Rainbow Numbers for Cycles in Plane Triangulations

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Received May 22, 2013; Revised March 26, 2014

Published online in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21803

Abstract: In the article, the existence of rainbow cycles in edge colored plane triangulations is studied. It is shown that the minimum number \(rb(T_n, C_3)\) of colors that force the existence of a rainbow \(C_3\) in any \(n\)-vertex plane triangulation is equal to \(\left\lfloor \frac{3n-4}{2} \right\rfloor\). For \(k \geq 4\) a lower bound and for \(k \in \{4, 5\}\) an upper bound of the number \(rb(T_n, C_k)\) is determined. © 2014 Wiley Periodicals, Inc. J. Graph Theory 00: 1–10, 2014

Keywords: edge coloring; rainbow number; rainbow subgraph; triangulation

Contract grant sponsor: Slovak Science and Technology Assistance Agency; Contract grant number: APVV-0023-10; Contract grant sponsor: Slovak VEGA; Contract grant number: 1/0652/12; Contract grant sponsor: P. J. Šafárik University within the project EXPERT; Contract grant number: ITMS 26110230056.

Journal of Graph Theory
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1. INTRODUCTION

We use [1] for terminology and notation not defined here and consider finite and simple graphs only. If \( G \) is edge colored in a given way and a graph \( H \subseteq G \) contains no two edges of the same color, \( H \) is called a rainbow subgraph of \( G \) or, in other words, a rainbow (copy of) \( H \). On the other hand, if all edges of \( H \) are colored with the same color, \( H \) is called monochromatic. Let \( f(G, H) \) denote the maximum number of colors in an edge coloring of \( G \) with no rainbow copy of \( H \). The number \( f(K_n, H) \) is called anti-Ramsey number and has been introduced by Erdős, Simonovits, and Sós in [2] (and denoted there by \( f(n, H) \)). It is closely related to the rainbow number \( \text{rb}(G, H) \) representing the minimum number \( c \) of colors such that any edge coloring of \( G \) with at least \( c \) colors contains a rainbow subgraph isomorphic to \( H \). Evidently, \( \text{rb}(G, H) = f(G, H) + 1 \).

Let \( H \) be a graph and \( G \) a class of graphs containing at least one graph \( G \) such that \( H \subseteq G \). The rainbow number of \( H \) in \( G \), in symbols \( \text{rb}(G, H) \), is the minimum number of colors \( c \) such that, if \( H \subseteq G \in G \), then any edge coloring of \( G \) with at least \( c \) colors contains a rainbow copy of \( H \).

For cycles the following result (which has been conjectured by Erdős, Simonovits, and Sós [2]) has been shown by Montellano-Ballesteros and Neumann-Lara [9].

**Theorem 1.** If \( n \geq k \geq 3 \), then \( \text{rb}(K_n, C_k) = \left\lfloor \frac{n}{k-1} \right\rfloor \left( \binom{k-1}{2} + (r) + \left\lfloor \frac{n}{k-1} \right\rfloor \right) \), where \( r \in \{0, \ldots, k-2\} \) is the residue of \( n \) modulo \( k-1 \).

Rainbow numbers for cycles with one or two additional edges have been studied in [4,5,8,10]. Jendrol’ et al. in [7] investigated the problem of finding the minimum number of colors that forces the existence of a rainbow face in any edge coloring of a fixed plane graph. A recent survey concerning rainbow numbers is given in [3].

In this article we study the numbers \( \text{rb}(T_n, C_k) \), where \( T_n \) is the class of all plane triangulations of order \( n \). Obviously, these numbers are defined for any \( n, k \) with \( n \geq k \geq 3 \).

For the computation of \( \text{rb}(K_n, H) \) often an induction can be applied by deleting a vertex \( v \in V(K_n) \) and considering the graph \( K_n-1 = K_n - v \). However, if \( G \in T_n \), then \( G - v \) may be out of \( T_n-1 \). So, we have to look for other proof methods.

2. RAINBOW 3-CYCLES

**Theorem 2.** If \( n \geq 4 \), then \( \text{rb}(T_n, C_3) = \left\lfloor \frac{3n-4}{2} \right\rfloor \).

**Proof.** We first construct a plane triangulation \( T_n^3 \in T_n \) for all \( n \geq 6 \) and its edge coloring with \( \left\lfloor \frac{3n-6}{2} \right\rfloor \) colors that contains no rainbow 3-cycle.

