Note on the circumference of a graph and its complement

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1. Results

We use [3] for terminology and notation not defined here and consider finite and simple graphs only.

In this note we will show a lower bound for the circumference of a graph and its complement. For the proof we will consider 2-edge colorings of the complete graph and make use of the Ramsey numbers for cycles.

Theorem 1. Let G be a graph of order n ≥ 6 and circumference c(G). Let \( \widehat{G} \) be the complement of G. Then

\[
\max\{c(G), c(\widehat{G})\} \geq \left\lceil \frac{2n}{3} \right\rceil
\]

and this bound is sharp.

Proof (of lower bound). We consider a 2-edge coloring of the complete graph \( K_n \) with colors red and blue such that all edges of \( G(\widehat{G}) \) are red (blue). Let R and B denote the subgraphs of \( K_n \) induced by the red and the blue edges, respectively. We consider and apply the Ramsey number \( r(C_r, C_s) \) for two even cycles \( C_r, C_s \). The following is known:

Theorem 2 ([4,7]). If \( 4 \leq s \leq r \) with \( s \) and \( r \) even, \( (r, s) \neq (4, 4) \), then

\[
r(r, s) = r(C_r, C_s) = r + \frac{1}{2}s - 1.
\]

It is easy and straightforward to verify the results for \( 6 \leq n \leq 9 \). We now consider three cases, and assume that \( n \geq 10 \).

Case \( n = 3k \)

Since \( r(2k, 2k) = 2k + k - 1 = 3k - 1 \leq n \); \( 3k = n \) it follows that there is a red \( C_{2k} \) or a blue \( C_{2k} \). Hence \( \max\{c(G), c(\widehat{G})\} \geq 2k = \left\lceil \frac{2n}{3} \right\rceil \).

Case \( n = 3k + 2 \)

Since \( r(2k + 2, 2k + 2) = 2k + 2 + k + 1 - 1 = 3k + 2 \) it follows that there is a red \( C_{2k+2} \) or a blue \( C_{2k+2} \). Hence \( \max\{c(G), c(\widehat{G})\} \geq 2k + 2 = \left\lceil \frac{2n}{3} \right\rceil \).

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Case \(n = 3k + 1\).
In this case \(\lfloor \frac{n}{k} \rfloor = 2k + 1\). We have \(r(2k + 2, 2k) = 2k + 2 + k - 1 = 3k + 1\). Assume there is no red \(C_{2k+2}\), but a blue \(C_{2k}\). Denote the blue cycle by \(C\).

Let us consider a graph \(G\) which has a maximum number of red edges. Thus, any additional red edge increases the circumference in \(R\) to \(2k + 1\). The property “\(G\) has circumference \(c(G)\)” is \(n\)-stable (see [2] and the Bondy–Chvátal closure concept [1]). Hence for every blue edge \(xy\) we have \(d^B(x) + d^B(y) \geq n\); where \(d^B(z)\) denotes the degree of \(z\) in the red graph \(R\).

Let \(H = \overline{C} - V(C)\). Then \(|V(H)| = k + 1\). We now distinguish three subcases.

Subcase 1: \(H\) contains a blue cycle

Then for every edge \(xy\) of this blue cycle \(C'\) there is a blue \(xy\)-path in \(H\) of order \(\geq 3\). Now let \(xy\) be an arbitrary edge of this blue cycle \(C'\).

Claim 1. \(d^B_i(x) + d^B_i(y) \leq k\)

Let the vertices of the blue cycle \(C\) be denoted by \(v_1, v_2, \ldots, v_{2k}\). Choose an orientation of \(C\) such that for every vertex \(v_i\) its successor is \(v_{i+1}\). If \(v_1 \in N^B(x) \cap N^B(y)\), then \(v_{k+1} \in N^B(x) \cup N^B(y)\), since \(C\) is a longest blue cycle. If \(v_1 \in N^B(x) \setminus N^B(y)\), then \(v_{k+1} \in N^B(y)\) and \(v_{k+1} \in N^B(x)\). Hence \(|N^B(x) \cap \{v_1, v_{k+1}, v_{k+2}, v_{k+3}\}| + |N^B(y) \cap \{v_1, v_{k+1}, v_{k+2}, v_{k+3}\}| \leq 2\) for every vertex \(v_i \in (V(C)).\) Therefore, \(d^B_i(x) + d^B_i(y) \leq \frac{2k}{2} = k\).

Thus \(d^B(x) + d^B(y) \leq k + 2k = 3k\). If \(d^B(x) + d^B(y) \geq 3k\), then \(d^B(x) + d^B(y) \geq 3k + 1\), a contradiction. So assume that \(d^B(x) + d^B(y) = 3k\). Then \(d^B_i(x) = d^B_i(y) = k\). Hence \(K_{k+1} - K_1 \subseteq H, \) (i.e. the edge \(xy\) is contained in a “blue book”). Let \(z \in N^B(x) \cap N^B(y)\). Then the edge \(xz\) is contained in a blue \(C_d\) for \(k \geq 3\). Now \(d^B(x) + d^B(z) \leq k\), since this follows immediately from the argument used in the previous claim for appropriate choice of \(x\) and \(z\). This implies \(d^B(x) + d^B(z) \leq k + (2k - 1) = 3k - 1\), and so \(d^B(x) + d^B(z) \geq 3k + 1\), a contradiction.

