Some Inequalities Among New Divergence Measures

Inder Jeet Taneja
Departamento de Matemática
Universidade Federal de Santa Catarina
88.040-900 Florianópolis, SC, Brazil.
e-mail: taneja@mtm.ufsc.br
http://www.mtm.ufsc.br/~taneja

Abstract

There are three classical divergence measures exist in the literature on information theory and statistics. These are namely, Jeffreys-Kullback-Leibler J-divergence, Burbea-Rao [1] Jensen-Shannon divergence and Taneja [8] arithmetic-geometric mean divergence. These three measures bear an interesting relationship among each other and are based on logarithmic expressions. The divergence measures like Hellinger discrimination, symmetric $\chi^2$-divergence, and triangular discrimination are also known in the literature and are not based on logarithmic expressions. Past years Dragomir et al. [3], Kumar and Johnson [7] and Jain and Srivastava [4] studied different kind of divergence measures. In this paper, we worked with inequalities relating these new measures with the previous know one. An idea of exponential divergence is also developed.

Key words: J-divergence; Jensen-Shannon divergence; Arithmetic-Geometric divergence; Csiszár's f-divergence; Information inequalities; Exponential divergence.

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1 Introduction

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \ldots, p_n) \bigg| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geq 2,$$

be the set of all complete finite discrete probability distributions. For all $P, Q \in \Gamma_n$, the following measures are well known in the literature on information theory and statistics:

- Hellinger Discrimination

$$h(P||Q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2,$$  \hspace{1cm} (1)


• Triangular Discrimination

\[ \Delta(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}, \]  

(2)

• Symmetric Chi-square Divergence

\[ \Psi(P||Q) = \chi^2(P||Q) + \chi^2(Q||P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2(p_i + q_i)}{p_iq_i}, \]  

(3)

where

\[ \chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1, \]

is the well-known \( \chi^2 \)-divergence.

• J-Divergence (Jeffreys [5]; Kullback-Leibler [6])

\[ J(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln \left( \frac{p_i}{q_i} \right). \]  

(4)

• Jensen-Shannon Divergence (Burbea and Rao [1])

\[ I(P||Q) = \frac{1}{2} \left[ \sum_{i=1}^{n} p_i \ln \left( \frac{2p_i}{p_i + q_i} \right) + \sum_{i=1}^{n} q_i \ln \left( \frac{2q_i}{p_i + q_i} \right) \right]. \]  

(5)

• Arithmetic-Geometric Mean Divergence (Taneja [8])

\[ T(P||Q) = \sum_{i=1}^{n} \left( \frac{p_i + q_i}{2} \right) \ln \left( \frac{p_i + q_i}{2\sqrt{p_iq_i}} \right). \]  

(6)

After simplification, we can write

\[ J(P||Q) = 4 \left[ I(P||Q) + T(P||Q) \right]. \]  

(7)

The author [8, 11] proved the following inequality among the above five symmetric divergence measures:

\[ \frac{1}{4} \Delta(P||Q) \leq I(P||Q) \leq h(P||Q) \leq \frac{1}{8} J(P||Q) \leq T(P||Q) \leq \frac{1}{16} \Psi(P||Q). \]  

(8)
We observe that we have an inequality among six divergence measures three of them are logarithmic and other three are non logarithmic. More studies on divergence measures can be seen in Taneja [9, 10] and Taneja and Kumar [12].

In this paper we shall study three different kind of non logarithmic divergence measures and establish an inequality among them.

2 Different Divergence Measures

In this section we shall consider different kind of measures and study their properties and inequalities among them.

2.1 First Divergence Measure

Let us consider a measure given by

\[ D^*(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^4}{\sqrt{(p_i q_i)^3}}, \quad P, Q \in \Gamma_n \]  

(9)

The above measure has been study by Dragomir et al. [3], where they prove that

\[ 0 \leq \frac{1}{2} J(P||Q) - \Delta(P||Q) \leq \frac{1}{12} D^*(P||Q), \]  

(10)

and

\[ 0 \leq \frac{1}{2} \Psi(P||Q) - J(P||Q) \leq \frac{1}{6} D^*(P||Q). \]  

(11)

Here below we shall unify and improve the inequalities given in (10) and (11).