In the first step, we construct a sequence of quadrangulations \( Q_r \) on \( r \geq 4 \) vertices starting with \( Q_4 \approx C_4 \). From \( Q_r \) we construct \( Q_{r+1} \) by choosing an arbitrary 4-face, inserting a new vertex in it and making it adjacent to two antipodal vertices of this 4-face. So \( Q_r \) has \( r \) vertices, \( 2r - 4 \) edges, \( r - 2 \) faces and we color its edges in a rainbow way. In the second step, we insert \( r - 2 \) vertices, one in each quadrangle of \( Q_r \) and make it...
adjacent to all vertices of that quadrangle. All these four edges are colored with a new color. An operation of this kind will be called in the sequel adding a monochromatic star.

In this way we obtain a plane triangulation $T^3_{2r-2}$ with $n = 2r - 2$ vertices whose edges are colored with $3r - 6 = \lfloor \frac{3n-6}{2} \rfloor$ colors. Adding a monochromatic star $K_{1,3}$ to one of the faces of $T^3_{2r-2}$ results in a triangulation $T^3_n \in \mathcal{T}_n$ with $n = 2r - 1$ vertices, whose edges are colored with $3r - 5 = \lfloor \frac{3n-5}{2} \rfloor$ colors.

Finally observe that $T^3_n$ contains no rainbow 3-cycle, which proves that $rb(T_n, C_3) \geq \lfloor \frac{3n-4}{2} \rfloor + 1$.

In the case $n = 4, 5$ we start from a 2-colored $K_3$ and (subsequently) add $n - 3$ monochromatic stars $K_{1,3}$. In this way we have found an edge coloring of $T_n \in \mathcal{T}_n$ with $n - 1$ colors, which means that $rb(T_n, C_3) \geq n = \lfloor \frac{3n-4}{2} \rfloor$.

For the proof of the upper bound we use the following

**Claim 1.** Let $C_k$ with $k \geq 4$ be a rainbow cycle in an edge colored graph $G$. If $G[C_k]$ has a chord, then there exists a rainbow cycle in $G$ of length smaller than $k$.

**Proof.** A chord in $C_k$ spans two cycles $C_{k_1}, C_{k_2}$ with $k_1 + k_2 = k + 2$, where $k_1, k_2 < k$. At least one of them is a rainbow cycle. ■

Let $T_n \in \mathcal{T}_n$ be edge colored using at least $\lfloor \frac{3n-4}{2} \rfloor$ colors and let $G = (V, E, F)$ be a rainbow map with $|E| = \lfloor \frac{3n-4}{2} \rfloor$ that originates from the given coloring of $T_n$. (From each of the $\lfloor \frac{3n-4}{2} \rfloor$ color classes one edge is taken to $E$.) Then, since $\lfloor \frac{3n-4}{2} \rfloor \geq n$, $G$ contains a (rainbow) cycle $C_k$ for some $k \geq 3$. If $k = 3$, we are done. Otherwise, we distinguish two cases.

**Case 1** $G$ has a cycle $\tilde{C} \cong C_k$ (for some $k \geq 4$) with no inner vertices. As $T_n[\tilde{C}]$ has a chord, using Claim 1 (repetitively, if necessary) we obtain a rainbow $C_3$ in $T_n$.

**Case 2** Every cycle $C_k$ in $G$ has an inner vertex. We are going to show that this assumption leads to a contradiction.

Let $G' = (V', E', F')$ be the graph obtained from $G$ by deleting all bridges. Denote by $\omega$ the number of components of $G$ and by $\omega'$ the number of components of $G'$. Evidently, $V = V', |F'| = |F'|$ and (an easy exercise) $\omega' = \omega + |E| - |E'|$. Then each face of $G'$ has a disconnected boundary.

**Claim 2.** $|F'| \leq \omega' - 1$.

**Proof.** For each face $f \in F'$ choose two vertices of $f$ belonging to distinct (topological) components of the boundary of $f$ and join them by an arc lying in $f$. The resulting plane graph has the number of components equal to $\omega' - |F'| \geq 1$.

