Insertible Vertices, Neighborhood Intersections, and Hamiltonicity

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ABSTRACT

Let $G$ be a simple undirected graph of order $n$. For an independent set $S \subseteq V(G)$ of $k$ vertices, we define the $k$ neighborhood intersections $S_i = \{v \in V(G) \setminus S ||N(v) \cap S|| = i\}$, $1 \leq i \leq k$, with $s_i = |S_i|$. Using the concept of insertible vertices and the concept of neighborhood intersections, we prove the following theorem.

**Theorem.** Let $G$ be a graph of order $n$ and connectivity $\kappa \geq 2$. Then $G$ is hamiltonian or there exists an independent set $X \subseteq V(G)$ of cardinality $t + 1$, $1 \leq t \leq \kappa$ such that

$$\sum_{i=1}^{t+1} w_i s_i \leq n - 1 - \sum_{j=2}^{t+1} |N_j(X)|$$

where $w_i$, $1 \leq i \leq t + 1$, are real numbers satisfying $0 \leq w_i \leq 1$, and for $1 < i_1 \leq i_2 \cdots \leq i_m \leq t + 1$ and $\sum_{j=1}^{m} i_j \leq t + 1$ we have $\sum_{j=1}^{m} (w_j - 1) \leq 1$; where $N_j(X)$ denotes the set of vertices whose nearest vertex in $X$ is at distance $j$.

This theorem improves or generalizes many well-known sufficient conditions for hamiltonian graphs. © 1995 John Wiley & Sons, Inc.

1. INTRODUCTION

In this paper we consider simple undirected graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$. For a set $S \subseteq V(G)$ let $N_j(S)$ denote the set of vertices whose nearest vertex in $S$ is at distance $j$. For simplicity we write $N(S)$ instead of $N_1(S)$ for the neighborhood of $S$. If $S$ contains at least 2 vertices we denote $n - \sum_{j \geq 2} |N_j(S)|$ by $n(S)$. For any independent set (pairwise nonadjacent) of $t$ vertices, $\sigma_t(G)$ is the minimum value of the degree-sum, respectively denote $\alpha(G)$ and $\kappa$ the independence number and (vertex-connectivity) number of $G$.

For an independent set $S \subseteq V(G)$ of $t + 1$ vertices we define the $t + 1$ neighborhood intersections $S_i = \{v \in V(G) \setminus S||N(v) \cap S| = i\}$, $1 \leq i \leq t + 1$. The cardinalities of $S_i$ will be denoted by $s_i$. In 1972 Chvátal and Erdős proved the following theorem.

**Theorem 1** [9]. Let $G$ be a graph with $n \geq 3$ vertices and $\alpha(G) \leq \kappa$. Then $G$ is hamiltonian.

Part of their proof has turned out to be a basic method and since then several sufficient conditions for hamiltonian graphs have been established; degree-conditions as well as conditions based on neighborhood unions. In this paper we show that the concept of neighborhood intersections [16,17] and the concept of insertible vertices [1,2,3] can be nicely combined. As a consequence, our main result (Theorem 2) is a sufficient condition for hamiltonian graphs generalizing or improving many previous results.

To present Theorem 2 we define an additive weight-function $w$ that assigns to each vertex $v \in V(G)$ a weight corresponding to an independent set $S \subseteq V(G)$ of cardinality $t + 1$. To be more precise, for a sequence of $t + 1$ non-negative real numbers $w_1, w_2, \ldots, w_{t+1}$ we define

$$w(v) = w_i, \text{ if } v \in S_i, 1 \leq i \leq t + 1 \text{ and } w(v) = 0 \text{ otherwise.} \quad (1)$$

In addition, we require (2) and (3) for $w_1, w_2, \ldots, w_{t+1}$

$$0 \leq w_1 \leq 1. \quad (2)$$

For $2 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq t + 1$ and

$$\sum_{j=1}^{p} i_j \leq t + 1, \text{ we have } \sum_{j=1}^{p} (w_{i_j} - 1) \leq 1. \quad (3)$$

**Remark 1.** It is easy to check that $w_i \leq 2$ for $2 \leq i \leq t + 1$.

**Theorem 2.** Let $G$ be a graph of order $n \geq 3$ and connectivity $\kappa \geq 2$. Then $G$ is hamiltonian or there exists an independent set $X \subseteq V(G)$ of
cardinality $t + 1$ such that

$$
\sum_{i=1}^{t+1} w_i s_i \leq n(X) - 1,
$$

where $w_i$, $1 \leq i \leq t + 1$ are real numbers satisfying (2) and (3).