Theorem 2.1. The following inequalities hold

\[ 0 \leq D_J\Delta(P||Q) \leq \frac{1}{4} D_{\Psi J}(P||Q) \leq \frac{8}{3} D_{\Psi T}(P||Q) \leq \frac{1}{24} D^*(P||Q), \]  

(12)

where

\[ D_J\Delta(P||Q) = \frac{1}{2} J(P||Q) - \Delta(P||Q), \]

\[ D_{\Psi J}(P||Q) = \frac{1}{2} \Psi(P||Q) - J(P||Q) \]
and
\[ D_{\Psi T}(P||Q) = \frac{1}{16} \Psi(P||Q) - T(P||Q). \]

The proof of the inequalities given in (12) is based on the following lemmas:

**Lemma 2.1.** If the function \( f : [0, \infty) \to \mathbb{R} \) is convex and normalized, i.e., \( f(1) = 0 \), then the \( f \)-divergence, \( C_f(P||Q) \) given by
\[ C_f(P||Q) = \sum_{i=1}^{n} q_i f \left( \frac{p_i}{q_i} \right), \tag{13} \]
is nonnegative and convex in the pair of probability distribution \((P, Q) \in \Gamma_n \times \Gamma_n\).

**Lemma 2.2.** Let \( f_1, f_2 : I \subset \mathbb{R}_+ \to \mathbb{R} \) two generating mappings are normalized, i.e., \( f_1(1) = f_2(1) = 0 \) and satisfy the assumptions:
(i) \( f_1 \) and \( f_2 \) are twice differentiable on \((a, b)\);
(ii) there exists the real constants \( m, M \) such that \( m < M \) and
\[
m \leq \frac{f_1''(x)}{f_2''(x)} \leq M, \quad f_2''(x) > 0, \quad \forall x \in (a, b),
\]
then we have the inequalities:
\[ m C_{f_1}(P||Q) \leq C_{f_1}(P||Q) \leq M C_{f_2}(P||Q). \tag{14} \]

The measure (13) is the well-known Csiszár’s \( f \)-divergence. The Lemma 2.1 is due to Csiszár [2] and the Lemma 2.2 is due to author [10].

**Proposition 2.1.** The following inequalities hold:
\[ 0 \leq D_{J\Delta}(P||Q) \leq \frac{1}{4} D_{\Psi J}(P||Q). \tag{15} \]

**Proof.** We can write
\[ D_{J\Delta}(P||Q) = \frac{1}{2} J(P||Q) - \Delta(P||Q) = \sum_{i=1}^{n} q_i f_{J\Delta} \left( \frac{p_i}{q_i} \right), \tag{16} \]
where
\[ f_{J\Delta}(x) = \frac{1}{2} f_J(x) - f_\Delta(x) = \frac{1}{2} (x - 1) \ln x - \frac{(x - 1)^2}{x + 1}, \quad x > 0. \tag{17} \]

This gives
\[
 f'_{J\Delta}(x) = \frac{1}{2} \left(1 - x^{-1} + \ln x\right) - \frac{(x - 1)(x + 3)}{(x + 1)^2}, \quad x > 0
\]
and
\[
 f''_{J\Delta}(x) = \frac{x + 1}{2x^2} - \frac{8}{(x + 1)^3} = \frac{(x - 1)^2(x^2 + 6x + 1)}{2x^2(x + 1)^3} \geq 0, \quad \forall x > 0. \tag{18}
\]

Thus we have \( f''_{J\Delta}(x) \geq 0 \) for all \( x > 0 \), and hence, \( f_{J\Delta}(x) \) is convex for all \( x > 0 \). Also, we have \( f_{J\Delta}(1) = 0 \). In view of this we can say that the measure \( D_{J\Delta}(P||Q) \) given by (13) is nonnegative and convex in the pair of probability distributions \((P, Q) \in \Gamma_n \times \Gamma_n\).