By Euler’s formula we have $|E'| = |V'| + |F'|- 1 - \omega'$. Since $G'$ has no 3-face, we have $2|E'| \geq 4|F'|$ and

$$2 + 2 \omega' = 2|V'| - 2|E'| + 2|F'| \leq 2|V'| - 2|E'| + |E'| = 2|V'| - |E'|,$$

hence

$$|F'| = |E'| - |V'| + 1 + \omega' \leq |V'| - 1 - \omega',$$

and, consequently (using Claim 2),

$$2|F'| - 2 - 2 \omega' \leq |F'| + |V'| - 3 - 3 \omega' \leq \omega' - 1 + |V'| - 3 - 3 \omega' = |V'| - 4 - 2 \omega'.$$

Journal of Graph Theory DOI 10.1002/jgt
Finally,
\[ |E| = |E'| + \omega' - \omega = |V'| + |F'| - 1 - \omega' + \omega' - \omega \]
\[ \leq |V'| + \frac{1}{2} (|V'| - 4 - 2 \omega') + \omega' - \omega = \frac{1}{2} (3n - 4) - \omega < \left\lceil \frac{3n - 4}{2} \right\rceil, \]
a contradiction.

3. RAINBOW 4-CYCLES

A. A lower bound

Theorem 3. If \( n \geq 42, r \in \{0, \ldots, 19\} \) and \( n \equiv 2 + r \mod 20 \), then
\( \text{rb}(T_n, C_4) \geq \frac{9}{5} (n - 2) - \frac{4}{5} r + 1. \)

Proof. Jendrol’ and Jucovič [6] have shown that for each integer \( t \geq 0 \) there exists a 3-connected plane map \( M'_t \) having only vertices of degree 4 and 6 and only 3-faces and 5-faces, in which no two faces of the same size share an edge. This map has \( n'_4 = 30 + 12t \) vertices of degree 4, \( n'_6 = 2t \) vertices of degree 6, \( f'_3 = 20 + 10t \) triangular faces and \( f'_5 = 12 + 6t \) pentagonal faces.

Then \( 3f'_3 = e' = 5f'_5 = 2n'_4 + 3 \cdot 2t \), which gives \( n'_4 = \frac{e'}{2} - 3t \). Now using Euler’s formula we obtain \( (f'_3 + f'_5) + (n'_4 + 2t) = e' + 2 \). Substituting from above we have \( (\frac{e'}{2} + \frac{e'}{5}) + (\frac{e'}{2} - 3t + 2t) = e' + 2 \). This leads to \( \frac{e'}{2} - 2 = t \) and, consequently, to \( n'_4 = \frac{2}{5} e' + 6 \).

Let us color \( M'_t \) in a rainbow way and add a monochromatic star \( K_{1,3} \) into each 5-face of \( M'_t \). The constructed plane triangulation \( T' \) has
\[ n'_4 + n'_6 + f'_5 = \left( \frac{2}{5} e' + 6 \right) + \left( \frac{e'}{15} - 4 \right) + \frac{e'}{5} = \frac{2}{5} e' + 2 = 20t + 42 \]
vertices. Finally, for \( n \equiv 2 + r \mod 20 \) with \( r \in \{0, \ldots, 19\} \) we subsequently add \( r \) monochromatic stars \( K_{1,3} \) to \( T' \) to obtain \( T'_n \in T_n \). Observe that we have used
\[ e' + f'_5 + r = \frac{6}{5} e' + r = \frac{9}{5} (n - r - 2) + r = \frac{9}{5} (n - 2) - \frac{4}{5} r \]
colors on the edges of \( T'_n \) and that \( T'_n \) has no rainbow 4-cycle. □

B. An upper bound

Let \( W_d \) be a wheel with a central vertex \( v \), rim vertices \( v_i \), spokes \( s_i = vv_i \), and rim edges \( r_i = v_i v_{i+1}, i = 1, \ldots, d \) (with indices modulo \( d \)). A cycle \( C \subseteq W_d \) is said to be central if it passes through the centre of \( W_d \). In each triangulation \( T \) on at least four vertices the subgraph of \( T \) induced by the closed neighborhood of \( v \in V(T) \) is a supergraph of the graph \( W(v) \cong W_d \) with central vertex \( v \) and \( d = \text{deg}(v) \). Moreover, for an edge coloring \( \varphi \) of \( T \) we denote by \( C_{\varphi}(v) \) the set of colors used by \( \varphi \) for the edges of \( W(v) \).

