Application of the WKB method to catenary-shaped slender structures

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Received 31 August 2006; received in revised form 14 July 2007; accepted 23 August 2007

Abstract

The purpose of this work is the derivation of closed form expressions for the linear vibration modes of catenary-shaped slender structures. The dynamic behaviour of the structure is described using the Euler–Bernoulli beam formulation with variable tension and angle terms. The desired expressions are obtained by treating the governing fourth-order partial differential equation of dynamic equilibrium using the WKB method [J.D. Logan, Applied Mathematics, second ed., Wiley Interscience, 1997].

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Keywords: WKB; Catenary; Slender structures; Eigenvalues

1. Introduction

Catenary-shaped slender structures are used in a variety of applications in offshore industry mainly as risers. These structures often suffer from the application of large dynamic bending stresses especially in the touch-down area. Thus, the solution of the dynamic problem is necessary for understanding the particulars of these unwanted impacts. The dynamic problem can be treated either fully numerically or analytically to a certain extent. In most of the cases the analytical approach to the desired solution relies on the method of the separation of variables. The successful implementation of a relevant technique provides the natural frequencies and the mode shapes as solutions to the reduced problem of ‘free vibrations’. It is very important that the mode shapes can provide by themselves an indication with regard to the mode of variation of the normal displacements and the bending moments along the catenary.

More specifically, the equation considered in the present is the following:

\[
\frac{\partial^2 y}{\partial \tau^2} = -K \frac{\partial^4 y}{\partial x^4} + \alpha(x) \frac{\partial^2 y}{\partial x^2} + \beta(x) \frac{\partial y}{\partial x}.
\]

(1)

The nondimensional equation (1) describes the uncoupled bending vibration of a slender catenary-shaped structure considering that the bending effects have an appreciable and nonnegligible contribution when compared to the tension.
terms. Eq. (1) is the linearized product of the governing equation in transverse direction and is derived from the complete nonlinear system of dynamic equilibrium in 2D space [2] after removing the nonlinear terms and omitting the components that describe the coupled axial and transverse vibrations. The various nondimensional terms which appear in Eq. (1) are given below:

\[ x = s/L, \quad y = q/L, \quad K = EI/(w_0 L^2), \quad \alpha(x) = T_0/(w_0 L), \quad \beta(x) = \sin(\phi_0), \quad \tau = t\sqrt{w_0/(ML)}. \]

Here \( L \) is the unstretched length of the structure, \( EI \) is the bending stiffness, \( w_0 \) is the effective weight, \( M \) is the effective mass, which in the case of a submerged structure is expressed as the summation of the mass \( m \) and the added mass \( m_a \), \( s \) is the unstretched Lagrangian coordinate and \( t \) is the time. All mass and weight terms are defined per unit unstretched length. In addition, \( q(s, t) \) denotes the transverse motion normal to the tangent along the structure and \( T_0 = T_0(s) \) and \( \phi_0 = \phi_0(s) \) are the functions that describe the variation of the static tension and static angle along the catenary. The angle is formed between the tangent on the line and the horizontal. \( T_0(s) \) and \( \phi_0(s) \) are obtained numerically through the solution of the static equilibrium problem.

For treating Eq. (1) only harmonic motions will be considered. Therefore, the solution will provide the eigenfrequencies and the mode shapes of the catenary which can be used subsequently for constructing a proper expression for the generalized solution of forced oscillations, with or without damping.

2. The WKB approximation

Some examples on the use of this method for assessing the dynamics of cables or risers are given in Refs. [3–7]. Triantafyllou [3,4] derived asymptotic solutions for the dynamics of taut inclined [3] and translating cables [4]. Cheng et al. [5] introduced the WKB method for analyzing nonuniform marine risers on the assumption that the properties are slowly varying within elements. Also, the WKB approximation was applied by Pesce et al. [6] for the derivation of closed form solutions for the general riser-like problem given the tension function along the length and by Pesce et al. [7] as a solution methodology for the eigenvalue problem for estimating the excitations at the touch-down point (TDP). Here the same method is extended on the dynamics of catenary-shaped slender structures with nonzero bending stiffness and variable tension and angle terms.