Again, we can write
\[
 D_{\Psi J}(P||Q) = \frac{1}{2} \Psi(P||Q) - J(P||Q) = \sum_{i=1}^{n} q_i f_{\Psi J} \left( \frac{p_i}{q_i} \right), \tag{19}
\]
where
\[
 f_{\Psi J}(x) = \frac{1}{2} f_{\Psi}(x) - f_{J}(x) = \frac{(x - 1)^2(x + 1)}{2x} - (x - 1) \ln x, \quad x > 0. \tag{20}
\]
This gives
\[
 f'_{\Psi J}(x) = \frac{(x - 1)(2x^2 + x + 1)}{2x^2} - (1 - x^{-1} + \ln x), \quad x > 0
\]
and
\[
 f''_{\Psi J}(x) = \frac{x^3 + 1}{x^3} - \frac{x + 1}{x^2} = \frac{(x - 1)^2(x + 1)}{x^3} \geq 0, \quad \forall x > 0. \tag{21}
\]

Thus we have \( f''_{\Psi J}(x) > 0 \) for all \( x > 0 \), and hence, \( f_{\Psi J}(x) \) is strictly convex for all \( x > 0 \). Also, we have \( f_{\Psi J}(1) = 0 \). In view of this we can say that the measure \( D_{\Psi J}(P||Q) \) given by (19) is nonnegative and convex in the pair of probability distributions \((P, Q) \in \Gamma_n \times \Gamma_n\).

Let us consider
\[
 g_{J\Delta_{\Psi J}}(x) = \frac{f''_{J\Delta}(x)}{f'_{\Psi J}(x)} = \frac{x(x^2 + 6x + 1)}{(x + 1)^4}, \quad x \in (0, \infty),
\]
where \( f''_{J\Delta}(x) \) and \( f'_{\Psi J}(x) \) are as given by (18) and (21) respectively.

Calculating the first order derivative of the function \( g_{J\Delta_{\Psi J}}(x) \) with respect to \( x \), one gets
\[
 g'_{J\Delta_{\Psi J}}(x) = -\frac{(x - 1)(x^2 + 10x + 1)}{2(x + 1)^5} \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} . \tag{22}
\]
In view of (22) we conclude that the function \( g_{J\Delta \Psi J}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence

\[
M = \sup_{x \in (0, \infty)} g_{J\Delta \Psi J}(x) = g_{J\Delta \Psi J}(1) = \frac{1}{4}. \tag{23}
\]

By the application of (14) with (23) we get (15).

Proposition 2.2. The following inequalities hold:

\[
D_{\Psi J}(P || Q) \leq \frac{32}{3} D_{\Psi T}(P || Q). \tag{24}
\]

Proof. We can write

\[
D_{\Psi T}(P || Q) = \frac{1}{16} \Psi(P || Q) - T(P || Q) = \sum_{i=1}^{n} q_i f_{\Psi T} \left( \frac{p_i}{q_i} \right), \tag{25}
\]

where

\[
f_{\Psi T}(x) = \frac{1}{16} f_{\Psi}(x) - f_{T}(x) = \frac{(x-1)^2(x+1)}{16x} - \left( \frac{x+1}{2} \right) \ln \left( \frac{x+1}{2\sqrt{x}} \right), \quad x > 0. \tag{26}
\]

This gives

\[
f'_{\Psi T}(x) = \frac{(x-1)(2x^2 + x + 1)}{16x^2} - \frac{1}{4x} \left( 2x \ln \left( \frac{x+1}{2\sqrt{x}} \right) + x - 1 \right), \quad x > 0
\]

and

\[
f''_{\Psi T}(x) = \frac{x^3 + 1}{16x^3} - \frac{x^2 + 1}{4x^2(x+1)} = \frac{(x-1)^2(x^2 + x + 1)}{8x^3(x+1)} \geq 0, \quad \forall x > 0. \tag{27}
\]

Thus we have \( f''_{\Psi T}(x) \geq 0 \) for all \( x > 0 \), and hence, \( f_{\Psi T}(x) \) is convex for all \( x > 0 \). Also, we have \( f_{\Psi T}(1) = 0 \). In view of this we can say that the measure \( D_{\Psi T}(P || Q) \) given by (25) is nonnegative and convex in the pair of probability distributions \( (P, Q) \in \Gamma_n \times \Gamma_n \).