The following two lemmas will be useful for our investigation.
Lemma 1. Let $T$ be a triangulation and $\varphi : E(T) \to A$ a surjection. Then

$$\sum_{v \in V(T)} |C_\varphi(v)| \geq 4|A|.$$ 

Proof. Each edge of $T$ belongs to exactly two 3-faces and therefore to exactly four wheels in $T$. Since $\varphi$ is a surjection, any color $c \in A$ appears in $C_\varphi(v)$ for at least four distinct vertices of $T$. Thus, we have

$$\sum_{v \in V(T)} |C_\varphi(v)| = \sum_{v \in V(T)} \sum_{c \in A} 1 = \sum_{c \in A} \sum_{v \in V(T)} 1 \geq \sum_{c \in A} 4 = 4|A|. \qed$$

Recall that $\text{rb}(W_d, C_k)$ denotes the rainbow number of the $k$-cycle in the wheel $W_d$.

Lemma 2. \[ \text{rb}(W_d, C_4) = \left\lceil \frac{4}{3} d \right\rceil + 1. \]

Proof. First, we construct a coloring $\tilde{\varphi}$ of $W_d$ that does not create a rainbow 4-cycle. Put for $i = 1, \ldots, d$

$$\tilde{\varphi}(s_i) = i$$

$$\tilde{\varphi}(r_i) = \begin{cases} 
\tilde{\varphi}(s_{i+1}), & \text{if } i \equiv 1 \pmod{3} \\
\tilde{\varphi}(s_i), & \text{if } i \equiv 2 \pmod{3} \\
d + \frac{i}{3}, & \text{if } i \equiv 0 \pmod{3}
\end{cases}$$

Since $\tilde{\varphi}$ uses $\left\lceil \frac{4}{3} d \right\rceil$ colors, we see that $\text{rb}(W_d, C_4) \geq \left\lceil \frac{4}{3} d \right\rceil + 1$.

To prove the opposite inequality suppose there is a surjection $\varphi : E(W_d) \to A$ with $|A| \geq \left\lceil \frac{4}{3} d \right\rceil + 1$ containing no rainbow $C_4$. Then in each of $d$ central cycles of length 4 there are two distinct edges having the same color. Therefore, if $A_i$ is the set of colors used $i$ times by $\varphi$, $q_j(c)$ the number of central 4-cycles containing $j$ edges of a color $c \in A$ and $q(c) = \sum_{j=2}^{4} q_j(c)$, then

$$d \leq \sum_{c \in A} q(c) = \sum_{i \geq 2} \sum_{c \in A_i} q(c). \quad (1)$$

Two edges colored with a color from $A_2$ can prevent at most one central 4-cycle from being rainbow, hence

$$\sum_{c \in A_2} q(c) \leq |A_2|. \quad (2)$$

Next, we show that $q(c) \leq i$ for every pair $i, c$ with $i \geq 3$ and $c \in A_i$. For this purpose let $q_j(c)$ denote the number of central 4-cycles containing $j$ edges of the color $c$ for $1 \leq j \leq 4$. Since every edge is contained in two central 4-cycles, we deduce that

$$2i = \sum_{j=1}^{4} jq_j(c) \geq 2 \sum_{j=2}^{4} q_j(c) = 2q(c),$$

and then

$$\sum_{i \geq 3} \sum_{c \in A_i} q(c) \leq \sum_{i \geq 3} \sum_{c \in A_i} i = \sum_{i \geq 3} i|A_i|. \quad (2)$$

Journal of Graph Theory DOI 10.1002/jgt
Now from (1) and (2) it follows $|A_2| + \sum_{i \geq 3} i|A_i| \geq d$ so that, using $\sum_{i \geq 1} i|A_i| = 2d$, we obtain $|A_1| + |A_2| \leq d$. On the other hand, we have

$$2d = \sum_{i \geq 1} i|A_i| \geq |A_1| + |A_2| + 3\left(\sum_{i \geq 1} |A_i| - |A_2| - |A_1|\right) \geq -2|A_1| - 2|A_2| + 3\left(\left\lfloor \frac{4}{3} d \right\rfloor + 1\right) \geq -2d + 3\left\lfloor \frac{4}{3} d \right\rfloor + 3 \geq 2d + 1,$$

a contradiction.