When only harmonic motions are considered, Eq. (1) is transformed into:

\[-K \frac{d^4y_0(x)}{dx^4} + \alpha(x) \frac{d^2y_0(x)}{dx^2} + \beta(x) \frac{dy_0(x)}{dx} + \omega^2 y_0(x) = 0, \quad (2)\]

where \( \omega \) is the nondimensional circular frequency.

The presence of functions \( \alpha(x) \) and \( \beta(x) \) precludes the derivation of a closed-form solution for satisfying Eq. (2). The latter remark applies even for linear functions \( \alpha(x) \) and \( \beta(x) \), which could be a case of a taut inclined structure. Nevertheless, a closed-form solution could be approximated to a certain extent using an efficient perturbation technique. To this end the WKB method (Wentzel–Kramers–Brillouin) [1] is implemented, making no assumption with regard to the variation of \( \alpha(x) \) and \( \beta(x) \).

The use of the WKB method requires the existence of a sufficiently small parameter \( \varepsilon \ll 1 \). Given the fact that for slender structures the nondimensional bending stiffness obtains very small values, it must be acknowledged that \( K \) can be used as a replacement for \( \varepsilon(K \approx \varepsilon \ll 1) \). Also, by defining a new spatial coordinate \( z = \varepsilon x \), Eq. (2) is recast into:

\[-\varepsilon^5 \frac{d^4y_0(z)}{dz^4} + \varepsilon^2 \alpha(z) \frac{d^2y_0(z)}{dz^2} + \varepsilon \beta(z) \frac{dy_0(z)}{dz} + \omega^2 y_0(z) = 0. \quad (3)\]

If we assume constant coefficients, say \( \alpha(z) = \alpha \) and \( \beta(z) = \beta \) then all four solutions of the above relation would be exponential functions with constant arguments. This suggests making the assumption of two possible solutions, i.e., \( y_0(z) = e^{\nu(z)/\varepsilon} \) and \( y_0(z) = e^{iu(z)/\varepsilon} \). Here we consider only the first expression. It is evident that the solution that corresponds to the second expression can be easily derived from the former.

After introducing \( y_0(z) = e^{u(z)/\varepsilon} \) into Eq. (3) and letting

\[ \nu = u' \]

(4)
the following is derived:
\[- \left( \varepsilon^4 v'''' + 4\varepsilon^3 v''' + 3\varepsilon^2 v'' + 6\varepsilon v' + \varepsilon v \right) + \alpha(z) \left( \varepsilon v' + v^2 \right) + \beta(z) v + \omega^2 = 0. \tag{5} \]

Here the primes denote differentiation with respect to \( z \). Next, a regular perturbation expansion is considered:
\[ v(z) = v_0(z) + \varepsilon v_1(z) + O(\varepsilon^2). \tag{6} \]

Eq. (6) is introduced into Eq. (5) and only the \( \varepsilon^0 \) and the \( \varepsilon^1 \) terms are retained. Thus, Eq. (5) is reduced to:
\[- \varepsilon v_0^4 + \varepsilon \alpha(z) v_0' + \alpha(z) v_0^2 + 2\varepsilon \alpha(z) v_0 v_1 + \beta(z) v_0 + \varepsilon \beta(z) v_1 + \omega^2 + O(\varepsilon^2) = 0. \tag{7} \]

The above equation produces at \( O(\varepsilon^0) \) and \( O(\varepsilon^1) \) the following expressions:

Order \( O(\varepsilon^0) \):
\[ \alpha(z) v_0^2 + \beta(z) v_0 + \omega^2 = 0 \tag{8} \]

and

Order \( O(\varepsilon^1) \):
\[ -v_0^4 + \alpha(z) v_0' + 2\alpha(z) v_0 v_1 + \beta(z) v_1 = 0. \tag{9} \]

Eqs. (8) and (9) are solved in terms of \( v_0 \) and \( v_1 \). Thus
\[ v_0 = \frac{-\beta(z) \pm \sqrt{\beta^2(z) - 4\omega^2\alpha(z)}}{2\alpha(z)} \tag{10} \]
\[ v_1 = \frac{v_0^4 - \alpha(z) v_0'}{2\alpha(z)v_0 + \beta(z)}. \tag{11} \]

