ALL RAMSEY NUMBERS FOR FIVE VERTICES AND SEVEN OR EIGHT EDGES

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For five vertices there are four graphs with seven edges and two graphs with eight edges. For all these six graphs the exact Ramsey numbers are given. Hence, for graphs with at most five vertices only the Ramsey number of the complete graph $K_5$ remains unknown.

For graphs $G$ and $H$ the Ramsey number $r(G, H)$ is defined to be the least number $p$ such that every 2-coloring (say green and red) of the edges of the complete graph $K_p$ contains either a subgraph $G$ with all its edges green, or a subgraph $H$ with all its edges red. The diagonal Ramsey numbers $r(G, G) = r(G)$ for $G = K_n$ are of particular interest. The values $r(K_3) = 6$, and $r(K_4) = 18$ are well known, but $r(K_5)$ is still unknown; all that is presently known is that $42 \leq r(K_5) \leq 55$.

If the numbers $r(G)$ are considered only for graphs $G$ with at most five vertices, then for all $G$ with at most six edges the Ramsey numbers are listed in [1]. For nine edges only one graph, $K_5 - e$, exists, and $r(K_5 - e) = 22$ recently was proved in [3]. For ten edges $r(K_5)$ remains open. Thus it remains to consider (except for $K_5$) only graphs with seven or eight edges (Figs. 1 and 2). It will be seen that the corresponding Ramsey numbers are:

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\begin{align*}
    r(H_1) &= 10, & r(H_2) &= 10, & r(H_3) &= 14, & r(H_4) &= 18, \\
    r(H_5) &= 15, & r(H_6) &= 18.
\end{align*}
$$

That $r(H_5) = 14$ is already known [8]. It is the purpose of this paper to give the proofs of the remaining five Ramsey numbers. It should be noted that Hendry [7] very recently has sent an unpublished table, where all but 9 values $r(G, H)$ for $G$ and $H$ with five vertices are listed without proofs, and which contain the above values. Moreover, in [5] a hint was given to an announcement of the result $14 \leq r(H_5) \leq 15$ of Bondy and Chvátal; however, a proof seems not to have been published so far.

A similar number in this context is $r_5(5)$, which denotes the least number $p$ such that every 2-coloring of the edges of $K_p$ contains some monochromatic subgraph of the set of all graphs with 5 vertices and $q$ edges. The values $r_5(5) = 6$, $r_4(5) = 10$, and $r_6(5) = 14$ were proved in [6].

**Theorem 1.** $r(H_1) = r(H_2) = 10$. 

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Since $H_1$ and $H_2$ both contain a $K_4 - e$ as a subgraph, $r(K_4 - e) = 10$ (see [2]) implies $r(H_1) \geq 10$, and $r(H_2) \geq 10$. The proof of "$\leq 10$" is partitioned into Lemmas 1 to 3.

**Lemma 1.** Any 2-coloring of $K_{10}$ with a monochromatic $K_4$ contains monochromatic graphs $H_1$ and $H_2$.

**Proof.** Let 1, 2, 3, 4 be the vertices of a green $K_4$ in a 2-coloring of $K_{10}$. Two green edges from one vertex of 5, 6, ..., 10 to the green $K_4$ yield a green $H_1$, and a green $H_2$. Thus, there are at least three red edges from each of vertices 5, 6, ..., 10 to the green $K_4$, so that one vertex, say 1, has at least 5 red edges, say to a subgraph $F$ with vertices 5, 6, ..., 9. No green 4-spoked wheel $W_5$ in $F$ can occur since $G_1$ and $G_2$ both are subgraphs of $W_5$. From this, it is immediate that $F$ contains a red $P_3$, say with red edges (5, 6) and (6, 7). Finally, one vertex of 2, 3, 4 exists with red edges to 5 and 7, and this yields a red $H_1$, and one vertex of 2, 3, 4 exists with red edges to 5 and 6, and this yields a red $H_2$.

In the following, $r(v)$ and $g(v)$ denote the numbers of red and green edges incident to a vertex $v$ in a 2-coloring of a graph. Moreover, $G_v$ and $R_v$ will denote the subgraphs induced by those vertices joined to $v$ by green and red edges, respectively.

**Lemma 2.** Any 2-coloring of $K_{10}$ with $r(v) \geq 6$, or $g(v) \geq 6$ for some vertex $v$ contains monochromatic graphs $H_1$ and $H_2$.

