On the approximate controllability of the stochastic Maxwell equations

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1. Introduction

The controllability of partial differential equations (PDEs) has been a very active research field since 1960s, with many important contributions (see, e.g. Bensoussan et al., 1993; Komornik, 1994; Lions, 1988a,b; Zuazua, 2006).

Let us consider the PDE in question as an infinite-dimensional dynamical system

\[ \begin{align*}
    u_t &= Lu + f, \\
    u(0) &= u_0
\end{align*} \]  

(1)

with state space \( H \) an appropriate functional space, where \( L \) is a linear operator and \( f \) is an external forcing that may be understood as a control variable. The concept of controllability in time \( T \) may be expressed in terms of the set of reachable states \( R(T; u_0) \), which is defined as

\[ R(T; u_0) = \{ u(T), \text{s.t. } u \text{ is a solution of (1) for some } f \} \]

where of course \( f \) is an acceptable external forcing, which may either be applied in the whole of the spatial domain of the PDE or in a certain part of it. Controllability has a number of different versions.

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The first one is exact controllability which requires that for every initial data \( u_0 \in H \), the set of reachable states \( R(T; u_0) \) coincides with \( H \). The second one is ‘null’ controllability which requires that for every initial data \( u_0 \in H \), the set of reachable states \( R(T; u_0) \) contains 0. One may easily show that for general reversible linear systems, the concept of exact and null controllability are equivalent. A third version is the concept of approximate controllability which states that for every initial data \( u_0 \), the set of reachable states \( R(T; u_0) \) is dense in \( H \), i.e. for every initial condition \( u_0 \), you can go as close as possible to any desired final state \( u(T) \) by the appropriate choice of the forcing function \( f \). While in finite-dimensional systems, the concepts of exact and approximate controllability coincide (see, e.g. Zuazua, 2006), this is not true in infinite-dimensional systems.

The very definition of controllability implies the idea of solution of the PDE as a final condition problem rather than as an initial condition problem. In fact, the problem of controllability is related to the solution of the backward adjoint problem

\[
- \phi_t + L^* \phi = 0, \\
\phi(T) = \tilde{\phi},
\]  

(2)

where by \( L^* \), we denote the adjoint of the operator \( L \). The connection of these two problems is the essence of the famous Hilbert uniqueness method (HUM), proposed by J.-L. Lions. Quoting Lagnese (1991), ‘the theoretical basis of HUM is, roughly speaking, the observation that if one has uniqueness of solutions of a linear evolutionary system in a Hilbert space it is possible to introduce a Hilbert space norm \( \| \cdot \|_F \) based on the uniqueness property in such a way that the dual system is exactly controllable to the dual space \( F' \). The exact controllability problem is thereby transferred to the problem of identifying (or otherwise characterizing) the couple \( F, F' \). The latter is essentially a problem in PDEs when the original evolutionary system is a distributed parameter system; can a priori estimates of \( \| \cdot \|_F \) be obtained in terms of norms in spaces which are both intrinsic to the given problem and which are readily identifiable?’ In particular, let us assume that the non-homogenous backward problem (2) admits a unique solution that attains the value \( \phi(0) \) at \( t = 0 \). We can therefore define the operator \( A \) mapping the data \( u(0) \) of the initial value problem (1) to \( \phi(0) \) and \( A u(0) = \phi(0) \). If we can find spaces where \( A \) is invertible for \( T \) large enough, then through the action of the inverse operator \( A^{-1} \) on the data of the problem, we can obtain appropriate initial conditions for the original system (1), such that the desired control is a functional of its solution. The proper definition of the operator \( A \) and its invertibility require the use of appropriate Hilbert spaces, the exact nature of which is determined by the uniqueness of the solutions of the backward adjoint problem (2). The exact dependence of \( \phi \) on \( u \) is highly problem dependent; concrete examples for certain differential operators such as the wave or the heat operator can be found in Lions (1988b), whereas results for Maxwell equations may be found in, e.g. Pignotti (1999) or Section 3. For a detailed discussion of HUM, see Lions (1988a,b). Through this adjoint problem, one may connect both controllability problems with minimization problems of appropriate functionals. The existence of solutions to these minimization problems is in turn related with certain inequalities, the so-called observability inequalities, for the dynamical system in question, which show that certain observed quantities of the solution (e.g. the integrals over \([0, T]\) of a norm of the restriction of the solution on particular subsets of the domain) uniquely determine the solution of the dynamical system. The observability inequalities are related to continuous dependence questions of the solution on the initial or the final data.

In a number of applications, the systems in question are subject to stochastic fluctuations arising either as a result of uncertain forcing (stochastic external forcing) or uncertainty of the governing laws of the system. A convenient model for such stochastic fluctuations is the Wiener process or its suitable
generalization in infinite-dimensional Hilbert spaces. A general stochastic linear system then assumes
the form
\[ u_t = Lu + f + \mathfrak{B} \dot{W}(t), \]
where \( W(t) \) is the Wiener process on a Hilbert space \( K \) and \( \mathfrak{B} \) is a suitable Hilbert–Schmidt operator
in \( L^2(K, H) \). Of course due to the pathological properties of the Wiener process, the derivative of \( W(t) \)
has to be interpreted in the weak sense, and the integrals have to be interpreted in the appropriate sense,
e.g. in the form of Itô’s stochastic integral (see, e.g. Da Prato & Zabczyk, 1992; Karatzas & Shreve,
1997).