Using the simple transformation (4), it can be shown that the general expressions that provide the independent solutions for the dimensionless transversal vibration \( y_0(z) \) will be calculated by:
\[ y_0(z) = \exp \left( \frac{i}{\varepsilon} \int_0^z v_0(\xi) d\xi + \int_0^z v_1(\xi) d\xi + O(\varepsilon) \right) \tag{12} \]
\[ y_0(z) = \exp \left( \frac{i}{\varepsilon} \int_0^z v_0(\xi) d\xi + i \int_0^z v_1(\xi) d\xi + O(\varepsilon) \right). \tag{13} \]

Eq. (10) provides two solutions for \( v_0 \) while \( v_1 \) depends directly on the former. Thus Eqs. (12) and (13), express the WKB approximation for all four linearly independent solutions of the governing equation (3) and the complete solution will be given by their linear combination. For simplifying the mathematical description of the mode shapes, two new variables are introduced:
\[ v_0^{(1)} = \frac{-\beta(z) + \sqrt{\beta^2(z) - 4\omega^2\alpha(z)}}{2\alpha(z)} \quad \text{and} \quad v_0^{(2)} = \frac{-\beta(z) - \sqrt{\beta^2(z) - 4\omega^2\alpha(z)}}{2\alpha(z)}. \tag{14} \]

Therefore, the associated \( v_1 \) values, given by Eq. (11), obtain the following forms:
\[ v_1^{(1)} = \frac{(v_0^{(1)})^4 - \alpha(z)(v_0^{(1)})'}{2\alpha(z)v_0^{(1)} + \beta(z)} \quad \text{and} \quad v_1^{(2)} = \frac{(v_0^{(2)})^4 - \alpha(z)(v_0^{(2)})'}{2\alpha(z)v_0^{(2)} + \beta(z)}. \tag{15} \]

Finally, the expression that provides the mode shapes is determined by forming the linear combination of all four linearly independent solutions. Thus, after restoring the coordinate transformation \( z/\varepsilon = x \) and removing the \( O(\varepsilon^2) \) terms, the following is derived:
\[ y_0(x) = C_1 \exp \left( \int_0^x v_0^{(1)}(\xi) d\xi + i \int_0^x v_1^{(1)}(\xi) d\xi \right) + C_2 \exp \left( \int_0^x v_0^{(2)}(\xi) d\xi + i \int_0^x v_1^{(2)}(\xi) d\xi \right) + C_3 \exp \left( i \int_0^x v_0^{(1)}(\xi) d\xi + i \varepsilon \int_0^x v_1^{(1)}(\xi) d\xi \right) + C_4 \exp \left( i \int_0^x v_0^{(2)}(\xi) d\xi + i \varepsilon \int_0^x v_1^{(2)}(\xi) d\xi \right), \tag{16} \]

where \( C_1, C_2, C_3 \) and \( C_4 \) are arbitrary constants.
Eq. (16) can be simplified significantly by removing the $O(\varepsilon)$ terms:

$$y_0(x) = C_1 \exp\left(\int_0^x -\beta(\xi) + \frac{\Delta(\xi)}{2\alpha(\xi)}\,d\xi\right) + C_2 \exp\left(\int_0^x -\beta(\xi) - \frac{\Delta(\xi)}{2\alpha(\xi)}\,d\xi\right) + C_3 \exp\left(i\int_0^x -\beta(\xi) + \frac{\Delta(\xi)}{2\alpha(\xi)}\,d\xi\right) + C_4 \exp\left(i\int_0^x -\beta(\xi) - \frac{\Delta(\xi)}{2\alpha(\xi)}\,d\xi\right)$$

where $\Delta(\xi) = \beta^2(\xi) - 4\omega^2\alpha(\xi)$.