**Corollary 1.** Let $G$ be a graph of order $n \geq 3$ and connectivity $\kappa \geq 2$. Suppose there exists some $t$, $1 \leq t \leq \kappa$, such that for any independent set $X \subset V(G)$ of cardinality $t + 1$ and for $1 \leq i \leq t + 1$, $w_i$ are real numbers satisfying (2) and (3), we have

$$
\sum_{i=1}^{t+1} w_i s_i > n(X) - 1.
$$

Then $G$ is hamiltonian.

**Corollary 2.** Let $G$ be a graph of order $n \geq 3$ and connectivity $\kappa \geq 2$. Suppose there exists some $t$, $1 \leq t \leq \kappa$, such that for any independent set $X \subset V(G)$ of cardinality $t + 1$ and for $1 \leq i \leq t + 1$, $w_i$ are real numbers satisfying (2) and (3), we have

$$
\sum_{i=1}^{t+1} w_i s_i > n - 1.
$$

Then $G$ is hamiltonian.

Theorem 2 will be proved in the next section. In Section 3 we show that Corollary 2 and thus Theorem 2 improves or generalizes many well-known sufficient conditions for hamiltonian graphs.

**2. PROOF OF THEOREM 2**

Let $C$ be a cycle of maximum length in a graph $G$ satisfying the hypothesis of the theorem. Suppose $C$ is not hamiltonian. Using Menger’s theorem for any arbitrary vertex $x_0 \in V(G) \setminus V(C)$ there exist $t \geq \kappa$ vertex disjoint (except $x_0$) paths from $x_0$ to $C$. Let $v_1, v_2, \ldots, v_t$ denote the contact vertices of these paths on $C$ and assume they appear in this order according to a fixed arbitrary orientation on $C$. For a vertex $v \in V(C)$ let $v^-$ and $v^+$ denote the vertex immediately preceding and following $v$ on $C$ according to the orientation, respectively. Set $y_i = v_i^+$ for $1 \leq i \leq t$ and $z_i = v_i^-$, for $1 \leq i \leq t - 1$ and $z_t = v_1^-$. It is well known (cf.[9]) that $\{x_0, y_1, \ldots, y_t\}$ is an independent set. Denote by $C_i$ the segment $[y_i, z_i]$ for $1 \leq i \leq t$. A vertex $u \in V(C_i)$ is said to be *insertible* if there exists $v, v^+ \in V(C) \setminus V(C_i)$
such that $uv, uv^+ \in E$. Similarly, a vertex $u \not\in V(C)$ is insertible if there exists $v, v^+ \in N(C)$ such that $uv, uv^+ \in E$.

**Lemma 1** [2]. For each $i$, $1 \leq i \leq t$, $C_i$ contains a noninsertible vertex.

According to the orientation of $C$ let $x_i$ be the first noninsertible vertex of $C_i$ for $i$, $1 \leq i \leq t$. Let $X = \{x_0, x_1, \ldots, x_t\}$.

**Lemma 2** [2].

(a) $X$ is an independent set.
(b) If $x_i, x_j \in X \setminus \{x_0\}$ are distinct, then there is no vertex $u$ in $[x_i, x_j]$ such that $x_iu^+, x_ju \in E$.

We now consider the following subsets of $V(G)$:

$$A = V(G) \setminus N(X), A_C = A \cap C, A_R = A \setminus C,$$

that is, $A$ is the set of vertices that are not adjacent to any vertex of $X$.

**Proposition 1** [2]. $X \subset A$. Therefore $x_0 \in A_R$

**Proposition 2** [2]. Two vertices of $X \setminus \{x_0\}$ have no common neighbor in $V(G) \setminus V(C) \cup \{x_0\}$.