The concept of exact and approximate controllability may be extended under appropriate modifications
to stochastic PDEs (see, e.g. Mahmudov, 2001). Furthermore, the general concept of the HUM
may be extended to the stochastic case. However, some care is needed when considering the generalization
of the backward adjoint problem in the stochastic case. The very nature of the stochasticity imposes
important conceptual and technical difficulties when considering the stochastic version of (2). An impor-
tant qualitative property of solutions of stochastic PDEs, that of the adaptivity of the solution with
respect to the filtration generated by the stochastic process modelling the fluctuations, will in general fail
when considering final value problems of stochastic PDEs, unless some care is taken to express the
problem correctly in terms of the theory of backward stochastic PDEs (BSPDEs). This is essential in
obtaining a correct description of the Pontryagin maximum principle for stochastic systems and being
able to propose feedback-type schemes for the steering of the system to the desired state. An alterna-
tive, so that we can bypass the admittedly interesting, however, rather technical theory of BSPDEs, is
to use the concept of approximate controllability which now is extended not only to drive the system
sufficiently close to the desired state (which is in this case a random variable) in terms of mean square
distance but also to a state which is consistent with the qualitative property of adaptivity of solutions.
This is the approach we adopt in this paper, thus extending the approach of Kim (2004) to the Maxwell
equations.

It is the aim of the present paper to contribute to this literature and in particular to the problem
of controllability of the Maxwell equations under stochastic forcing. The Maxwell equations consti-
tute an interesting problem as they govern the evolution of electromagnetic fields in a variety of media
and allow for important theoretical investigations as well as for important applications. The problem of
controllability for the deterministic Maxwell equations has been addressed by a number of authors in
various publications and by a number of different methods, using either extensions of the HUM or more
specialized techniques based on microlocal analysis or generalizations of Carleman-type inequalities.
Most of the work on the Maxwell equations has been done for dielectric media. Nicaise (2000) has
studied the problem of exact boundary controllability in heterogeneous media using the HUM, through
the use of appropriate energy estimates in function spaces. Weck (2000) has studied a similar problem
that of null controllability of the Maxwell equations by controlling the lateral boundary conditions and
provided interesting connections between the topology of the domain and the controllability of the sys-
tem. Eller & Masters (2002) proved exact controllability results for Maxwell equations with variable
coefficients in bounded domains using duality arguments and observability inequalities for the adjoint
system, removing the geometric restriction on the domain. In a similar vein, Pignotti (1999) considered
boundary and internal controllability of the system using extensions of the HUM. Lagnese (1989) has
also used similar arguments for the problem of boundary controllability. Kapitonov (1994) has treated
the controllability of the system using general Leontovich boundary conditions using appropriate energy
decay estimates. Finally, connections with the problem of stabilization with linear and non-linear bound-
ary feedbacks have been addressed in Eller et al. (2002). Recently, the problem of controllability has
been studied for complex electromagnetic media (e.g. chiral media; see, Courilleau et al. 2006, 2007, and references therein). In Courilleau & Molinaro (2005), the controllability of Maxwell’s equations in media with constitutive relations of the Drude–Born–Fedorov (DBF) type or general linear constitutive relations has been addressed and shown to fail for certain types of constitutive relations. In Courilleau et al. (2007), several issues related to controllability are treated for general non-local dispersive constitutive relations, whereas in Courilleau et al. (2006), the time-harmonic problem, which reduces to elliptic equations, is studied. Similar problems for media modelled by DBF-type constitutive relations have also been treated in Ciarlet & Legendre (2007).

While several versions of the controllability problem for Maxwell equations have been treated in the literature, to the best of our knowledge, only the problem of controllability of the deterministic Maxwell equations has been treated so far. However, noise due to experimental uncertainties, intrinsic randomness of the media, etc. plays an important role in electromagnetics, thus calling for the inclusion of stochastic terms in the Maxwell equations. In this direction, there has been work on the mathematical modelling of electromagnetic media with the use of stochastic PDEs or stochastic integrodifferential equations (see, e.g. Liaskos et al., 2010).

In the present work, we address the question of the approximate controllability of the stochastic Maxwell equations for a bounded domain with the perfect conductor boundary conditions for which we give a positive answer and a construction of the necessary control protocol. Our approach is largely inspired by that of Kim (2004) for the wave equation. The proposed approach may be extended to study the problems of boundary controllability, to include linear complex media in electromagnetics, or the effects of more general noise terms such as spatially correlated noise, Lévy-type noise, etc. The problem of exact controllability requires a different approach based on the theory of BSPDEs. Furthermore, there is interest on related questions for non-linear stochastic Maxwell equations, using methods similar to those described in the recent monograph by Coron (2007). Such studies are currently under investigation by the present authors and will be reported separately.

2. The model and some necessary results

Consider the stochastic Maxwell equations in the following form:

$$d(A\mathcal{E}(t)) = (\mathcal{M}\mathcal{E}(t) + J(t) + Bu(t))dt + \sum_{j=1}^{N} g_j \, dW_j(t),$$

$$\mathcal{E}(0) = \mathcal{E}_0,$$  \hspace{1cm} (3)

where $\mathcal{E} = (E, H)^\text{tr}$ (the superscript tr denotes transposition) is the electromagnetic field and $\mathcal{M}$ is the Maxwell operator, i.e.

$$\mathcal{M} := \begin{pmatrix} \mathbb{O}_{3\times3} & -\text{curl} \\ \text{curl} & \mathbb{O}_{3\times3} \end{pmatrix}. \hspace{1cm} (4)$$

By $W_j(t)$, we denote the components of the $N$-dimensional Wiener process$^1$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We will use the notation $\mathbb{F} = \{\mathcal{F}_t\}$. Furthermore, $g_j = (g_j^{(1)}, g_j^{(2)})^\text{tr} \in \mathbb{R}^6$ is a vector function of the spatial variables multiplying the noise terms such that

$$\text{div}g_j^{(k)} = 0 \quad \text{for } k = 1, 2, \quad j = 1, \ldots, N,$$

$^1$The case where $N = \infty$ may be considered with minor technical modifications.
$J$ is the vector of currents and

$$A = \begin{pmatrix} \varepsilon I_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \mu I_{3\times 3} \end{pmatrix},$$

where $\varepsilon$ is the electric permittivity and $\mu$ is the magnetic permeability of the medium. The model is assumed to hold in a bounded domain $D \subset \mathbb{R}^3$ with sufficiently smooth boundary $\partial D$. The term $u$ is a control field that is assumed to be exercised at the whole of the domain $D$ (to avoid possible complications related to the choice of the minimum time needed for the control procedure), while $B$ is a matrix that quantifies the effect of the control field $u$ on the state of the system. Furthermore, we assume the perfect conductor boundary condition

$$n \times E(t,x) = 0, \quad x \in \partial D,$$

where $n$ denotes the outward unit normal vector to $\partial D$. The electromagnetic field is assumed to be divergence free, i.e.

$$\text{div} E = \text{div} H = 0$$

in $D$.