Let us consider

\[
g_{\Psi J \Psi T}(x) = \frac{f''_{\Psi J}(x)}{f''_{\Psi T}(x)} = \frac{8(x+1)^2}{x^2 + x + 1}, \quad x > 0,
\]

where \( f''_{\Psi J}(x) \) and \( f''_{\Psi T}(x) \) are as given by (21) and (27) respectively.

Calculating the first order derivative of the function \( g_{J\Delta \Psi J}(x) \) with respect to \( x \), one gets
\[ g'_{\Psi J\Psi T}(x) = -\frac{8(x - 1)(x + 1)}{(x^2 + x + 1)^2} \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \] (28)

In view of (28) we conclude that the function \( g_{\Psi J\Psi T}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence

\[ M = \sup_{x \in (0, \infty)} g_{\Psi J\Psi T}(x) = g_{\Psi J\Psi T}(1) = \frac{32}{3}. \] (29)

By the application of (14) with (29) we get (24).

\[ D_{\Psi T}(P||Q) \leq \frac{1}{64} D^*(P||Q). \] (30)

**Proof.** We can write

\[ D^*(P||Q) = \sum_{i=1}^{n} q_i f_{D^*} \left( \frac{p_i}{q_i} \right), \] (31)

where

\[ f_{D^*}(x) = \frac{(x - 1)^4}{x^{3/2}}, \quad x > 0. \] (32)

This gives

\[ f'_{D^*}(x) = \frac{(x - 1)^3(5x + 3)}{2x^{5/2}}, \quad x \in (0, \infty), \]

and

\[ f''_{D^*}(x) = \frac{3(x - 1)^2(5x^2 + 6x + 5)}{4x^{7/2}} \geq 0, \quad \forall x > 0. \] (33)

Thus we have \( f''_{D^*}(x) \geq 0 \) for all \( x > 0 \), and hence, \( f_{D^*}(x) \) is convex for all \( x > 0 \). Also, we have \( f_{D^*}(1) = 0 \). In view of this we can say that the measure \( D^*(P||Q) \) given by (9) is nonnegative and convex in the pair of probability distributions \( (P, Q) \in \Gamma_n \times \Gamma_n \).

Let us consider

\[ g_{\Psi T,D^*}(x) = \frac{f''_{\Psi T}(x)}{f''_{D^*}(x)} = \frac{(x^2 + x + 1)\sqrt{x}}{6(5x^2 + 6x + 5)(x + 1)}, \quad x > 0, \]

where \( f''_{\Psi T}(x) \) and \( f''_{D^*}(x) \) are as given by (27) and (33) respectively.
Calculating the first order derivative of the function \( g_{\Psi_{T,D^*}}(x) \) with respect to \( x \), one gets
\[
g'_{\Psi_{T,D^*}}(x) = \frac{(x-1)(5x^4 + 9x^3 + 12x^2 + 9x + 5)}{12(5x^2 + 6x + 5)^2(x + 1)\sqrt{x}} \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \quad (34)
\]

In view of (34) we conclude that the function \( g_{\Psi_{T,D^*}}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\[
M = \sup_{x \in (0, \infty)} g_{\Psi_{T,D^*}}(x) = g_{\Psi_{T,D^*}}(1) = \frac{1}{64}. \quad (35)
\]

By the application of (14) with (35) we get (30).

Thus in view of Propositions 2.1 to 2.3 we get the proof of the Theorem 2.1.

### 2.2 Second Divergence Measure

Expression (32) after simplification can be written as
\[
f_{D^*}(x) = \frac{(x-1)^4}{x^{3/2}} = \frac{(x^2 - 1)^2}{x^{3/2}} - \frac{4(x-1)^2}{x^{1/2}}. \quad (36)
\]

In view of expression (31), we have
\[
D^*(P||Q) = \sum_{i=1}^{n} q_i f_{D^*} \left( \frac{p_i}{q_i} \right) = 2\Psi_M(P||Q) - 4K_0(P||Q), \quad (37)
\]
where
\[
\Psi_M(P||Q) = \frac{1}{2} \sum_{i=1}^{n} \frac{(p_i^2 - q_i^2)^2}{\sqrt{(p_iq_i)^3}}, \quad (38)
\]
and
\[
K_0(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{\sqrt{p_iq_i}}. \quad (39)
\]