**Proof.** Without loss of generality, $g(v) \geq 6$ can be assumed.

(H2): In the subgraph [1, 2, ..., 6] of $G_v$ either a red $H_2$, or a green $P_3$ exists, say (1, 2, 3), where (1, 3) is red (Lemma 1). In [4, 5, 6] at least one edge is red
(Lemma 1). Vertices 4, 5, 6 are connected only in red to 1 and 3 to avoid a green $H_2$. Then, however, a red $H_2$ occurs.

(H1): In any 2-coloring of $K_{10}$ consider a subgraph $F = [1, 2, \ldots, 7]$ containing at least 6 vertices of $G_u$.

Case 1. There exists in $F$ a green cycle $C_i, 3 \leq i \leq 7$.

(1.1) $i = 4$: A green $C_4$ in $F$ yields a green $H_1$.

(1.2) $i = 3$: A green $C_3 = [1, 2, 3]$ with $(v, 2)$ and $(v, 3)$ green forces $(v, 1)$ red (Lemma 1). Then only red edges from 1 to $[4, 5, 6, 7]$ avoid a green $H_1$. In $[4, 5, 6, 7]$ either a green $C_4$ occurs, or a red $P_3$ with red edges $(4, 5)$ and $(5, 6)$ can be assumed. Lemma 1 forces $(4, 6)$ to be green. If both edges from 2 to 3 and 4 and 6 are red, then a red $H_1$ occurs, otherwise a green $H_1$ is guaranteed.

(1.3) $i = 5$: If neither a green $C_3$ nor a green $C_4$ occur in $F$, then a green $C_5$ forces a red $C_3$ as its complement; moreover, each of the remaining two vertices of $F$ has at least 4 red edges to the vertices of the green and red $C_5$. Then a red $H_1$ is guaranteed.

(1.4) $i = 6$: In $F$ with neither a green $C_3$, $C_4$ nor $C_5$ all diagonals of a green $C_6$, so as at least 5 edges from the vertices of this $C_6$ to the remaining vertex of $F$, have to be red. Thus $F$ contains a red $H_1$.

(1.5) $i = 7$: All diagonals of a green $C_7$ in $F$ with neither a green $C_3$, $C_4$ nor $C_5$ have to be red, so that a red $H_1$ occurs.

Case 2. No green cycle occurs in $F$.

One vertex, say 1, exists with $g(1) \leq 1$. Let $[2, 3, 4, 5, 6]$ be connected only red to 1. No green $C_3$ and no green $C_5$ imply a red $C_3$ in $[2, 3, 4, 5, 6]$. Together with vertex 1 a red $K_4$ occurs, and a monochromatic $H_1$ follows by Lemma 1.

Lemma 3. Any 2-coloring of $K_{10}$ with $g(v) \leq 5$ and $r(v) \leq 5$ for all vertices $v$ contains monochromatic graphs $H_1$ and $H_2$.

Proof. There are \(\binom{10}{3} - 10 \cdot 5 \cdot 4 \cdot \frac{1}{2} = 20\) monochromatic triangles in any 2-coloring of $K_{10}$, if $g(v) = 5$ and $r(v) = 4$, or $g(v) = 4$ and $r(v) = 5$ for all vertices $v$. Thus one vertex $w$ exists which is a vertex of at least 6 monochromatic triangles, which means that the numbers of green edges in $G_w$ and of red edges in $R_w$ are together at least 6. It can be assumed $g(w) = 5$, $r(w) = 4$, $G_w = [1, 2, 3, 4, 5]$, and $R_w = [6, 7, 8, 9]$.

Case 1. There are at most 3 green edges in $G_w$, and therefore at least 3 red edges in $R_w$.

(1.1) $R_w$ contains a red $C_4$ or a red $C_5$: A red $C_3$ implies a red $K_4$, and Lemma 1 guarantees monochromatic graphs $H_1$ and $H_2$. A red $C_4$ implies a red wheel $W_5$ with red subgraphs $H_1$ and $H_2$. 

(1.2) \( R_w \) contains a green triangle, and a red \( K_{1,3} \): let 7, 8, 9 be the vertices of the green triangle in \( R_w \).

In \( G_w \) a green \( P_3 \) exists, since otherwise a red \( H_1 \), and a red \( H_2 \) occur. Let (1, 2) and (2, 3) be green edges, then (1, 3) is red, or Lemma 1 guarantees monochromatic graphs \( H_1 \) and \( H_2 \).