The problem we wish to address in this paper is the following. **Question** Given an initial condition $\xi_0 \in L^2(\mathbb{P}; \mathcal{F}_0; \mathcal{H})$, where $\mathcal{H}$ is an appropriately chosen Hilbert space, and a given final state $\xi_T \in L^2(\mathbb{P}; \mathcal{F}_T; \mathcal{H})$, can we find an adapted control $u$ such that the system (3) is driven $\varepsilon$-close to the final condition $\xi_T$ in the chosen time period?

The choice of the Hilbert space $\mathcal{H}$ is crucial for our problem. Here, we will choose

$$\mathcal{H} := (\mathbb{H}_0(\text{curl}) \cap \mathbb{H}(\text{div} = 0)) \times (\mathbb{H}(\text{curl}) \cap \mathbb{H}_0(\text{div} = 0)),$$

where

$$\mathbb{H}(\text{curl}) := \{X : D \to \mathbb{R}^3, X \in (L^2(D))^3, \text{curl} X \in (L^2(D))^3\},$$

$$\mathbb{H}_0(\text{curl}) := \{X \in \mathbb{H}(\text{curl}), X \times n = 0 \text{ on } \partial D\},$$

$$\mathbb{H}(\text{div} = 0) := \{X : D \to \mathbb{R}^3, X \in (L^2(D))^3, \text{div} X = 0\},$$

$$\mathbb{H}_0(\text{div} = 0) := \{X \in \mathbb{H}(\text{div} = 0), X \cdot n = 0 \text{ on } \partial D\}.$$ 

Furthermore, we will use the space

$$\mathcal{S} = \mathbb{H}(\text{div} = 0) \times \mathbb{H}(\text{div} = 0).$$

For details on the proper functional setting of the Maxwell equations in the deterministic case, one can consult, e.g. Cessenat (1996) and Monk (2003).

We need the following assumptions:

**Assumption 1** Assume that

1. $\varepsilon$ and $\mu$ are positive constants (hence $A$ is an invertible matrix).
2. $B$ is a positive-definite matrix.
3. $J \in L^2((0,T), L^2(\mathbb{P}, \mathcal{F}) \otimes \mathcal{H})$ (i.e. $\mathbb{E} \left[ \int_0^T \| J(t) \|^2_{\mathcal{H}} \, dt \right] < \infty$).
The solvability of final value problems of the form (2) for stochastic differential equations is not straightforward, on account of problems regarding the adaptivity of the solutions to the filtration $\mathbb{F}$. To bypass such problems (without resorting the theory of BSPDEs), we will approximate the stochastic final value problem with a series of forward and backward deterministic problems, with properly selected random initial and final data and finally with a forward stochastic differential equation. The major technical result, essential in our work, is a generalization of the martingale representation theorem in finite-dimensional spaces (see, e.g. Karatzas & Shreve, 1997) that is summarized in the following lemma from Kim (2004):

**Lemma 1** Given an $\mathcal{F}_T$-measurable random variable $\xi$, for any $\epsilon > 0$, there exists $\tau > 0$ and a $\mathcal{F}_{T-\tau}$-measurable random variable $\xi_\epsilon$ such that for some $\epsilon = \epsilon(\tau) > 0$, $\mathbb{E}[\|\xi - \xi_\epsilon\|^2_{\mathbb{H}}] < \epsilon$ and $\mathbb{E}[\|\xi_\epsilon\|^2_{\mathbb{H}}] \leq \mathbb{E}[\|\xi\|^2_{\mathbb{H}}]$.

**Remark 1** The choice of $\tau$ can be made explicit using the Clark–Ocone form for the martingale representation theorem (Da Prato, 2007), in the case where $\xi$ is Malliavin differentiable; estimates for $\tau$ can be established in terms of the Malliavin derivative of $\xi$.

The following proposition settles the question of mild well posedness of the problem. Further regularity conditions on the solution may be obtained by straightforward modifications of the Faedo–Galerkin method proposed for the wave equation in Kim (2004) (see also Remark 2.4 in Kim, 2004).

**Proposition 1** Under Assumption 1, the stochastic PDE (3) is mildly well posed in $\mathbb{H}$ for every $u \in L^2((0, T), L^2(\mathbb{P}, \mathbb{F}) \otimes \mathbb{H})$.

**Proof.** Let the operator $L = A^{-1}M$ be defined on the domain $D(L) = \mathbb{H}$. This operator is closed and its domain is dense in $\mathbb{H}$. Furthermore, $L$ is skew adjoint, i.e. $L^* = -L$ (see, e.g. Zhao, 2002; Eller & Masters, 2002), and therefore by Stone’s theorem (Engel & Nagel, 2000), $L$ is the generator of a unitary $C_0$ group on $\mathbb{H}$. Therefore, by standard results, Da Prato & Zabczyk (1992), the solvability is ascertained and the solution can be expressed as

$$\mathcal{E}(t) = \exp(Lt)\mathcal{E}(0) + \int_0^t \exp(L(t - s))(J(s) + Bu(s))ds + \sum_{j=1}^N \int_0^t \exp(L(t - s))g_j dW_j(s). \quad (8)$$

The variation of constants formula given in (8) may be used for the necessary estimates in various norms of the solution in terms of the data. \square

### 3. Approximate controllability of the stochastic Maxwell equations

#### 3.1 An abstract approach

One can treat the problem of approximate controllability of the stochastic Maxwell equations in view of an abstract approach in the spirit of Mahmudov (2001). The work of Mahmudov links the controllability (approximate or exact) of abstract linear stochastic systems with additive noise with the relevant notions for the corresponding deterministic system.