Measure (38) has been studied by Kumar and Johnshon [7] where they proved that
\[
\Psi(P||Q) \leq \Psi_M(P||Q). \quad (40)
\]

The measure (39) has been studied by Jain and Srivastava [4] where they proved that
\[
T(P||Q) \leq \frac{1}{8} K_0(P||Q) \leq \frac{1}{16} \Psi(P||Q). \quad (41)
\]
Thus combining the inequalities given in (40) and (41), we have

\[ T(P||Q) \leq \frac{1}{8} K_0(P||Q) \leq \frac{1}{16} \Psi(P||Q) \leq \frac{1}{16} \Psi_M(P||Q). \tag{42} \]

Also as a consequence of (37), we have

\[ \frac{1}{2} D^*(P||Q) \leq \Psi_M(P||Q). \tag{43} \]

The inequalities given in (43) give a relationship among the first and second divergence measure. Let us consider below the third divergence measure

### 2.3 Third Divergence Measure

Let us consider the following new measure

\[ F_N(P||Q) = \sum_{i=1}^{n} \frac{(\sqrt{p_i} - \sqrt{q_i})^4}{p_i + q_i}. \tag{44} \]

Let us prove the convexity of the measure (44).

We can write

\[ F_N(P||Q) = \sum_{i=1}^{n} \frac{(\sqrt{p_i} - \sqrt{q_i})^4}{p_i + q_i} = \sum_{i=1}^{n} q_i f_{F_N} \left( \frac{p_i}{q_i} \right), \]

where

\[ f_{F_N}(x) = \frac{(\sqrt{x} - 1)^4}{x + 1} \]

This gives

\[ f'_{F_N}(x) = \frac{(\sqrt{x} - 1)^3(x + \sqrt{x} + 2)}{(x + 1)^2 \sqrt{x}}, \quad x > 0 \]

and

\[ f''_{F_N}(x) = \frac{(\sqrt{x} - 1)^2(2x^{5/2} + 2x^{3/2} + x^3 + 6x^2 + x)}{(x + 1)^3 x^{5/2}} \geq 0, \quad \forall x > 0 \tag{45} \]

Thus we have \( f''_{F_N}(x) \geq 0 \) for all \( x > 0 \), and hence, \( f_{F_N}(x) \) is convex for all \( x > 0 \). Also, we have \( f_{F_N}(1) = 0 \). In view of this we can say that the measure \( F_N(P||Q) \) given by (44) is nonnegative and convex in the pair of probability distributions \( (P, Q) \in \Gamma_n \times \Gamma_n \).
After simplification we can write

\[
\sum_{i=1}^{n} \left( \frac{\sqrt{p_i} - \sqrt{q_i}}{p_i + q_i} \right)^4 = \sum_{i=1}^{n} \left( \frac{\sqrt{p_i} - \sqrt{q_i}}{p_i + q_i} \right)^2 - \frac{1}{2} \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}
\]

\[
= 2 \sum_{i=1}^{n} \frac{\left( \frac{\sqrt{p_i} - \sqrt{q_i}}{2} \right)^2}{p_i + q_i} - \frac{1}{2} \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}
\]

\[
= 2 \left[ \sum_{i=1}^{n} \frac{\left( \frac{\sqrt{p_i} - \sqrt{q_i}}{2} \right)^2}{p_i + q_i} - \frac{1}{4} \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i} \right]
\]

\[
= 2 \left( h(P||Q) - \frac{1}{4} \Delta(P||Q) \right).
\]

Thus we have

\[
F_N(P||Q) = 2 \left( h(P||Q) - \frac{1}{4} \Delta(P||Q) \right) = 2D_h\Delta(P||Q).
\]

Now we shall establish an inequality among the measures given by (9), (38) and (44).