At least one edge from 2 to 7, 8, 9, say (2, 7), is red, otherwise a green \( K_4 \) occurs (Lemma 1). If (2, 6) is red, then a red \( H_2 \) is fixed and \( r(6) \leq 5 \) implies (1, 6) and (1, 3) green, which means a green \( H_1 \). If, however, (2, 6) is green, then \( g(2) \leq 5 \) forces (2, 8) or (2, 9) red and hence a green \( H_2 \).

(1.3) \( R_w \) contains a green \( P_4 \) and a red \( P_4 \): A red \( H_2 \) is fixed at once.

As in (1.2) a green \( P_3 \) exists in \( G_w \), and again it can be assumed (1, 2) and (2, 3) green and (1, 3) red. If there is no third green edge in \( G_w \), or if the third green edge is (4, 5), then a red \( H_1 \) exists in \( G_w \). If the 3 green edges of \( G_w \) are those of a \( K_{1,3} \), then a red \( K_4 \) occurs in \( G_w \) (Lemma 1). It remains that the 3 green edges are those of a \( P_4 \), and (3, 4) can be assumed to be the third green edge. If now 2 and 3 are both connected red to 7 and 8, then avoidance of a red \( H_1 \) forces a green \( K_4 = [2, 3, 6, 9] \) (Lemma 1). Thus (2, 7) can be assumed to be green, which implies (4, 7) red, (3, 7) green, and (1, 7) red, if monochromatic graphs \( H_1 \) are avoided. At last (5, 7) red yields a red \( K_4 \) (Lemma 1), and (5, 7) green determines a green \( H_1 \) in \([w, 2, 3, 5, 7]\).

Case 2. There are at least 4 green edges in \( G_w \).

(2.1) \( G_w \) contains a green cycle \( C_i \), \( 3 \leq i \leq 5 \): A green \( C_3 \) yields a green \( K_4 \) (Lemma 1). A green \( C_4 \) determines a green \( H_1 \) and a green \( H_2 \). A green \( C_5 \) in \( G_w \) forces a green \( H_2 \) at once, and avoidance of a green \( C_4 \) also forces a red \( C_5 \) in \( G_w \).

Let (1, 2), (2, 3), (3, 4), (4, 5), (1, 5) be the edges of the green \( C_5 \).

At least one edge in \( R_w \), say (6, 7), is red. Any vertex of \( R_w \) is connected by at most 2 green edges to 2 vertices of \( G_w \) which are connected by a green edge, since otherwise a green \( H_1 \) occurs. Vertices 6 and 7 each has at most 3 red edges to \( G_w \) if a red \( H_1 \) is avoided. If a red \( K_4 \) (Lemma 1) and a red \( H_1 \) are avoided, it can be assumed (1, 6), (2, 6), (3, 7), (4, 7) are green, and the remaining edges from 6 and 7 to \( G_w \) are red. Since \( r(6) \leq 5 \), it follows that (6, 8) and (6, 9) are both green. To avoid a red \( H_1 \) the edges (5, 8) and (5, 9) are green. If one edge of the triangle \([1, 8, 9]\) is green, then a green \( H_1 \) occurs in \([1, 5, 6, 8, 9]\). Finally a red triangle \([1, 8, 9]\) yields a red \( H_1 \) in \([w, 1, 7, 8, 9]\).

(2.2) The green subgraph of \( G_w \) is a tree: It follows that there are exactly 4 green edges in \( G_w \).

(2.2.1) If the green tree is a \( K_{1,4} \), then a red \( K_4 \) occurs (Lemma 1).

(2.2.2) If the green tree contains a \( K_{1,3} \), say (1, 2), (1, 3) and (1, 4) are green, then (4, 5) can also be assumed to be green, and the remaining edges of \( G_w \) are red. Then a green \( H_2 \) is already fixed. Since \( r(5) \geq 4 \), it can be assumed (5, 6) is red. This yields (4, 6) green, or a red \( H_1 \) occurs. To avoid a red \( K_4 \) (Lemma 1) either (2, 6) or (3, 6) has to be green, and in both cases a green \( H_1 \) is forced.
It remains, that the green tree is a $P_5$, say $(1, 2)$, $(2, 3)$, $(3, 4)$ and $(4, 5)$ are green. A green $H_5$ is guaranteed. Since $g(3) \geq 4$, it can be assumed $(3, 6)$ is green. Avoiding a green $H_1$, it follows $(1, 6)$ and $(5, 6)$ are red, and either $(2, 6)$ or $(4, 6)$ is red. In both cases, however, a red $H_1$ occurs.