**Theorem 4.** \(rb(T_n, C_4) \leq 2(n - 2) + 1\) for all \(n \geq 4\).

**Proof.** Consider \(T \in T_n\) and suppose there is a surjection \(\varphi : E(T) \to A\) with \(|A| \geq 2(n - 2) + 1\) that does not create a rainbow \(C_4\). Note that then no rainbow \(C_4\) is present in the restriction of \(\varphi\) to \(W(v), v \in V(T)\). Therefore, by Lemmas 1 and 2,

$$4|A| \leq \sum_{v \in V(T)} |C_\varphi(v)| \leq \sum_{v \in V(T)} (rb(W(v), C_4) - 1) \leq \sum_{v \in V(T)} \frac{4}{3} \deg(v) = \frac{4}{3} \cdot 2 |E(T)| = \frac{8}{3} (3n - 6) = 8(n - 2),$$

a contradiction. So, \(rb(T_n, C_4) \leq 2(n - 2) + 1\).

## 4. RAINBOW 5-CYCLES

### A. A lower bound

**Theorem 5.** If \(n \geq 20\), \(r \in \{0, \ldots, 17\}\) and \(n - 2 \equiv r \pmod{18}\), then

$$rb(T_n, C_5) \geq \frac{19}{9}(n - 2) - \frac{10}{9}r + 1.$$  

**Proof.** We start our construction with a plane triangulation \(M'\) having \(n'\) vertices, \(e'\) edges and \(f' = 2n' - 4\) faces. Replace each edge \(xy \in E(M')\) by a configuration given in Fig. 1. For the resulting map \(M''\) we have \(n'' = |V(M'')| = n' + 5e' = 16n' - 30\), and \(e'' = |E(M'')| = 12e' = 36n' - 72\).

We color the edges of \(M''\) in a rainbow way and add a monochromatic star \(K_{1,9}\) into each 9-face of \(M''\). The triangulation created so far has \(n'' + f'' = (16n' - 30) + (2n' - 4) = 18n' - 34 \geq 20\) vertices and we used in it \(e'' + f'' = (36n' - 72) + (2n' - 4) =
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rainbow way
and rim edges of
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using \( \left\lfloor \frac{d}{3} \right\rfloor \) additional colors that form monochromatic
P_3s (and one monochromatic
P_4 if
d
is odd).

The proof of the upper bound is similar to the proof of Lemma 2. Suppose there is a
surjection
φ : E(W_d) \to A
with |A| ≥ \( \left\lfloor \frac{3i}{2} \right\rfloor \) + 1 that does not create a rainbow
C_5 and let
A_i
be the set of colors used
i
times by
φ.
For a color
c \in A
let
p_j(c)
be the number of
j
central 5-cycles in
W_d
containing
j
edges colored
c,
p(c) = \sum_{j=2}^{5} p_j(c), s(c) the number of
speaks colored
c
and
r(c)
the number of rim edges colored
c.
Note that each spoke is
in exactly two central 5-cycles, each rim edge is in exactly three central 5-cycles and if
\( c \in A_i \), then
s(c) + r(c) = i.
Because of that for
\( c \in A_i \)
we have
\[ 2p(c) \leq 2p(c) + p_1(c) \leq \sum_{j \geq 1} j p_j(c) = 2s(c) + 3r(c) \leq 3i \]
and
\[ p(c) \leq \left\lfloor \frac{3i}{2} \right\rfloor. \]
Moreover, it is easy to see that
\[ p(c) \leq 2 \text{ for } c \in A_2. \]
Each of
\( d \) central 5-cycles of
W_d
contains a color
c
with
p(c) ≥ 1.
Therefore,
\[ d \leq \sum_{i \geq 2} \sum_{c \in A_i} p(c) \leq 2|A_2| + 4|A_3| + \sum_{i \geq 4} \frac{3i}{2} |A_i|, \]
and, consequently,
\[ \frac{2d}{3} \leq \frac{4}{3} |A_2| + \frac{8}{3} |A_3| + \sum_{i \geq 4} i |A_i|. \]
Since
\[ 2d = \sum_{i \geq 1} i |A_i|, \]
from the last inequality we obtain
\[ \frac{4d}{3} \geq |A_1| + \frac{2}{3} |A_2| + \frac{1}{3} |A_3| = \sum_{i=1}^{3} \frac{4 - i}{3} |A_i| \]
and then
\[ \left\lfloor \frac{3}{2} d \right\rfloor + 1 \leq |A| = \sum_{i=1}^{3} |A_i| + \sum_{i \geq 4} i |A_i| \leq \sum_{i=1}^{3} |A_i| + \sum_{i \geq 4} \frac{i}{4} |A_i| \]
\[ = \sum_{i=1}^{3} \left( 1 - \frac{i}{4} \right) |A_i| + \frac{1}{4} \sum_{i \geq 1} i |A_i| = \sum_{i=1}^{3} \frac{4 - i}{4} |A_i| + \frac{1}{4} \cdot 2d \]
\[ = \frac{3}{4} \sum_{i=1}^{3} \frac{4 - i}{3} |A_i| + \frac{d}{2} \leq \frac{3}{4} \cdot \frac{4d}{3} + \frac{d}{2} = \frac{3d}{2}, \]
a contradiction. \[ \blacksquare \]
In the next auxiliary result we bound from above the number $v_3(T)$ of 3-vertices in a triangulation belonging to $\mathcal{T}_n$.