Let $\alpha_1$ and $\alpha_2$ be two consecutive vertices of $A$ on $C$. As $X \subset A$, we can suppose that $\alpha_1$ and $\alpha_2$ belong to a segment $[y_i, y_{i+1}]$ with $1 \leq i \leq t - 1$ or $[y_i, y_1]$. Then, by Lemma 2,

$$N(x_i), N(x_{i-1}), \ldots, N(x_1), N(x_t), N(x_{t-1}), \ldots, N(x_{i+1}), N(x_0)$$

form consecutive segments of $C$ in $(\alpha_1, \alpha_2)$ (possibly empty) that can have only their extremities in common. Furthermore, by (4) and the choice of $X \setminus \{x_0\}$ we know that

$$|N(x_j) \cap V(\alpha_1, \alpha_2)| \leq 1$$

for $0 \leq j \leq t, j \neq i, i + 1, (i + 1 \text{ modulo } t)$

where $\{\alpha_1, \alpha_2\} \subset (x_i, x_{i+1}) (i + 1 \text{ modulo } t)$ for some $i$ with $1 \leq i \leq t$. Set $m = |V((\alpha_1, \alpha_2))| \geq 0$ and let $p$ denote the number of vertices of $V((\alpha_1, \alpha_2))$, each vertex belonging to $S_i$ with $i \geq 2$. Each vertex of $(\alpha_1, \alpha_2)$ belongs to a set $S_i$ for some $i$ with $1 \leq i \leq t + 1$. Let $1 < i_1 \leq i_2 \leq \cdots \leq i_p \leq t + 1$ be the indices of the neighborhood intersections $S_i$ with
$i \geq 2$ in a nondecreasing ordering. Then, remembering (5) we have

$$\sum_{j=1}^{p} i_j \leq t + 1. \quad (6)$$

The vertices of $C$ may now be considered as the union of segments of type $[\alpha_1, \alpha_2)$. If $\alpha_2 = \alpha_1^+$ then $w(V([\alpha_1, \alpha_2))) = w(\alpha_1) = 0$. Next we consider the case $\alpha_2 \neq \alpha_1^+$. If $(\alpha_1, \alpha_2)$ contains a vertex $v \in S_t \cup S_{t+1}$, then by (4) and (5)

$$V((\alpha_1, \alpha_2)) \setminus \{v\} \subset S_1 \quad (7)$$

and thus

$$w(V([\alpha_1, \alpha_2])) \leq 0 + 1 \cdot 2 + (m - 1) \cdot 1 = m + 1 = |V([\alpha_1, \alpha_2])|. \quad (8)$$

If $V([\alpha_1, \alpha_2)) \cap (S_t \cup S_{t+1}) = \emptyset$, then we have inequality in (6), i.e., $\sum_{j=1}^{p} i_j \leq t + 1$ and thus by (3) $\sum_{j=1}^{p} (w_{i_j} - 1) \leq 1$. Then by (1)

$$w(V([\alpha_1, \alpha_2])) = 0 + m + \sum_{j=1}^{p} (w_{i_j} - 1) \leq m + 1 = |V([\alpha_1, \alpha_2])|. \quad (9)$$

Therefore in any case

$$w(V([\alpha_1, \alpha_2])) \leq |V([\alpha_1, \alpha_2])|. \quad (10)$$

By Proposition 2 we have $U = (V(G) \setminus (V(C) \cup \{x_0\})) \subseteq A_R \cup S_1$. Hence $w(u) \leq 1$ for each vertex $u \in U$ and thus

$$w(U) \leq |U|. \quad (11)$$

Finally, $x_0 \in A_R \subset A$ and hence $w(x_0) = 0$. Combining this with (8) and (9) we obtain

$$w((V)) = w(V(C)) + w(U) + w(x_0) \leq |V(C)| + |U| + 0 = n - 1. \quad (12)$$

Furthermore, if $\alpha_1 \in \bigcup_{j\geq 2} N_j(X)$, then $\alpha_1, \alpha_1^+ \in A$. As already mentioned, we have $w(\alpha_1) = 0$ if $\alpha_2 = \alpha_1^+$ and thus an amount of 1 can be substracted in the estimation above. The same holds for any vertex of $A_R \cap (\bigcup_{j\geq 2} N_j(X))$. Thus we obtain

$$\sum_{i=1}^{i+1} w_i \cdot s_i = w((V)) = w(V(C)) + w(U) + w(x_0) - \sum_{j>2} |N_j(X)| \leq n(X) - 1 \quad (13)$$

for our set $X$. This completes the proof of Theorem 2. \qed
3. NEW AND OLD SUFFICIENT CONDITIONS