Here, we avoid providing details regarding the complete analytical elaboration of Eq. (16) or even its reduced form given by Eq. (17). The reason is that the application of the boundary conditions, say for a hinged-hinged structure,

$$y_0(0) = y''_0(0) = y_0(1) = y''_0(1) = 0$$

(18)

derives very lengthy expressions for both models which cannot be written in the present short text. The fact is that the use of the boundary conditions, as required for obtaining the eigenfrequencies and the mode shapes, produces a linear $4 \times 4$ homogeneous system in terms of the unknown arbitrary constants $C$. As usual, the eigenfrequencies are calculated by the requirement that the determinant of the $4 \times 4$ matrix, which is formed by the coefficients of the unknown constants $C$, should be zero. Next, the mode shapes are obtained with the aid of the equivalent truncated $3 \times 3$ linear inhomogeneous system which yields the three of the unknown constants in terms of the fourth. Here, all numerical calculations were performed using the MATLAB software package using the generalized complex forms for both eigenfrequencies and the unknown constants $C$. Nevertheless, the numerical predictions comply with the requirement that the eigenfrequencies and the corresponding eigenmodes cannot be complex. In particular, in all cases the calculated results indicate that the imaginary parts of the eigenfrequencies and the real parts of $C$ constants are zero.

2.1. Simplified model

Eq. (16) can be recast into a reduced model by making a simple and valid assumption for the practical applications of the original equation (2). A quick inspection of Eq. (2) and of the associated nondimensional coefficients reveals that the second term $\alpha(x) d^2y_0(x)/dx^2$ is influenced by the variation of the static tension along the structure while the third term $\beta(x)dy_0(x)/dx$ is mainly connected with the configuration of the structure in the static equilibrium position. If we assume that the latter is one order of magnitude smaller than the former, then Eq. (3) can be written as

$$-\varepsilon^4 \frac{d^4y_0(z)}{dz^4} + \varepsilon^2 \alpha(z) \frac{d^2y_0(z)}{dz^2} + \frac{\omega^2 y_0(z)}{\omega^2} = 0. \quad (19)$$

Eq. (19) could represent the case of a near neutrally buoyant structure or the case of a riser with an extreme pretension applied at the top. Under these conditions, the relation that corresponds to Eq. (5) becomes

$$-\left(\varepsilon^4 u'''' + 4\varepsilon^3 u''u + 3\varepsilon^2 u'v' + 6\varepsilon^2 u'v'' + \varepsilon v'' \right) + \alpha(z) \left(\omega u' + v^2\right) + \beta(z) u + \omega^2 = 0. \quad (20)$$

For the derivation of the above, the first type of solution $y_0(z) = e^{\omega(z)/\varepsilon}$ was considered. Using the perturbation expansion (6) and separating terms at $O(\varepsilon^0)$ and $O(\varepsilon^1)$, we obtain the following simplified forms for $v_0$ and $v_1$:

$$u_0 = \pm i\omega/\sqrt{\alpha(z)} \quad (21)$$

$$u_1 = \frac{1}{4} \alpha'(z) - \frac{1}{2} \beta(z) + \frac{1}{2} \alpha^2(z)/\sqrt{\alpha(z)} \quad (22)$$

The final WKB approximation of the desired solution will be given by:

$$y_0(z) = C_1 \exp\left(i\Re(\xi) + i\Im(\xi)\right) + C_2 \exp\left(i\Re(\xi) - i\Im(\xi)\right) + C_3 \exp\left(i\Re(\xi) - \Im(\xi)\right) + C_4 \exp\left(i\Re(\xi) + \Im(\xi)\right)$$

(23)
where
\[
\zeta = \frac{1}{4} \ln (\alpha(\xi)) \bigg|_0^z - \frac{1}{2} \int_0^z \beta(\xi) \frac{d\xi}{\alpha(\xi)} + i \left( \frac{\omega}{\varepsilon} \int_0^z \frac{d\xi}{\sqrt{\alpha(\xi)}} - \frac{\omega^3}{2} \int_0^z \frac{d\xi}{\alpha''(\xi)\sqrt{\alpha(\xi)}} \right)
\]  
(24)
and \(\Re\) and \(\Im\) denote the real and the imaginary parts of the complex argument \(\zeta\).

