and its inner product are defined by

$$H^k(\Omega,\Lambda) = \{ U : U \in L^2(\Omega,\Lambda), D^i U \in L^2(\Omega,T^{i+1}), \quad i = 1, \ldots, k \}, \quad (A.9)$$

$$(U, V)_{H^k(\Omega,\Lambda)} = \sum_{i=0}^k (D^i U, D^i V)_{L^2(\Omega,T^{i+1})}, \quad \forall U, V \in H^k(\Omega,\Lambda). \quad (A.10)$$

In particular, $H^0(\Omega,\Lambda) = L^2(\Omega,\Lambda)$.

(9a) The Lie bracket $[X,Y]$ of two vector fields $X$ and $Y$ is the vector field defined by

$$[X,Y](f) = X[Y(f)] - Y[X(f)], \quad (A.11)$$

where $f$ is a $C^2$ function on $M$.

(9b) The exterior derivative

$$d : \Lambda^k(M) \to \Lambda^{k+1}(M) \quad (A.12)$$

is defined as follows [29, p. 53]: Let $\eta \in \Lambda^k(M)$; then

$$d\eta(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} X_i[\eta(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})]$$

$$+ \sum_{i<j} (-1)^{i+j} \eta([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{k+1}), \quad (A.13)$$

where $X_1, \ldots, X_{k+1}$ are vector fields. A caret over a term means that this term is omitted. According to this definition, we have:

(i)

$$f \in \Lambda^0(M) \quad \Rightarrow \quad df(X) = X(f) = \langle Df, X \rangle. \quad (A.14)$$
(ii) \[ \eta \in \Lambda^1(M) \implies d\eta(X, Y) = X[\eta(Y)] - Y[\eta(X)] - \eta([X, Y]). \] (A.15)

(iii) (Local expression of the exterior differential) Let \((\Omega, \phi)\) be a local chart, \(x^1, \ldots, x^n\) the corresponding coordinates on \(\Omega\), \(\{\partial/\partial x^i\}, i = 1, \ldots, n\), the \(n\) vector fields of the natural basis, and \(\{dx^i\}\) the dual basis. The differential \(k\)-form \(\eta\) is written as ([24, p. 63], [29, p. 53])

\[ \eta = \sum_{j_1 < j_2 < \cdots < j_k} a_{j_1 \ldots j_k}(x) dx^{j_1} \wedge \cdots \wedge dx^{j_k}, \] (A.16)

where \(a_{j_1 \ldots j_k}\) are differentiable real-valued functions. Then ([24, p. 66], [29, p. 54]),

\[ d\eta = \sum_{j_1 < j_2 < \cdots < j_k} da_{j_1 \ldots j_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}, \] (A.17)

\(da\) denoting the differential of the scalar function \(a\). From this expression, we see that ([24, p. 67], [26, p. 183], [29, p. 54]),

\[ d(\eta \wedge \xi) = d\eta \wedge \xi + (-1)^k \eta \wedge d\xi, \] (A.18)

and then

\[ d^2 = d \circ d = 0. \] (A.19)

(10) There is a first-order differential operator [24, p. 162]

\[ \delta: \Lambda^{k+1}(\Omega) \to \Lambda^k(\Omega) \] (A.20)

which is the formal adjoint of \(d\), and is characterized by

\[ (d\alpha, \beta)_{L^2(\Omega, \Lambda^{k+1})} = (\alpha, \delta\beta)_{L^2(\Omega, \Lambda^k)} \quad \text{for } \alpha \in \Lambda^k(\Omega), \beta \in \Lambda^{k+1}(\Omega), \] (A.21)

with compact support. We set

\[ \delta = 0 \quad \text{on 0-forms.} \] (A.22)

Formula (A.22) implies

\[ \delta^2 = 0, \] (A.23)

\(\delta\) on 1-forms [24, p. 164]. Let \(X\) be a vector field and \(\omega\) be the 1-form corresponding to \(X\) under the given metric \(g: g(Y, X) = \langle Y, \omega \rangle\); see point (1) above \(\Lambda(M) = \mathcal{X}(M)\). Then [24, p. 164]

\[ \delta\omega = - \text{div} \, X. \] (A.24)
This identity is equivalent to

\[(X, \text{grad} u)_{L^2(M)} = -(\text{div} X, u)_{L^2(M)},\]

and the definition of \( \delta \) as the formal adjoint of \( d \).

Moreover, if \( \omega \) is a 2-form in \( \mathbb{R}^2 \) of the type \( \omega = f \, dx \wedge dy \) [as in (3.1.33), (3.1.34)], then [26, p. 228]

\[\delta_0 \omega = -\frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial y} dx.\]  

(A.25)

(11) The Hodge Laplacian operator on \( k \)-forms

\[\Delta : C^\infty(\Omega, \Lambda^k) \rightarrow C^\infty(\Omega, \Lambda^k)\]  

(A.26)

is defined by ([24, p. 162], [26, p. 226, with opposite sign])

\[-\Delta = (d + \delta)^2 = d\delta \pm \delta d,\]  

(A.27)

recalling (A.23) and (A.19). Consequently,

\[A.28(-\Delta u, v)_{L^2(\Omega, \Lambda^k)} = (du, dv)_{L^2(\Omega, \Lambda^{k+1})} + (\delta u, \delta v)_{L^2(\Omega, \Lambda^{k-1})}\]  

for \( u, v \in C^\infty_0(\Omega, \Lambda^k).\)  

(A.28)

Since \( \delta = 0 \) on \( \Lambda^0(\Omega) \), see (A.22), we have

\[-\Delta = \delta d \quad \text{on} \quad \Lambda^0(\Omega).\]  

(A.29)

A.2. Shell

(a) A shell is a body in \( \mathbb{R}^3 \). Assume that the middle surface of the shell occupies a bounded region \( \Omega \) of the smooth orientable surface \( M \) in \( \mathbb{R}^3 \). Then the shell of thickness \( h > 0 \) (“small”) is defined by

\[S = \left\{ p : p = x + zN(x), \ x \in \Omega, \ -\frac{h}{2} < z < \frac{h}{2} \right\}.\]  

(A.30)

Here \( N(x) \) is the unit normal field on \( \Omega \).

(b) The second fundamental form \( \Pi \) of \( M \) is the 2-covariant tensor, defined by

\[\Pi(X, Y) = \langle \tilde{D}X N, Y \rangle, \quad \forall X, Y \in \mathcal{X}(M),\]  

(A.31)

where \( \tilde{D} \) is the covariant differential of \( \mathbb{R}^3 \) in the usual dot product. For any vector fields \( X, Y \) on \( M \), since the manifold \( M \) is the submanifold of \( \mathbb{R}^3 \), we then obtain

\[\tilde{D}X Y = D_X Y + \langle \tilde{D}X Y, N \rangle N = D_X Y - \Pi(X, Y) N,\]

where \( D \) is the covariant differential of the surface \( M \) with respect to the metric \( g \).

(c) The mean curvature \( H \) of the surface \( M \) and the Gaussian curvature \( \kappa \) are given by ([2, (3.1.31)], [30, p. 142])

\[\kappa = \kappa_1, \kappa_2, \quad H = \text{tr} \Pi = \kappa_1 + \kappa_2,\]  

(A.32)

where \( \kappa_1, \kappa_2 \) are the principal curvatures.
References