Theorem 2 generalizes or improves many sufficient conditions for hamiltonian graphs: degree-conditions as well as conditions based on neighborhood unions. In order to compare a degree-condition with a condition based on neighborhood unions, it has turned out to be very useful to write both conditions in terms of neighborhood intersections. Let \( S = \{v_1, \ldots, v_p\} \) be an independent set of \( p \geq 1 \) vertices. Then the following equations hold.

\[
\sum_{i=1}^{p} d(v_i) = \sum_{i=1}^{p} i s_i \quad \text{and} \quad |N(S)| = \sum_{i=1}^{p} s_i. \tag{11}
\]

Let \( i \) be an integer, \( 1 \leq i \leq t + 1 \), \( \phi_i \), and \( \beta \) be real such that \( \beta \geq 1/(t + 1) \) and \( \phi_i = 3/2 + (i - (t + 1)/2)\beta \). As it is difficult to exhibit a weight-function \( w \) satisfying conditions (2) and (3), we find it useful to state the following weaker version of Theorem 2.

**Theorem 3.** Let \( G \) be a \( \kappa \)-connected graph on \( n \geq 3 \) vertices. Suppose there exists some \( t, 1 \leq t \leq \kappa \), such that for every independent set \( X \subset V(G) \) of \( t + 1 \) vertices we have

\[
\sum_{i=1}^{t+1} w_i s_i > n(X) - 1,
\]

where \( w_i = 1 \) if either \( \theta_i \leq 1 \) or \( i = 1 \), \( w_i = 2 \) if either \( \theta_i \geq 2 \) or \( i \in \{t, t + 1\} \), \( w_i = \theta_i \) otherwise. Then \( G \) is hamiltonian.

**Proof.** It suffices to prove that \( w_i \) satisfies conditions (2) and (3).}

**Remark 2.** If \( t = 1, 2 \) then the weight-function is best possible in a sense that there exists no weight-function \( w' \) satisfying (2) and (3) and having at least one component bigger than its corresponding component in \( w \). If we drop \( \sum_{j>2} |N_j(X)| \) and set \( t = 1 \), then \( w = (1, 2) \), and Theorem 3 reduces to the well-known theorem of Ore.

**Corollary 3** [14]. Let \( G \) be a graph with \( n \geq 3 \) vertices. If \( \sigma_2 \geq n \), then \( G \) is hamiltonian.

This result is itself an improvement of an earlier result by Dirac.

**Corollary 4** [10]. Let \( G \) be a graph with \( n \geq 3 \) vertices. If \( \sigma_1 \geq n/2 \), then \( G \) is hamiltonian.

If we drop \( \sum_{j>2} |N_j(X)| \) and set \( t = 2 \), then \( w = (1, 2, 2) \) and Theorem 3 reduces to a Theorem of Flandrin, Jung, and Li.
Corollary 5 [12]. Let $G$ be a 2-connected graph of order $n$ such that
\[d(u) + d(v) + d(w) \geq n + |N(u) \cap N(v) \cap N(w)|\]
for any independent set $u, v, w$. Then $G$ is hamiltonian.

For $t \geq 3$ it is possible to derive several simple weight-functions. Each one is used to state a sufficient condition for hamiltonian graphs.

**Theorem 4.** Let $G$ be a $\kappa$-connected graph on $n \geq 3$ vertices. Suppose there exists some $t$, $1 \leq t \leq \kappa$, such that for every independent set $X \subset V(G)$ of $t + 1$ vertices we have
\[\sum_{i=1}^{t+1} w_i s_i > n(X) - 1,\]
where $w_1 = 1$, $w_t = w_{t+1} = 2$, $w_i = 1 + (i/(t + 1))$, $2 \leq i \leq t - 1$. Then $G$ is hamiltonian.

**Proof.** Replace $\beta$ by $1/(t + 1)$ and use Theorem 3.  

**Corollary 6.** Let $G$ be a $\kappa$-connected graph on $n \geq 3$ vertices. Suppose there exists some $t$, $1 \leq t \leq \kappa$, such that for every independent set $X \subset V(G)$ of $t + 1$ vertices we have
\[\sum_{x_j \in X} |N(X \setminus \{x_j\})| + \sum_{x_j \in X} d(x_j) > (t + 1) (n(X) - 1),\]
then $G$ is hamiltonian.