**Proposition 2** Under Assumption 1, the control system (3) is approximately controllable.

**Proof.** Based on Theorem 4.1(a) of Mahmudov (2001), it suffices to prove the positivity of the operator $\Pi_0^T: L^2(\mathbb{W}, \mathcal{F}_T) \to L^2(\mathbb{W}, \mathcal{F}_T)$, defined by
\[ \Pi_T \xi \geq C \int_0^T \mathcal{U}(T-t) \mathcal{U}^*(T-t) \mathbb{E}[\xi | \mathcal{F}_t] \, dt \]

and the positivity of \( \Pi_T \) follows by the positivity of the conditional expectation operator. \( \square \)

Eventhough the above result settles the problem of approximate controllability of the stochastic Maxwell equations, it is proved by abstract arguments which do not provide an easy to use insight on the construction of the control procedure which will drive the system to the desired state. It is evident that in applications, a constructive approach is necessary. Furthermore, this constructive approach may be generalized to the case of multiplicative noise or to non-linear generalizations of the Maxwell system. This is beyond the scope of the present paper and will be reported elsewhere.

### 3.2 A constructive approach

In this section, we provide a constructive proof of the above approximate controllability result for the Maxwell equations. Our method is inspired by the related work of Kim (2004) for the stochastic wave equation. For such an approach, we need to introduce certain auxiliary problems, necessary for the use of the ideas of the HUM method, and the construction of the control procedure.

In order to provide a proof of existence for the control, which provides at the same time a constructive scheme, we need to introduce the following auxiliary problems.

In what follows, we assume that Assumption 1 holds. We also assume without loss of generality that \( \epsilon = \mu = 1 \), i.e. that \( A = I_{6 \times 6} \).

#### 3.2.1 The backward adjoint problem

Consider the deterministic problem

\[ - \frac{\partial}{\partial t} (Av) = M^* v, \]

\[ v(T) = a, \quad (9) \]

which is a final value problem for some \( a \in \mathcal{H} \). Here, \( M^* \) is the adjoint of the Maxwell operator in \( D(L) = \mathcal{H} \). We may consider \( a \) to be a random variable (see Remark 2).

For the backward problem (9), we wish to find an estimate of the form

\[ \|a\|_\mathcal{H}^2 \leq C \rho_1(\tau) \int_{T-\tau}^T \|v(t)\|_\mathcal{H}^2 \, dt \quad (10) \]

for some function \( \rho_1 \) of \( \tau \).

**Proposition 3** The solution to the backward problem (9) satisfies an estimate of the form (10) where \( \rho_1(\tau) \) blows up as \( \tau \to 0 \).
Proof. Recall that for the choice of $H$ in this paper, the operator $M$ has the property that $M^* = -M$, so that the backward adjoint system assumes the form

$$\partial_t (Av) = Mv,$$

$$v(T) = \alpha.$$  \hfill (11)

Therefore, we may use results on backward continuation of the Maxwell system. In fact, since the Maxwell operator generates a $C_0$ group, these results are straightforward adaptations of the results for the forward system. Now, let us denote $\alpha = (\alpha_1, \alpha_2)^T$ $v := (v_1, v_2)^T$. By a simple change of variables, we may assume that $\epsilon = \mu = 1$. Then, we have

$$\frac{\partial^2}{\partial t^2}v_i + \text{curl}\text{curl}v_i = 0, \quad i = 1, 2.$$  \hfill (12)

We will use the following known fact (see, e.g. Nédélec, 2001): there exist $C_1 > 0$ and $C_2 > 0$, that depend only on $D$, such that for any $w_1 \in H_0(\text{curl}) \cap H(\text{div} = 0)$, one has

$$\|w_1\|_{H_0(\text{curl}) \cap H(\text{div} = 0)} \leq C_1 \|\text{curl} w_1\|_{L^2(D)}$$  \hfill (13)

and for any $w_2 \in H(\text{curl}) \cap H_0(\text{div} = 0)$, one has

$$\|w_2\|_{H(\text{curl}) \cap H_0(\text{div} = 0)} \leq C_2 \|\text{curl} w_2\|_{L^2(D)}.$$  \hfill (14)

Closely following Haraux (1991), we multiply the identity

$$\frac{\partial^2}{\partial t^2}v_1 + \text{curl}\text{curl}v_1 = 0$$

by $\rho(t)v_1$, where $\rho(t) := (T - \tau - t)^2(T - t)^2$ and integrate over $(T - \tau, T) \times D$. We get

$$\int_{T-\tau}^{T} \int_{D} \rho'(t)v_1 \frac{\partial}{\partial t}v_1 \, dx \, dt + \int_{T-\tau}^{T} \int_{D} \rho(t) \left| \frac{\partial}{\partial t}v_1 \right|^2 \, dx \, dt = \int_{T-\tau}^{T} \int_{D} \rho(t)|\text{curl}v_1|^2 \, dx \, dt.$$

Note that

$$\int_{T-\tau}^{T} \int_{D} \rho'(t)v_1 \frac{\partial}{\partial t}v_1 \, dx \, dt = \int_{T-\tau}^{T} \int_{D} \frac{\rho'}{\sqrt{\rho}} \sqrt{\rho} \, v_1 \frac{\partial}{\partial t}v_1 \, dx \, dt$$

and using standard inequalities, we see that for all $\lambda > 0$

$$\int_{T-\tau}^{T} \int_{D} \rho(t)|\text{curl}v_1|^2 \, dx \, dt \leq \int_{T-\tau}^{T} \int_{D} \rho(t) \left| \frac{\partial}{\partial t}v_1 \right|^2 \, dx \, dt$$

$$+ \frac{\lambda}{2} \sup_{T-\tau \leq t \leq T} \int_{T-\tau}^{T} \int_{D} \frac{\rho'(t)^2}{\rho(t)} \left| \frac{\partial}{\partial t}v_1 \right|^2 \, dx \, dt$$