Case 2 \(H\) contains an induced blue forest

Let \(x\) and \(y\) be endvertices of a path in the blue forest. As in the previous case we obtain \(|N^B(x) \cap \{v_1, v_{k+1}, v_{k+2}\}| + |N^B(y) \cap \{v_1, v_{k+1}, v_{k+2}\}| \leq 2\) for every vertex \(v_i \in (V(C))\). Hence \(d^B(x) + d^B(y) = d^B_i(x) + d^B_i(y) \geq d^B_i(x) + d^B_i(y) \leq \frac{2k}{2} + k + 1\); 3k; a contradiction.

Case 3 \(H\) contains no blue edges

Claim 2. Consecutive vertices \(v_i, v_{i+1} \in (V(C))\), cannot both have blue neighbors in \(H\).

Suppose there are two consecutive vertices \(x_1, x_2 \in (V(C))\) and two vertices \(y_1, y_2 \in (V(H))\) such that \(x_1y_1, x_2y_2 \in (E(B))\).
Since \(C\) is a longest cycle, we have (using an Ore type argument [6]) \(d^B_i(x_1) + d^B_i(y_2) \leq 2k\), and \(d^B_i(x_2) + d^B_i(y_1) \leq 2k\). Since \(C\) is a longest cycle we have \(N^B_i(x_1) \cap N^B_i(x_2) = \emptyset\). Hence \(d^B_i(x_1) + d^B_i(x_2) + d^B_i(y_1) + d^B_i(y_2) \leq 2(2k) + k + 1 = 5k + 1\); 6k. Therefore, \(d^B(x_1) + d^B(y_1) \geq 3k\) or \(d^B(x_2) + d^B(y_2) \geq 3k\), a contradiction.

Hence, there are at least \(k\) vertices of the cycle \(C\) which have no blue neighbors in \(H\). But then we can find a red \(C_{2k+1}\) using these \(k\) vertices along with the \(k + 1\) vertices of \(H\), a contradiction.

2. Examples and conjectures

Example 1. Consider a complete graph \(K_n\) of order \(n = p(k + 1)\) with vertex set \(V(K_n) = \cup_{i=1}^{k+1} V_i\), with \(|V_i| = p\) for each \(i\).
Color all edges of \(G(V_i)\) with color \(i\) for \(1 \leq i \leq k\) and all edges of \(G(V_{k+1})\) with color \(k\). For every pair \(i, j\) with \(1 \leq i \neq j \leq k + 1\) all edges between \(V_i\) and \(V_j\) are colored with color \(i\). Denote this \(k\)-edge colored graph by \(F_k(n)\).

The edges of the graph \(F_k(n)\) are colored with \(k\) colors, and the largest monochromatic cycle in \(F_k(n)\) has order \(2n\). For \(k = 2\), this implies the bound in Theorem 1 is sharp. For \(k = 3\), this implies that one cannot expect a monochromatic cycle of order greater than \(n/2\) in a 3-edge colored \(K_n\). Andras Gyárfás drew our attention to the following example.

Example 2. For \(k \geq 2\) the affine plane of index \(k\) and order \(k^2\) induces a \((k + 1)\)-coloring of the edges of a complete graph \(K_{k^2} + k + 1\) subgraphs \(H_i\) of color \(i\) for \(1 \leq i \leq k + 1\) such that each \(H_i = kK_k\). Let \(G_k(n)\) be the graph of order \(n\) obtained by replacing each vertex of \(K_{k^2} + k + 1\) by a \(K_{n/(k^2)}\), extending the coloring of \(K_{k^2} + k + 1\) to \(G_k(n)\), and arbitrarily coloring the edges in each of the \(K_{k^2} + k + 1\).

The edges of the graph \(G_k(n)\) are colored with \(k + 1\) colors, and the largest monochromatic cycle in \(F_k(n)\) has order \(n/k\). This leads to the following conjecture.

Conjecture 1. For \(k \geq 2\) let \(K_n\) be a \((k + 1)\)-edge colored graph and let \(G_i\) be the graph induced by color \(i\) for \(1 \leq i \leq k + 1\).

Then \(\max\{c(G_1), c(G_2), \ldots, c(G_{k+1})\} \geq \frac{n}{k}\).

Some support for the conjecture in the case when \(k = 2\) is given by the following theorem of Gyárfás, Ruszinkó, Sárközy, and Szemerédi, which implies there is a monochromatic path with at least \(n/2\) vertices in a 3-edge colored \(K_n\).

Theorem 3 ([5]). For \(m\) sufficiently large, \(r(P_m, P_m, P_m) < 2m\).
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