**Proof.** Using (11), the condition is equivalent to
\[t \sum_{i=1}^{t+1} s_i + \sum_{i=2}^{t+1} s_i + \sum_{i=1}^{t+1} is_i > (t + 1) (n(X) - 1)\]
\[s_1 + \sum_{i=2}^{t+1} \left(1 + \frac{i}{t + 1}\right)s_i > n(X) - 1\]
and the conclusion follows from Theorem 4.  

**Remark 3.** Corollary 6 can also be derived from a result obtained in [3].
"Let $G$ be a $\kappa$-connected graph on $n \geq 3$ vertices. Suppose there exists some $t$, $1 \leq t \leq \kappa$, such that for every independent set $X \subset V(G)$ of $t + 1$ vertices there is a vertex $u \in X$ such that
\[|N(X \setminus \{u\})| + d(u) \geq n(X),\]
then $G$ is hamiltonian."

**Corollary 7** [16]. Let $G$ be a graph of order $n$ and connectivity $\kappa$. Suppose there exists some $t, t \leq \kappa$ such that for every independent set $S \subset V(G)$ of $t + 1$ we have

$$\sum_{i=1}^{t+1} \left(1 + \frac{i-1}{t}\right)s_i > n - 1.$$ 

Then $G$ is hamiltonian.

**Proof.** Clearly $(i-1)/t \leq i/(t+1)$ since $i \leq t + 1$. It is also possible to derive this corollary from Theorem 3 by setting $\beta = 1/t$. $\blacksquare$

**Corollary 8** [5]. Let $G$ be a $\kappa$-connected graph with $n \geq 3$ vertices. If

$$\sigma_{\kappa+1} \geq \frac{(\kappa + 1)(n - 1) + 1}{2}$$

then $G$ is hamiltonian.

**Proof.** Using (11), the condition of this corollary reduces to

$$\sum_{i=1}^{\kappa+1} (2i/(\kappa + 1))s_i > n - 1.$$ Clearly $(2i/(\kappa + 1)) \leq 1 + (i/\kappa + 1)$ and the proof follows from Theorem 4. $\blacksquare$

**Theorem 5.** Let $G$ be a $\kappa$-connected graph on $n \geq 3$ vertices. Suppose there exists some $t, 1 \leq t \leq \kappa$, such that for every independent set $X \subset V(G)$ of $t + 1$ vertices we have

$$\sum_{i=1}^{t+1} w_is_i > n(X) - 1,$$

where $w_i = 1$ if $i = 1$ or $3i \leq t + 1$, $w_i = 2$ if $i = t$, $t + 1$, or $3i \geq 2(t + 1)$ and $w_i = (3i/(t + 1))$ otherwise. Then $G$ is hamiltonian.

**Proof.** Replace $\beta$ by $(3/(t + 1))$ and use Theorem 3. $\blacksquare$

**Remark 4.** Corollary 8 is easily derived from Theorem 5.

**Corollary 9** [17]. Let $G$ be a graph of order $n$ and connectivity $\kappa \geq 2$ such that for every independent set $S \subset V(G)$ of cardinality $\kappa + 1$ we have

$$\sum_{i=1}^{\kappa+1} w_is_i > n - 1,$$
where \( w_i = \min(2, 3(i/(\kappa + 1))) \), then \( G \) is hamiltonian.

**Theorem 6.** Let \( G \) be a \( \kappa \)-connected graph on \( n \geq 3 \) vertices. Suppose there exists some \( t, 2 \leq t \leq \kappa \), such that for every independent set \( X \subset V(G) \) of \( t + 1 \) vertices we have

\[
\sum_{i=1}^{t+1} w_i s_i > n(X) - 1,
\]

where \( w_{t+1} = 2, w_i = 1 + ((i - 1)/(t - 1)) \) if \( 1 \leq i \leq t \). Then \( G \) is hamiltonian.

**Proof.** Replace \( \beta \) by \( 1/(t - 1) \) and use Theorem 3.  ■

**Remark 5.** Corollaries 3, 5, 7, and 8 are easily derived from Theorem 6.

**Theorem 7.** Let \( G \) be a \( \kappa \)-connected graph on \( n \geq 3 \) vertices. Suppose there exists some \( t, 2 \leq t \leq \kappa \), such that for every independent set \( X \subset V(G) \) of \( t + 1 \) vertices we have

\[
\sum_{i=1}^{t+1} w_i s_i > n(X) - 1,
\]

where \( w_i = 1 \) if \( 3(i - 1) \leq (t - 1) \), \( w_i = 2 \) if \( 3(i - 1) \geq 2(t - 1) \) and \( w_i = (3(i - 1)/(t - 1)) \) otherwise. Then \( G \) is hamiltonian.