$$+ \frac{1}{2\lambda} \int_{T-\tau}^{T} \int_{D} \rho(t)|v_1|^2 \, dx \, dt.$$  \hfill (16)
Using then (13), which implies that there exists a constant $C_3$ such that $\|v_1\|_{L^2(D)} \leq C_3 \|\text{curl} v_1\|_{L^2(D)}$ (see, e.g. corollary 3.51 in Monk, 2003) by choosing $\lambda$ large enough

$$\int_{T-\tau}^{T} \int_{D} \rho(t) |\text{curl} v_1|^2 \, dx \, dt \leq C \left( \int_{T-\tau}^{T} \int_{D} \rho(t) \left| \frac{\partial}{\partial t} v_1 \right|^2 \, dx \, dt \right)^{1/2} \left( \sup_{T-\tau \leq t \leq T} \rho(t) \int_{T-\tau}^{T} \left| \frac{\partial}{\partial t} v_1 \right|^2 \, dx \, dt \right)^{1/2},$$

(17)

for some $C$ which does depend only on $D$.

Next, we establish that

$$\int_{T-\tau}^{T} \int_{D} \rho(t) \left( \left| \frac{\partial}{\partial t} v_1 \right|^2 + |\text{curl} v_1|^2 \right) \, dx \, dt = \|(\text{curl} \alpha_1, \text{curl} \alpha_2)\|_{L^2(D)}^2 \int_{T-\tau}^{T} \rho(t) \, dt, \tag{18}$$

Clearly, the same result holds for $v_2$. Let us remark that (18) is formally deducible from the fact that the energy of the Maxwell system (written as described above in the form (12)) is conserved.

Indeed, by multiplying (15) by $\Psi(t) \frac{\partial}{\partial t} v_1$, where $\Psi(t) := \int_{T-\tau}^{T} \rho(t') \, dt'$ and integrating over $[T-\tau, T] \times D$, we get that $I_1 + I_2 = 0$, where

$$I_1 = \int_{T-\tau}^{T} \int_{D} \Psi(t) \left( \frac{\partial}{\partial t} v_1 \right)^2 \, dx \, dt, \quad I_2 = \int_{T-\tau}^{T} \int_{D} \Psi(t) \left( \frac{\partial}{\partial t} v_1 \right) \text{curl} \text{curl} v_1 \, dx \, dt.$$

Observe that $\Psi(t)$ is such that $\Psi(T-\tau) = 0$, $\Psi(T) = \int_{T-\tau}^{T} \rho(t') \, dt'$ and $\Psi'(t) = \rho(t)$. In $I_1$ integrate over $t$ by parts to obtain

$$I_1 = \int_{D} \Psi(T) \left( \frac{\partial}{\partial t} v_1 \right)^2 \bigg|_{t=T} \, dx - \int_{T-\tau}^{T} \int_{D} \Psi'(t) \left( \frac{\partial}{\partial t} v_1 \right)^2 \, dx \, dt - \int_{T-\tau}^{T} \int_{D} \Psi(t) v_1 \frac{\partial^2}{\partial t^2} v_1 \, dx \, dt$$

$$= \Psi(T) \|\text{curl} \alpha_2\|_{L^2(D)}^2 - \int_{T-\tau}^{T} \int_{D} \rho(t) \left| \frac{\partial}{\partial t} v_1 \right|^2 \, dx \, dt - I_1$$

since by Maxwell’s equations, we have that $|\text{curl} v_2|^2 = \left| \frac{\partial}{\partial t} v_1 \right|^2$. This gives that

$$I_1 = \frac{1}{2} \Psi(T) \|\text{curl} \alpha_2\|_{L^2(D)}^2 - \frac{1}{2} \int_{T-\tau}^{T} \int_{D} \rho(t) \left| \frac{\partial}{\partial t} v_1 \right|^2 \, dx \, dt.$$

In $I_2$ integrate by parts over $x$ to obtain

$$I_2 = \int_{T-\tau}^{T} \int_{D} \Psi(t) \left( \frac{\partial}{\partial t} \text{curl} v_1 \right) \text{curl} v_1 \, dx \, dt = \frac{1}{2} \int_{T-\tau}^{T} \Psi(t) \frac{\partial}{\partial t} |\text{curl} v_1|^2 \, dx \, dt,$$

which is then integrated by parts over $t$ to give

$$I_2 = \frac{1}{2} \Psi(T) \|\text{curl} \alpha_1\|_{L^2(D)}^2 - \frac{1}{2} \int_{T-\tau}^{T} \rho(t) |\text{curl} v_1|^2 \, dx \, dt.$$

Then, $I_1 + I_2 = 0$ gives (18).
Adding \( \int_{T-\tau}^{T} \int_{D} \rho(t) |\nabla \psi_1|^2 \, dx \, dt \) on both sides of (17) and using (18), we obtain the inequality

\[
\| (\nabla \alpha_1, \nabla \alpha_2) \|_{L^2(D)}^2 \int_{T-\tau}^{T} \rho(t') \, dt' \leq C \left( \int_{T-\tau}^{T} \int_{D} \left| \frac{\partial}{\partial t} \psi_1 \right|^2 \, dx \, dt \right) + \sup_{T-\tau \leq t \leq T} \frac{\rho'(t)^2}{\rho(t)} \int_{T-\tau}^{T} \left( \int_{D} \left| \frac{\partial}{\partial t} \psi_1 \right|^2 \, dx \right) \right) \\
+ \int_{T-\tau}^{T} \int_{D} \rho(t) |\nabla \psi_1|^2 \, dx \, dt \\
\leq C \int_{T-\tau}^{T} \int_{D} (|\nabla \psi_1|^2 + |\nabla \psi_2|^2) \, dx \, dt
\]