**Lemma 4.** If $T \in \mathcal{T}_n$ with $n \geq 5$, then $v_3(T) \leq \lceil \frac{2(n-2)}{3} \rceil$.

**Proof.** The average degree of $T$ is

$$\frac{\sum_{v \in V(T)} \deg(v)}{n} = \frac{2(3n-6)}{n} = \frac{6 - \frac{12}{n}}{n} \geq 3.$$

Thus, removing all $v_3$ vertices of degree 3 from $T$ ($n \geq 5$ implies that they form an independent set) we obtain a (nonempty) triangulation $T'$ with $n'$ vertices and $f' = 2n' - 4$ faces. Then $n' = n - v_3$ and $v_3 \leq f' = 2(n - v_3) - 4$, which implies the desired result.

Now we are able to prove

**Theorem 6.** If $n \geq 5$, then $\text{rb}(\mathcal{T}_n, C_5) \leq \frac{5}{2} (n - 2) + 1$.

**Proof.** Proceeding by the way of contradiction suppose that for some $T \in \mathcal{T}_n$ there is $A$ with $|A| > \frac{5}{2} (n - 2)$ and a surjection $\varphi : E(T) \to A$ with no rainbow 5-cycle. For each $v \in V(\tilde{T})$ the restriction of $\varphi$ to $W(v)$ contains no rainbow $C_5$. Therefore if $d = \deg(v) \geq 4$, by Lemma 3 we see that $|C_\varphi(v)| \leq \text{rb}(W_d, C_5) - 1 = \lceil \frac{3d}{2} \rceil$. On the other hand for $d = 3$ we have $|C_\varphi(v)| \leq |E(W(v))| = 6$. Using Lemmas 1 and 4 then

$$4|A| \leq \sum_{v \in V(T)} |C_\varphi(v)| \leq 6v_3(T) + \sum_{v \in V(T), \deg(v) \geq 4} \left[ \frac{3 \deg(v)}{2} \right] \leq 3v_3(T) + \frac{3}{2} \sum_{v \in V(T)} \deg(v) \leq \frac{3}{2} \cdot \frac{2(n-2)}{3} + \frac{3}{2} \cdot 2(3n-6) = 10(n-2),$$

which implies $|A| \leq \frac{5}{2} (n - 2)$, a contradiction.

\section{Rainbow Cycles of Length at Least 6}

**Theorem 7.** If $6 \leq k \leq n$, then $\text{rb}(\mathcal{T}_n, C_k) \geq (3n-6) \cdot \frac{k-3}{k-2} - \frac{k-5}{k-2}$.