Again, Eq. (23) can be simplified significantly by retaining only the \(O(1/\varepsilon)\) terms with respect to the \(z\) coordinate, or equivalently, by removing the \(O(\varepsilon)\) terms with respect to \(x\). The resulting product can be alternatively expressed by the following suitable form:
\[
y_0(x) = C_1 \cos \left( \omega \int_0^x \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) + C_2 \sin \left( \omega \int_0^x \frac{d\xi}{\sqrt{\alpha(\xi)}} \right)
+ C_3 \cosh \left( \omega \int_0^x \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) + C_4 \sinh \left( \omega \int_0^x \frac{d\xi}{\sqrt{\alpha(\xi)}} \right).
\]  
(25)

It is immediately apparent that the bending stiffness terms do not participate in the above formulation and only the static tension terms are retained. This makes Eq. (25) suitable only for highly tensioned structures where the variation of the static tension is nearly linear. Nevertheless, the simplified form of Eq. (25) makes the application of the boundary conditions (18) more convenient. After some mathematical manipulations, it can be shown that the relations that provide the eigenfrequencies and the mode shapes for the simplified model and infinitesimal bending stiffness are given by the following Eqs. (26) and (27) respectively.
\[
\sin \left( \omega_j \int_0^1 \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) \left[ \frac{\omega_j^2}{\alpha(1)} \sinh \left( \omega_j \int_0^1 \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) - \frac{\omega_j \alpha'(1)}{2 \alpha(1) \sqrt{\alpha(1)}} \cosh \left( \omega_j \int_0^1 \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) \right]
- \sinh \left( \omega_j \int_0^1 \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) \left[ -\frac{\omega_j^2}{\alpha(1)} \sin \left( \omega_j \int_0^1 \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) - \frac{\omega_j \alpha'(1)}{2 \alpha(1) \sqrt{\alpha(1)}} \cos \left( \omega_j \int_0^1 \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) \right] = 0
\]  
(26)
\[
y_0(x) = A_j \left[ \sin \left( \omega_j \int_0^x \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) - \frac{\sin \left( \omega_j \int_0^1 \frac{d\xi}{\sqrt{\alpha(\xi)}} \right)}{\sinh \left( \omega_j \int_0^1 \frac{d\xi}{\sqrt{\alpha(\xi)}} \right)} \sinh \left( \omega_j \int_0^x \frac{d\xi}{\sqrt{\alpha(\xi)}} \right) \right].
\]  
(27)

It should be mentioned that Eqs. (26) and (27) were obtained by using the assumption \(d\alpha(0)/dx = 0\). Practically speaking, this means that the angle which is formed at the touch-down point is zero. Eq. (26) has an infinite number of real roots for the eigenfrequencies \(\omega_j\). The corresponding arbitrary constant \(A_j\) is determined by the initial conditions while in the case of time-dependent forced motion, \(A_j\) becomes a function of time and depends on the properties of the excitation.

3. Numerical implementation

This section presents calculations for all mode shape models which were developed in the present, i.e., the complete model (Eq. (16)), the complete model with \(\varepsilon \approx 0\) (Eq. (17)), the simplified model (Eq. (23)) and the simplified model with \(\varepsilon \approx 0\) (Eq. (27)). For validating the efficacy of the above described methodology, comparative calculations were performed using the numerical method developed by Triantafyllou et al. [8], which was reproduced for supporting the objectives of the present contribution. The method treats the linearized system of dynamic equilibrium that describes the coupled in-plane tangential and transversal vibrations of a catenary cable including the extensibility effect. The bending stiffness of the cable was set equal to zero and the desired numerical solution was derived using an efficient centered differences numerical scheme. The calculations refer to a catenary riser with the following properties [9]: length 1500 m, effective mass \((M = m + m_a)\) 305.66 kg/m, submerged weight 1008.2 N/m, bending stiffness 74 515 kN m², elastic stiffness \(5.4428 \times 10^6\) kN, outer diameter 0.355 m and wall thickness 0.025 m. The installation depth was considered equal to 1300 m and the pretension at the top was 1500 kN. Comparative results for the eigenfrequencies are given in Table 1.
Table 1
Comparative results for the five initial eigenfrequencies (rad/s)

<table>
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<tr>
<th>Mode number</th>
<th>Simplified model, ( \varepsilon \approx 0 ), Eq. (27)</th>
<th>Complete model, ( \varepsilon \approx 0 ), Eq. (17)</th>
<th>Simplified model, ( \varepsilon \approx 0 ), Eq. (23)</th>
<th>Complete model, ( \varepsilon \approx 0 ), Eq. (16)</th>
<th>Numerical solution</th>
</tr>
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</table>

Fig. 1. Mode shapes for the out-of-plane motions calculated by the simplified model of Eq. (27) assuming \( \varepsilon \approx 0 \).