**Theorem 2.** $r(H_5) = 15.$

**Proof.** The lower bound $r(H_5) > 14$ follows from the 2-coloring in Fig. 3, where only the green edges are drawn. It is easily seen that no green wheel $W_5 = H_5$ occurs, since for every vertex $v$ the green subgraph of $G_v$ is a cycle $C_6$, and since the red subgraph of $R_v$ is a cycle $C_7$ together with both diagonals of order 2 incident to one vertex of that $C_7$.

To prove $r(H_5) \leq 15$, notice first that a 2-coloring of $K_{15}$ with $g(v) = r(v) = 7$ for all vertices $v$ is impossible. Thus for at least one vertex $w$ it can be assumed that $g(w) \geq 8$. The rest of the proof of $r(H_5) \leq 15$ is partitioned into the following Lemmas 4 and 5.

**Lemma 4.** The green subgraph of any 2-coloring of $K_8$ without a green $C_4$, and without a red $H_5$ is isomorphic to the graph in Fig. 4.

**Proof.**

**Case 1.** $g(v) \leq 1$ for one vertex $v$ of $K_8$.

Since $R_v$ has at least 6 vertices and $r(C_4, C_4) = 6$ (see [2]), a green $C_4$, or with $v$ a red $H_5$ is guaranteed.

Fig. 3. The green subgraph of a 2-coloring of $K_{14}$ without a monochromatic wheel $W_5 = H_5$.
Case 2. $g(v) \geq 4$ for one vertex $v$ of $K_8$.

Let $[1, 2, 3, 4]$ be a subgraph of $G_v$. Any green $P_3$ in $[1, 2, 3, 4]$ yields a green $C_4$, and thus $(1, 2)$, $(2, 3)$, $(3, 4)$ and $(1, 4)$ can be assumed to be red. More than one, or no green edge from each of the vertices $5, 6, 7$ to $[1, 2, 3, 4]$, forces a green $C_4$ or a red $H_5$, respectively. More than one green edge from 1, 2, 3, or 4 to $[5, 6, 7]$ forces a red $H_5$. Therefore $(1, 5)$, $(2, 6)$ and $(3, 7)$ can be assumed to be all possible green edges from $[1, 2, 3, 4]$ to $[5, 6, 7]$. To avoid a red $H_5$ in $[1, 2, 3, 4, 7]$ it follows that $(1, 3)$ is green. Then $(5, 7)$ green determines a green $C_4$ in $[1, 3, 5, 7]$, and $(5, 7)$ red forces a red $H_5$ in $[2, 3, 4, 5, 7]$.

Case 3. $g(v) = 2$ for all vertices of $K_8$.

To avoid a green $C_4$, only a green $C_8$ is possible, or a green $C_5$ together with a vertex disjoint $C_3$, and in both cases a red $H_5$ is easily found.

Case 4. $g(v) = 3$ for at least one vertex of $K_8$.

Then at least 2 vertices, say 1 and 2, exist with $g(1) = g(2) = 3$.

First it is assumed that for every pair $u, w$ with $g(u) = g(w) = 3$ the graphs $G_u$ and $G_w$ are vertex disjoint. Then $G_1$ and $G_2$ each contain a red $P_3$ if no green $C_4$ occurs. Each vertex of $G_1$ or $G_2$ is connected by exactly one green edge to the 3 vertices of $G_2$ or $G_1$, respectively, since 3 red edges force a red $H_5$ and 2 green edges force a green $C_4$. Then the third edges in $G_1$ and $G_2$ are also red, since otherwise 2 vertices $v$ and $w$ exist with $g(v) = g(w) = 3$, and with a common vertex of $G_v$ and $G_w$. Then, however, in $G_1$ together with $G_2$ red graphs $H_5$ occur.