**Proof.** Replace \( \beta \) by \( 3/(t - 1) \) and use Theorem 3.  ■

**Theorem 8.** Let \( G \) be a \( \kappa \)-connected graph on \( n \geq 3 \) vertices. Suppose there exists some \( t, 1 \leq t \leq \kappa \), such that for every independent set \( X \subset V(G) \) of \( t + 1 \) vertices we have

\[
\sum_{i=1}^{t+1} w_i s_i > n(X) - 1,
\]

where \( w_1 = 1, w_i = w_{i+1} = 2, \) and \( w_{i+1} = 2 \) if \( i > 1 \) and \( w_i = 1 + (1/(l(t - 1)/2)) \) otherwise, then \( G \) is hamiltonian.

**Proof.** Obvious.  ■

Theorem 8 is equivalent to the following Theorem 9.

**Theorem 9 [1].** Let \( G \) be a graph of order \( n \) and connectivity \( \kappa \). Suppose there exists some \( t, 1 \leq t \leq \kappa \), such that for every independent set \( X \subset V(G) \) of \( t + 1 \) vertices we have

\[
\sum_{i=1}^{t+1} w_i s_i > n(X) - 1,
\]

where \( w_1 = 1, w_i = w_{i+1} = 2, \) and \( w_{i+1} = 2 \) if \( i > 1 \) and \( w_i = 1 + (1/(l(t - 1)/2)) \) otherwise, then \( G \) is hamiltonian.
$V(G)$ of cardinality $t + 1$, we have
\[
\sum_{i=1}^{t} s_i + \sum_{i=1}^{t+1} \frac{s_i}{(t + 1)/2} + \sum_{i=2}^{t} \frac{(t - 1)/2}{(t + 1)/2} > n(X) - 1.
\]

Then $G$ is hamiltonian.

The next corollaries are straightforward.

**Corollary 10** [1]. Let $G$ be a graph of order $n$ and connectivity $\kappa$. Suppose there exists some $t$, $t \leq \kappa$, such that for every independent set $S \subseteq V(G)$ of cardinality $t$ we have
\[
|N(S)| = \sum_{i=1}^{t} s_i > \frac{t}{(t + 1)} (n - 1) - \sum_{i=2}^{t} \frac{s_i}{(t + 1)} \left(\left\lfloor \frac{t}{2}\right\rfloor + \left\lfloor \frac{t + 1}{2}\right\rfloor\right)
\]
\[
- \frac{s_t}{t + 1} \left\lfloor \frac{t - 1}{2}\right\rfloor.
\]

Then $G$ is hamiltonian.

**Corollary 11** [1]. Let $G$ be a graph of order $n$ and connectivity $\kappa$. Suppose there exists some $t$, $1 < t \leq \kappa$, such that for every independent set $S \subseteq V(G)$ of cardinality $t$ we have
\[
|N(S)| = \sum_{i=1}^{t} s_i > \frac{t}{(t + 1)} (n - 1) - \sum_{i=2}^{t} \frac{s_i}{t + 1} \left(\left\lfloor \frac{t}{2}\right\rfloor + \left\lfloor \frac{t + 1}{2}\right\rfloor\right).
\]

Then $G$ is hamiltonian.

**Corollary 12** [1]. Let $G$ be a graph of order $n$ and connectivity $\kappa$. Suppose there exists some $t$, $1 < t \leq \kappa$, such that for every independent set $X \subseteq V(G)$ of cardinality $t + 1$ we have
\[
s_1 + \sum_{i=2}^{t+1} \left(1 + \frac{1}{(t + 1)/2}\right)s_i > n - 1
\]

Then $G$ is hamiltonian.

**Corollary 13** [1]. Let $G$ be a graph of order $n$ and connectivity $\kappa$. Suppose there exists some $t$, $1 < t \leq \kappa$, such that for every independent set $X \subseteq V(G)$ of cardinality $t + 1$ we have
\[
\sum_{i=1}^{t+1} s_i + 2s_t + 2s_{t+1} > n - 1.
\]
Then $G$ is hamiltonian.