(19)

for some constant \( C \) that depends on the suprema of the functions \( \rho \) and \( \rho' \). In the above, we used the fact that \( \left| \frac{\partial \psi_1}{\partial t} \right|^2 = |\nabla \psi_2|^2 \), as dictated by Maxwell’s equations, and we estimated the terms \( \int_{T-\tau}^{T} \int_{D} \rho(t') |\nabla \psi_1|^2 \, dx \, dt \), \( i = 1, 2 \) by

\[
\int_{T-\tau}^{T} \int_{D} \rho(t') |\nabla \psi_i|^2 \, dx \, dt \leq \sup_{T-\tau \leq t \leq T} \rho(t) \int_{T-\tau}^{T} |\nabla \psi_i|^2 \, dx \, dt, \quad i = 1, 2.
\]

Combining (19) with (13), (14) gives the desired result for \( \rho_1(t) = \left( \int_{T-\tau}^{T} \rho(t') \, dt' \right)^{-1} \). □

**Remark 2** In Proposition 3, we may assume \( \alpha \) to be a random variable \( \alpha \in L^2(\mathbb{P}, \mathcal{F}_{T-\tau}) \). As the backward adjoint problem does not contain a stochastic integral, the proof proceeds a.s. in \( \omega \) in a straightforward manner and then using standard arguments (see, e.g. Kim, 2004), we may pass to the \( L^2(\mathbb{P}, \mathcal{F}_{T-\tau}) \) setting. The same comment applies for Propositions 4 and 5.

### 3.2.2 The forward deterministic problem.

Next, consider the forward deterministic problem

\[
\frac{\partial^2 w}{\partial t^2} + \nabla \nabla w = \hat{B} \frac{\partial v}{\partial t},
\]

\[
w(T - \tau) = 0, \quad \frac{\partial w}{\partial t}(T - \tau) = 0,
\]

(20)

where \( \psi \) is a solution of the backward problem (9) and the matrix \( \hat{B} \) is defined in terms of the matrix \( B \) by

\[
\hat{B} = \Theta B \Theta^{-1} + B,
\]

where

\[
\Theta = \begin{pmatrix}
O_{3 \times 3} & -I_{3 \times 3} \\
I_{3 \times 3} & O_{3 \times 3}
\end{pmatrix}.
\]

**Remark 3** From the properties of the backward problem (9), one can easily see that the right-hand side of (20) can be written in the equivalent form \( -\hat{B} \Theta \nabla \psi \). Furthermore, this equation has an equivalent
first-order formulation in the general form (recall that for simplicity, we have assumed \( A = I_{6 \times 6} \))

\[
\frac{\partial}{\partial t} (A w) = M w + B v + h,
\]

\[
w(T - \tau) = 0,
\]

where \( h \) is a solution of the homogeneous system

\[
\frac{\partial h}{\partial t} = -M h,
\]

\[ h(T - \tau) = -B v(T - \tau). \tag{22} \]

This system is well posed by standard results on the Maxwell system (see, e.g. Cessenat, 1996).

The following estimate for (20) will be needed.

**Proposition 4** Let \( \beta := (w_1(T), w_2(T)) \). Then,

\[
\| \beta \|^2_H \leq C \int_{T-\tau}^T \int_D \sum_{i=1}^2 \left| \frac{\partial v_i}{\partial t} \right|^2 \, dx \, dt.
\]

**Proof.** To avoid unnecessary cumbersome notation, in the proof \( C \) is used as a proxy for a constant, the value of which may vary from estimate to estimate.

In a similar fashion as for the wave equation (see, e.g. Evans, 1998), multiply the equation with \( \frac{\partial w_i}{\partial t} \) and integrate over \( D \). This gives

\[
\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial w_i}{\partial t} \right\|_{L^2(D)}^2 + \frac{1}{2} \frac{d}{dt} \| \text{curl} w_i \|^2_{L^2(D)} = \int_D \sum_{j=1}^2 \hat{B}_{ij} \frac{\partial v_j}{\partial t} \frac{\partial w_i}{\partial t} \, dx \]

\[
\leq C \left( \left\| \frac{\partial w_i}{\partial t} \right\|_{L^2(D)}^2 + \sum_{i=1}^2 \left\| \frac{\partial v_i}{\partial t} \right\|_{L^2(D)}^2 \right). \tag{23} \]

Define

\[
\eta_i(t) := \left\| \frac{\partial w_i}{\partial t} \right\|_{L^2(D)}^2 + \| \text{curl} w_i \|^2_{L^2(D)}, \quad i = 1, 2.
\]

Then, (23) implies that

\[
\eta'_i(t) \leq C \left( \eta_i(t) + \sum_{j=1}^2 \left\| \frac{\partial v_j}{\partial t} \right\|_{L^2(D)}^2 \right), \quad T - \tau \leq t \leq T, \quad i = 1, 2
\]

and a straightforward application of Gronwall’s inequality gives that

\[
\sup_{T-\tau \leq t \leq T} \eta_i(t) \leq C \int_{T-\tau}^T \sum_{j=1}^2 \left\| \frac{\partial v_j}{\partial t} \right\|_{L^2(D)}^2 \, dt
\]

By the definition of \( \eta_i(t) \) and the equivalence of the norm \( \| w \|_H \) with \( \| \text{curl} w \|_{L^2(D)} \), we obtain the stated result. \( \square \)
3.2.3 Find the proper final condition for the backward adjoint system (9). We consider the map 
\( A : H \rightarrow H \) defined by

\[ \alpha \mapsto v \mapsto w \mapsto w(T) = A(\alpha), \]

where \( v \) is the solution of the backward deterministic equation (9) and \( w \) is the solution of the forward equation (20).

The following property of \( A \) is essential for what follows:

**Proposition 5** There exists an \( \alpha^* \) such that \( A(\alpha^*) = \xi \), for every \( \mathcal{F}_{T-\tau} \)-measurable random variable \( \xi \) with values in \( H \).

**Proof.** We split the proof into four steps.

1. First of all observe that the operator \( A : H \rightarrow H \) is a linear operator. This follows easily by the linearity of the adjoint system and the linearity of the forward system.

2. We next observe the the operator \( A : H \rightarrow H \) is continuous. Indeed, by definition, \( A(\alpha) = w(T) \).