It is reminded that the simplified models assume that the term \( \beta(x)dy_0(x)/dx \) is one order of magnitude smaller than \( \alpha(x)d^2y_0(x)/dx^2 \). From the values listed in Table 1, it is immediately apparent that the comparison with the results of the numerical solution is favorable (more than 95%) only for the models that include the \( O(\varepsilon) \) terms, while the removal of the these terms from the WKB approximation of the mode shapes, leads to notable discrepancies as far as the eigenfrequencies are concerned. This is due to the important contribution of the in-plane static geometric shape (which in the present formulation is represented by the term \( \beta(x)dy_0(x)/dx \)) on the in-plane transversal modes of the structure. Given the fact that the static geometric shape is important only for the in-plane modes, it is defensible to say that the reduced models (Eqs. (17) and (27)) can represent the out-of-plane motions of the structure. Apparently, the tension variation geometric rigidity term \( \alpha(x)d^2y_0(x)/dx^2 \) is important for both the in-plane and the out-of-plane motions of the catenary.

The mode shapes for the out-of-plane motions are depicted in Figs. 1 and 2, while the mode shapes associated with the in-plane motions are shown in Figs. 4 and 5. Figs. 1 and 4 correspond to the simplified models which ignore the term \( \beta(x)dy_0(x)/dx \). When this term is removed, the mode shapes (Figs. 1 and 4), exhibit a node shift towards the touch-down region, while no worth mentioning variation of the transversal motions is encountered along the structure. This can be traced back to the fact that the structure has been considered sufficiently stretched, while the nonsymmetrical dispersion of nodes originates from the inclination.

On the contrary, a different behaviour is encountered when the structure has a fully developed catenary configuration or equivalently when the complete models are used for calculating the mode shapes. Figs. 2 and 5 show that the catenary will experience strong variations of the out-of-plane and the in-plane normal motions along the structure, which will be characterized by their extreme amplification at the lower portion of the catenary. In addition, for increasing consecutive mode number, the location of the maximum displacement goes towards the touch down. The fact that the modes of vibration are associated with the physical properties and the installation characteristics of the structure implies that the maximum out-of-plane (Fig. 2) and the maximum in-plane (Fig. 5) transversal motion and the associated maximum curvatures (Figs. 3 and 6 respectively) should be expected to occur close to the touch-
Fig. 2. Mode shapes for the out-of-plane motions calculated by the complete model of Eq. (17), assuming $\varepsilon \approx 0$.

Fig. 3. Mode shapes for the out-of-plane normalized curvature calculated using the complete model of Eq. (17) assuming $\varepsilon \approx 0$.

Fig. 4. Mode shapes for the in-plane motions calculated by the simplified model of Eq. (23).

down point without reference to the orientation of the excitation at the top. Apparently, the above discussion concerns only the linear vibration problem.
For comparative purposes and for showing what would happen to the equivalent catenary cable, Fig. 7 is given which depicts the initial in-plane mode shapes for the transversal vibration along the cable. Here the calculations
were performed using the aforementioned centered differences numerical scheme. The mode shapes in Fig. 7 are compared with those depicted in Figs. 4 and 5 for the in-plane transversal motions along the riser under consideration. Although the coincidence of the eigenfrequencies is favorable (Table 1) and the associated mode shapes exhibit apparent similarities, what it is not captured by the present WKB approximation is the first small cross-over of all mode shapes which occurs very close to the lower end. The latter remark must be seen in correlation with the contribution of the higher-order bending stiffness terms very close to the touch-down point. The contribution of these terms, which were not considered in the present analytical formulation, is very important but only locally, i.e. close to the touch-down point where the static curvature obtains large values. Thus, independently of how much small is the value of the bending stiffness (here $K = 2.189 \times 10^{-5}$), higher-order terms may contribute considerably in capturing the characteristics of the dynamic behaviour of risers close to the touch-down point.

Acknowledgement

The author wishes to thank the anonymous referee who through his comments and his personal work guided him to expand the calculations and to provide a proper interpretation of the results presented in this work. This support is greatly acknowledged.

References