In the remaining case 2 vertices, say 1 and 2, exist with $g(1) = g(2) = 3$ and $G_1 = [3, 4, 5]$, $G_2 = [5, 6, 7]$, since at least 2 common vertices force a green $C_4$. It follows that 8 is connected by at most one green edge to each of $G_1$ and $G_2$ (no green $C_4$), and together with $g(8) \geq 2$ it follows $(5, 8)$ is red, and it can be assumed $(4, 8)$ and $(6, 8)$ are red, and $(3, 8)$ and $(7, 8)$ are green. No red $H_5$ yields $(4, 6)$ green, and then no green $C_4$ implies $(3, 6)$, $(4, 7)$, $(5, 6)$ and $(4, 5)$ are red. If $(3, 5)$ and $(5, 7)$ are both green, then a green $C_4$ occurs. Thus by symmetry it can be assumed that $(3, 5)$ is red. Then a red $H_5$ is avoided by $(3, 4)$ green, and $g(3) \leq 3$ implies $(3, 7)$ red. The remaining 2 edges $(5, 7)$ and $(6, 7)$ are neither both red (a red $H_5$), nor both green $(g(7) < 3)$. At last both green subgraphs with $(5, 7)$ green, or with $(6, 7)$ green are isomorphic to that graph of Fig. 4 (unlabelled).
**Lemma 5.** Any 2-coloring of $K_{15}$ with $g(w) \geq 8$ for one vertex $w$ contains a monochromatic graph $H_5$.

**Proof.** Consider 8 vertices $1, 2, \ldots, 8$ in $G_w$. The induced green subgraph can be assumed to be the graph of Fig. 4 (Lemma 4). Next consider any vertex $x$ of the graph $F = [9, 10, \ldots, 14]$.

If $(1, x)$ and $(5, x)$ are both red, and no red $H_5$ occurs, then green edges from $x$ to $2, 4, 6, 8$ are forced, and also $(3, x)$ or $(7, x)$, say $(3, x)$, has to be green. Then $[w, 2, 3, 4, x]$ contains a green $H_5$. Thus $(1, x)$ can be assumed to be green.

If $(2, x)$ and $(8, x)$ are both red, and no red $H_5$ occurs, then $(3, x)$ and $(7, x)$, and $(4, x)$ or $(6, x)$, say $(4, x)$, have to be green. Then $[w, 3, 4, 7, x]$ contains a green $H_5$. Thus $(2, x)$ can be assumed as green edge.

It follows by symmetry, that every vertex of $F$ is connected in green to both vertices of at least one of the edges $(1, 2), (4, 5), (5, 6)$ and $(1, 8)$. It can be assumed that 2 vertices of $F$, say $a$ and $b$, are both connected by green edges to $1$ and $2$. If no green $H_5$ occurs, then all edges from $3, 4$ and $8$ to $a$ and $b$ have to be red, and $[3, 4, 8, a, b]$ contains a red $H_5$.

**Theorem 3.** $r(H_4) = r(H_6) = 18$.

**Proof.** Since $H_4$ and $H_6$ both contain a subgraph $K_4$, the Ramsey number $r(K_4) = 18$ (see [2]) immediately ensures $r(H_6) \geq 18$. Since $H_6$ is a subgraph of $H_6$, it remains to prove $r(H_6) \leq 18$.

Any 2-coloring of $K_{18}$ contains a monochromatic (say green) $K_4 = [1, 2, 3, 4]$, since $r(K_4) = 18$. Every vertex of $F_1 = [5, 6, \ldots, 18]$ is connected by at least 3 red edges to $[1, 2, 3, 4]$, if no green $H_6$ occurs. Altogether there are at least 42 red edges between $F_1$ and $[1, 2, 3, 4]$, so that at least one vertex of $[1, 2, 3, 4]$, say $1$, is connected by red edges to at least 11 vertices of $F_1$, say to $5, 6, \ldots, 15$. Since $r(K_{13} + e, K_4) = 10$ [2], either these 11 vertices yield a green $K_4$, or together with 1 they yield a red $H_6$.

Again every vertex of $F_2 = [9, 10, \ldots, 15]$ is connected by at least 3 red edges to $[5, 6, 7, 8]$, if no green $H_6$ occurs. Altogether there are at least 21 red edges between $F_2$ and $[5, 6, 7, 8]$, so that at least one vertex of $[5, 6, 7, 8]$, say $5$, is connected by red edges to at least 6 vertices of $F_2$, say to $9, 10, \ldots, 14$. At last either a red edge in $[9, 10, \ldots, 14]$ determines a red $H_6$, or $[9, 10, \ldots, 14]$ is a green $K_6$, which contains a green $H_6$.

Hence, the Ramsey numbers are now known for all graphs with at most 5 vertices, excluding the complete graph $K_5$.

**References**