   By Proposition 4, concerning the solutions of the forward problem (20), we have that

   \[ \| A(\alpha) \|_H^2 \leq \int_{T-\tau}^T \int_D \sum_{i=1}^2 \left| \frac{\partial v_i}{\partial t} \right|^2 dx \, dt. \]  
   
   Working as for the proof of (18) but choosing \( \Psi(t) = t - (T - \tau) \), we obtain the identity

   \[ \int_{T-\tau}^T \int_D \left( \left| \frac{\partial v_i}{\partial t} \right|^2 + |\text{curl}v_i|^2 \right) dx \, dt = \| (\text{curl}\alpha_1, \text{curl}\alpha_2) \|_{L^2(D)}^2, \quad i = 1, 2, \]

   which gives the estimate

   \[ \int_{T-\tau}^T \int_D \left| \frac{\partial v_i}{\partial t} \right|^2 dx \, dt \leq C, \| \alpha \|_{H}^2. \]  

   Combining (24) and (25), we obtain that

   \[ \| A(\alpha) \|_H \leq C \| \alpha \|_H, \]

   which proves the continuity of the operator \( A \).

3. Multiply the forward equation (20) by \( \frac{\partial v_i}{\partial t} \), \( i = 1, 2 \) and integrate over \([T - \tau, T] \times D \). This gives

   \[ \int_{T-\tau}^T \int_D \frac{\partial^2 w_1}{\partial t^2} \frac{\partial v_1}{\partial t} dx \, dt + \int_{T-\tau}^T \int_D \frac{\partial v_1}{\partial t} \text{curl}\text{curl}w_1 dx \, dt \]

   \[ = \hat{B}_{11} \int_{T-\tau}^T \int_D \left| \frac{\partial v_1}{\partial t} \right|^2 dx \, dt + \hat{B}_{12} \int_{T-\tau}^T \int_D \frac{\partial v_1}{\partial t} \frac{\partial v_2}{\partial t} dx \, dt. \]  

   Integration of the first term on the left-hand side (LHS) by parts over \( t \) gives

   \[ I_1 := \int_{T-\tau}^T \int_D \frac{\partial^2 w_1}{\partial t^2} \frac{\partial v_1}{\partial t} dx \, dt = \int_D \frac{\partial v_1}{\partial t} (T) \frac{\partial v_1}{\partial t} (T) dx - \int_{T-\tau}^T \int_D \frac{\partial^2 v_1}{\partial t^2} \frac{\partial w_1}{\partial t} dx \, dt. \]
The second term on the LHS gives

\[ I_2 := \int_{T-\tau}^T \int_D \frac{\partial v_1}{\partial t} \text{curl} \text{curl} w_1 \, dx \, dt \]

\[ = \int_D v_1 \text{curl} \text{curl} w_1 \bigg|_{t=T-\tau}^T - \int_{T-\tau}^T \int_D \text{curl} \text{curl} \frac{\partial w_1}{\partial t} \, dx \, dt \]

\[ = \int_D \text{curl} v_1 \text{curl} w_1 \bigg|_{t=T-\tau}^T - \int_{T-\tau}^T \int_D \text{curl} \text{curl} v_1 \frac{\partial w_1}{\partial t} \, dx \, dt \]

\[ = \int_D \text{curl} \alpha_1 \text{curl} \beta_1 \, dx - \int_{T-\tau}^T \int_D \text{curl} \text{curl} v_1 \frac{\partial w_1}{\partial t} \, dx \, dt, \quad (28) \]

where we first integrated by parts over \( t \) and then we integrated by parts over \( x \) (once for the first integral and twice for the second).

Combining (26–28) and using the fact the \( v_1 \) solves the adjoint equation (9), we obtain that

\[ \int_D \frac{\partial w_1}{\partial t}(T) \frac{\partial v_1}{\partial t}(T) \, dx + \int_D \text{curl} \alpha_1 \text{curl} \beta_1 \, dx = \hat{B}_{11} \int_{T-\tau}^T \int_D \left\| \frac{\partial v_1}{\partial t} \right\|^2 \, dx \, dt \]

\[ + \hat{B}_{12} \int_{T-\tau}^T \int_D \frac{\partial v_1}{\partial t} \frac{\partial v_2}{\partial t} \, dx \, dt. \quad (29) \]

We remark that the first term in the LHS of the above relation can be expressed as a linear combination of \( \text{curl} \alpha_1 \text{curl} \beta_1 \) and \( \text{curl} \alpha_2 \text{curl} \beta_2 \), which may be recombined to form the inner product \( (\alpha, \beta)_H \). Indeed, observe that by (21), \( \frac{\partial}{\partial t} w(t = T) \) is equal to a linear combination of \( \text{curl} \alpha_i \), \( \text{curl} \beta_i \) and \( h_i \), \( i = 1, 2 \), where \( h_i \) are the components of \( h \). Since \( h \) solves (22), by the conservation of the \( L^2 \)-norm for the solution of (22) combined with the conservation of the \( L^2 \)-norm for the solution of (9), we see that the \( L^2 \)-norm of \( h \) is related to the \( L^2 \)-norm of \( \alpha \) which in turn is bounded above by the \( L^2 \)-norm of \( \text{curl} \alpha \).