**Proof.** Let $b = \lceil \frac{n-2}{k-2} \rceil - 1 \geq 1$. We first construct an auxiliary plane triangulation $\tilde{T}$ on $2b + 4$ vertices as follows: $V(\tilde{T}) = X \cup Y$, where $X = \{x_1, x_2, x_3\}$, $Y = \{y_i : 1 \leq i \leq 2b + 1\}$, and

$$E(\tilde{T}) = \{x_1x_2, x_1x_3, x_2x_3\} \cup \{y_iy_{i+1} : 1 \leq i \leq 2b\} \cup \{y_{2i-1}y_{2i+1} : 1 \leq i \leq b\} \cup \{x_1y_{2i+1} : 0 \leq i \leq b\} \cup \{x_3y_i : 4 \leq i \leq 2b + 1\}$$

Expressing $n$ in the form

$$n = (k - 2)(b + 1) + 2 - r, \ 0 \leq r \leq k - 3 \quad (3)$$

it is possible to insert $v_0 \in [k-5, k-4]$ vertices in the face $x_1x_2x_3$ and $v_i \leq k-4$ vertices in the face $y_{2i-1}y_{2i}y_{2i+1}, 1 \leq i \leq b$, in such a way that $\sum_{i=0}^{b} v_i = n - |V(\tilde{T})|$.

Journal of Graph Theory DOI 10.1002/jgt
Indeed, for that we need
\[ k - 5 \leq \sum_{i=0}^{b} v_i \leq (k - 4)(b + 1); \]
since \( n - |V(\tilde{T})| = (k - 2)(b + 1) + 2 - r - (2b + 4) = (k - 4)(b + 1) - r \), the required inequalities follow from \( b \geq 1 \) and from \( 0 \leq r \leq k - 3 \). We now triangulate the face with \( v_i \) inserted vertices using \( 3v_i \) additional edges, \( 0 \leq i \leq b \), to obtain a triangulation \( T_n^k \in T_n \). To be sure that \( T_n^k \) contains \( C_k \) we triangulate the face \( x_1x_2x_3 \) in such a way that there is a path \( P \) of length \( v_0 + 3 \) from \( x_1 \) to \( x_2 \) traversing all vertices of \( \{x_3\} \cup V_0 \), where \( V_0 \) is the set of \( v_0 \) vertices inserted in \( x_1x_2x_3 \). Then \( P \) together with either the path \( x_2y_2x_1 \) (if \( v_0 = k - 4 \)) or the path \( x_2y_2y_1x_1 \) (if \( v_0 = k - 5 \)) forms a cycle of length \( k \) in \( T_n^k \).

We color all the edges of \( T_n^k \) joining \( X \) to \( Y \) with the same color and all remaining edges of \( T_n^k \) in a rainbow way without repeating the “frequent” color. Evidently, no rainbow \( C_k \) has been created, and, because of (3), the number of colors is
\[
p = \sum_{i=0}^{b} (3v_i + 3) + 1 = 3(b + 1)(k - 3) - 3r + 1 = (3n - 6) \cdot \frac{k - 3}{k - 2} - \frac{3r}{k - 2} + 1.
\]
As a consequence from \( r \leq k - 3 \) we obtain \( \text{rb}(T_n^k, C_k) \geq p + 1 \geq (3n - 6) \cdot \frac{k - 3}{k - 2} - \frac{3r}{k - 2} \).

If \( k = n \geq 6 \), then Theorem 7 yields \( \text{rb}(T_n, C_n) \geq (3n - 6) \cdot \frac{n - 3}{n - 2} - \frac{n - 5}{n - 2} = 3n - 10 + \frac{3}{n - 2} \), hence \( \text{rb}(T_n, C_n) \geq 3n - 9 \). This bound can be improved as follows:

**Proposition 1.** If \( n \geq 4 \), then \( \text{rb}(T_n, C_n) \geq 3n - 7 \).

**Proof.** It is easy to see that for each \( n \geq 4 \) there is \( H_n \in T_n \) that is Hamiltonian and contains a vertex \( x \) of degree 3. In any edge coloring of \( H_n \), that uses \( 3n - 8 \) colors in such a way that all edges incident to \( x \) have the same color, there is no rainbow (Hamiltonian) cycle \( C_n \). Thus, \( \text{rb}(T_n, C_n) \geq 3n - 7 \).

Finally, notice that for bounding \( \text{rb}(T_n, C_k) \) from above we have only the trivial bound \( 3n - 6 \).

**REFERENCES**


*Journal of Graph Theory* DOI 10.1002/jgt