**Corollary 14** [18]. Let $G$ be a graph of order $n$ and connectivity $\kappa$. Suppose there exists some $t$, $t \leq \kappa$, such that for every independent set $X = \{x_1, x_2, \ldots, x_{t+1}\}$ we have

$$\sum_{i=1}^{t+1} |N(X \setminus \{x_i\})| > t(n - 1).$$

Then $G$ is hamiltonian.

**Proof.** The condition of this corollary is equivalent to

$$s_1 + \sum_{i=2}^{t+1} (1 + 1/t)s_i > n - 1$$

and the conclusion follows. \[\square\]

**Corollary 15** [7]. Let $G$ be a 2-connected graph with $n \geq 3$ vertices. If

$$d(u) + d(v) + 2|N(u) \cup N(v)| \geq (2n - 1)$$

for all $uv \notin E(G)$, then $G$ is hamiltonian.

**Corollary 16** [13]. Let $G$ be a $\kappa$-connected graph with $n \geq 3$ vertices. Suppose there exists some $t$, $t \leq \kappa$, such that for every independent set $S \subset V(G)$ of cardinality $t$ we have

$$|N(S)| > \frac{t(n - 1)}{t + 1}.$$ 

Then $G$ is hamiltonian.

**Theorem 10.** Let $G$ be a $\kappa$-connected graph on $n \geq 3$ vertices. Suppose there exists some $t$, $1 \leq t \leq \kappa$, such that for every independent set $X \subset V(G)$ of $t + 1$ vertices we have

$$\sum_{i=1}^{t+1} w_is_i > n(X) - 1,$$

where $w_i = 1$ if $i < (t + 1)/2$, $w_i = 3/2$ if $i = (t + 1)/2$, and $w_i = 2$ otherwise. Then $G$ is hamiltonian.

**Proof.** Obvious. \[\square\]
Theorem 11. Let $G$ be a $\kappa$-connected graph on $n \geq 3$ vertices. Suppose there exists some $t$, $1 \leq t \leq \kappa$, such that for every independent set $X \subseteq V(G)$ of $t + 1$ vertices we have

$$\sum_{i=1}^{t+1} w_i s_i > n(X) - 1,$$

where $w_i = 1$ if $i \leq (t + 1)/3$, $w_i = 2$ if $i \geq 2(t + 1)/3$, and $w_i = 3/2$ otherwise. Then $G$ is hamiltonian.

Proof. If $t \leq 2$, then conditions (2) and (3) hold. Assume now that $t \geq 3$. If either $p \leq 2$ or $i_{p-2} \leq (t + 1)/3$, then $\sum_p^i (w_{ij} - 1) \leq w_{ip} + w_{ip-2} - 2 \leq 2(3/2 - 1) \leq 1$ since $i_p \leq t - 1$ and hence $w_{ip} = 3/2$. If $i_{p-2} \geq (t + 2)/3$, then $\sum_p^i i_j \geq i_{p-2} + i_{p-1} + i_p > t + 1$, a contradiction.

Recently Chen and Shelp obtained the following:

Theorem 12 [8]. Let $G$ be a graph of order $n \geq 3$ and connectivity $\kappa$. If there exists a sequence of $\kappa + 1$ real numbers $w_1, w_2, \ldots, w_{\kappa+1}$ satisfying (2) and the following condition (12) such that for every independent set $S \subseteq V(G)$ of cardinality $\kappa + 1$ we have

$$\sum_{i=1}^{\kappa+1} w_i s_i > n - 1.$$

Then $G$ is hamiltonian.

Condition (12).

(i) For $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq \kappa + 1$ and $\sum_{j=1}^m (i_j - 1) \leq \kappa - 1$ we have $\sum_{j=1}^m (w_{ij} - 1) \leq 1$

(ii) For each $i$, $2 \leq i \leq \kappa - 1$, we have $w_1 + 2w_{\kappa+2-i} \leq 5$.

Since (12) is a stronger hypothesis than (3), Corollary 2 and thus Theorem 2 hold whenever Theorem 12 does.

Remark 6. It is easy to check that Theorem 12 remains true if we only require (i) of condition (12). This proves a conjecture of Chen and Schelp [8].

Final Remark. Using the same arguments as in [2] (Theorems 3, 4, and 5) we can easily derive from Theorem 2 sufficient conditions for a graph to be hamiltonian-connected, 1-hamiltonian and traceable.
References


Received August 4, 1992