We now integrate the equation for \( w_2 \) by \( \frac{\partial v_2}{\partial t} \) and perform the same steps to obtain

\[ \int_D \frac{\partial w_2}{\partial t}(T) \frac{\partial v_2}{\partial t}(T) \, dx + \int_D \text{curl} \alpha_2 \text{curl} \beta_2 \, dx = \hat{B}_{22} \int_{T-\tau}^T \int_D \left\| \frac{\partial v_2}{\partial t} \right\|^2 \, dx \, dt \]

\[ + \hat{B}_{21} \int_{T-\tau}^T \int_D \frac{\partial v_1}{\partial t} \frac{\partial v_2}{\partial t} \, dx \, dt. \quad (30) \]

We now add (29) and (30) to get the following estimate

\[ (\alpha, \beta)_H \geq C \int_{T-\tau}^T \int_D \left( \left\| \frac{\partial v_1}{\partial t} \right\|^2 + \left\| \frac{\partial v_2}{\partial t} \right\|^2 \right) \, dx \, dt. \quad (31) \]

The LHS of (31) can be estimated by the Cauchy–Schwartz inequality by

\[ (\alpha, \beta)_H \leq \|\alpha\|_H \|\beta\|_H. \quad (32) \]
By the backward uniqueness estimate of Proposition 3, the right-hand side of (31) can be estimated by
\[ \int_{T-\tau}^{T} \int_{D} \left( \left| \frac{\partial v_1}{\partial t} \right|^2 + \left| \frac{\partial v_1}{\partial t} \right|^2 \right) dx \, dr = \int_{T-\tau}^{T} \int_{D} (|\text{curl} v_1|^2 + |\text{curl} v_2|^2) dx \, dr \]
\[ \geq C' \| \alpha \|^2_{\mathcal{H}}. \]  (33)

Combining (31–33), we obtain
\[ \| \alpha \|_{\mathcal{H}} \| \beta \|_{\mathcal{H}} \geq C \| \alpha \|_{\mathcal{H}}^2, \]
which readily provides the estimate
\[ \| \beta \|_{\mathcal{H}} = \| A(\alpha) \|_{\mathcal{H}} \geq C \| \alpha \|_{\mathcal{H}} \]  (34)
for all \( \alpha \in \mathcal{H} \).

4. The inequality (34) along with continuity property guarantees the invertibility of the operator \( A \), using the standard arguments of the Lax–Milgram lemma.

5. The above arguments hold a.s. in \( \omega \). Using standard arguments (see Remark 2 and e.g. Kim, 2004), the analysis is transferred to the \( L^2(\bar{\mathbb{P}}, \mathcal{F}_{T-\tau}) \) setting. This concludes the proof. \( \square \)

3.2.4 Construction of the control. The following procedure may now be used for the construction of the approximate control.

1. Choose an \( \epsilon > 0 \) and a desired final state \( \xi_T \) for system (3).
2. Solve the forward uncontrolled system (3) with \( u = 0 \) and find the solution \( \mathcal{E}(T) = \Xi \). This is an \( \mathcal{F}_T \)-measurable random variable.
3. Find the distance of the uncontrolled solution from the desired state \( \xi_T \) and \( \xi = \xi_T - \Xi \). This is an \( \mathcal{F}_T \)-measurable random variable.
4. Approximate \( \xi \) by the \( \mathcal{F}_{T-\tau} \)-measurable random variable \( \xi_\epsilon \).
5. Find the solution of equation \( A(\alpha^*) = \xi_\epsilon \).
6. Find the solution \( v^* \) of (9) with final condition \( \alpha = \alpha^* \).
7. Solve the initial value problem (22) to obtain \( h \).
8. Set \( U^*(t) = (B v^*(t) + h(t)) \mathbf{1}_{[T-\tau,T]}(t) \). This is the desired control.

The following result ensures the validity of this approach.

PROPOSITION 6 Let \( \mathcal{E}^* \) be the solution of the forward stochastic system
\[ d(\mathcal{A}\mathcal{E}^*(t)) = (\mathcal{M}\mathcal{E}^*(t) + J(t) + U^*(t))dt + \sum_{j=1}^{N} g_j \, dW_j(t), \]  (35)
\[ \mathcal{E}^*(0) = \mathcal{E}_0, \]  (36)
where \( U^*(t) \) is as above. For every \( \epsilon > 0 \), there exists \( \tau > 0 \) (cf. Lemma 1) such that
\[ \mathbb{E}[\| \mathcal{E}^*(T) - \xi_T \|^2_{\mathcal{H}}] \leq \epsilon. \]
Proof. The solution to (3) with the above choice of $U(t)$ is

$$
\mathcal{E}^*(t) = \exp(\mathcal{L}t)\mathcal{E}_0 + \int_0^t \exp(\mathcal{L}(t-s))J(s)ds + \sum_{j=1}^N \int_0^t \exp(\mathcal{L}(t-s))g_j(s)dW_j(s), \quad t < T - \tau
$$

and

$$
\mathcal{E}^*(t) = \exp(\mathcal{L}(t - (T - \tau)))\mathcal{E}^*(T - \tau) + \int_{T-\tau}^t \exp(\mathcal{L}((t - (T - \tau)) - s))(J(s) + U^*(s))ds
$$

$$
+ \sum_{j=1}^N \int_{T-\tau}^t \exp(\mathcal{L}((t - (T - \tau)) - s))g_j(s)dW_j(s), \quad T - \tau \leq t \leq T.
$$

Let $\mathcal{E}(t)$ be the solution of the uncontrolled system in $[0, T]$. The above representation shows that $\mathcal{E}(t) = \mathcal{E}^*(t)$ in $[0, T - \tau)$. Define $e(t) := \mathcal{E}^*(t) - \mathcal{E}(t)$ to be the difference of the solutions of the controlled system (35) and the uncontrolled system (3) with $u \equiv 0$. This solves the deterministic system

$$
\frac{\partial}{\partial t}(Ae) = Me + U^*,
$$

$$
e(T - \tau) = 0,
$$

which is of the same form as (21) (see Remark 3). By the choice of $U^*$, we have that $e(T) = \xi_\epsilon$. Therefore,

$$
e(T) := \mathcal{E}^*(T) - \mathcal{E}(T) = \mathcal{E}^*(T) - \Xi = \xi_\epsilon.
$$

Then,

$$
\mathcal{E}^*(T) - \xi_\epsilon T = \mathcal{E}^*(T) - \Xi + \Xi - \xi_\epsilon T = \xi_\epsilon - (\xi_\epsilon T - \Xi) = \xi_\epsilon - \xi
$$

which by the construction of $\xi_\epsilon$ guarantees that $\mathbb{E}[\|\mathcal{E}^*(T) - \xi_\epsilon T\|_{\mathcal{H}}]$ is less than $\epsilon$. This concludes the proof. \qed

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