In view of (8), we have

\[
h(P||Q) \leq \frac{1}{8} J(P||Q).
\]

This gives

\[
h(P||Q) - \frac{1}{4} \Delta(P||Q) \leq \frac{1}{8} J(P||Q) - \frac{1}{4} \Delta(P||Q),
\]

i.e.,

\[
2 F_N(P||Q) = D_h\Delta(P||Q) \leq \frac{1}{4} D_J\Delta(P||Q).
\]

(46)

Applying the Theorem 2.1, we have

\[
2 F_N(P||Q) \leq \frac{1}{96} D^*(P||Q)
\]

(47)

Combining (43) and (47) we have

\[
F_N(P||Q) \leq \frac{1}{192} D^*(P||Q) \leq \frac{1}{96} \Psi_M(P||Q).
\]

(48)

Thus the above inequality (48) establishes a relationship with these three measures presented in this section.
2.4 Forth Divergence Measure

Let us consider the measure $K_0(P||Q)$ given in (39) as forth measure. Here below we shall present some interesting inequalities having this measure.

In view of (42) we have

$$0 \leq \frac{1}{8} K_0(P||Q) - T(P||Q) \leq \frac{1}{16} \Psi(P||Q) - T(P||Q) \leq \frac{1}{16} \Psi_M(P||Q) - T(P||Q),$$

i.e.,

$$0 \leq D_{K_0T}(P||Q) \leq D_{\Psi T}(P||Q) \leq D_{\Psi_M T}(P||Q).$$

In view of (8) and (49) we have

$$0 \leq \frac{1}{8} K_0(P||Q) - T(P||Q) \leq \frac{1}{8} K_0(P||Q) - h(P||Q),$$

i.e.,

$$0 \leq D_{K_0T}(P||Q) \leq D_{K_0h}(P||Q).$$

We shall now combine the inequalities (49) and (50)6. In order to do so, we need to establish an inequality between $D_{K_0h}(P||Q)$ and $D_{\Psi T}(P||Q)$. It is given in the proposition below.

**Proposition 2.4.** The following inequalities hold:

$$D_{K_0h}(P||Q) \leq \frac{1}{2} D_{\Psi T}(P||Q)$$

**Proof.** We can write

$$D_{K_0h}(P||Q) = \frac{1}{8} K_0(P||Q) - h(P||Q) = \sum_{i=1}^{n} q_i f_{K_0h} \left( \frac{p_i}{q_i} \right),$$

where

$$f_{K_0h}(x) = \frac{(x-1)^2}{8 \sqrt{x}} - \frac{(\sqrt{x} - 1)^2}{2}$$

This gives us

$$f'_{K_0h}(x) = \frac{3x^2 - 8x^{3/2} + 6x - 1}{16x^{3/2}}, \ x > 0$$
and
\[ f''_{K_0h}(x) = \frac{3(x-1)^2}{32x^{5/2}}. \] (53)

Thus we have \( f''_{K_0h}(x) \geq 0 \) for all \( x > 0 \), and hence, \( f_{K_0h}(x) \) is convex for all \( x > 0 \). Also, we have \( f_{K_0h}(1) = 0 \). In view of this we can say that the measure \( D_{K_0h}(P||Q) \) given by (52) is nonnegative and convex in the pair of probability distributions \((P, Q) \in \Gamma_n \times \Gamma_n\). The convexity of the measure \( D_{T \Psi}(P||Q) \) is already given in the Proposition 2.2.

Let us consider now the following function
\[ g_{K_0h \Psi T}(x) = \frac{f''_{K_0h}(x)}{f''_{\Psi T}(x)} = \frac{3(x+1)\sqrt{x}}{4(x^2 + x + 1)}, \quad x > 0, \]
where \( f''_{K_0h}(x) \) and \( f''_{\Psi T}(x) \) and are as given by (53) and (27) respectively.

Calculating the first order derivative of the function \( g_{K_0h \Psi T}(x) \) with respect to \( x \), one gets
\[ g'_{K_0h \Psi T}(x) = -\frac{3(x-1)(x^2 + 3x + 1)}{8(x^2 + x + 1)^2 \sqrt{x}} \]
\[ \quad \left\{ \begin{array}{ll}
> 0, & x < 1 \\
< 0, & x > 1
\end{array} \right. \] (54)

In view of (54) we conclude that the function \( g_{K_0h \Psi T}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\[ M = \sup_{x \in (0, \infty)} g_{K_0h \Psi T}(x) = g_{K_0h \Psi T}(1) = \frac{1}{2}. \] (55)

By the application of (14) with (55) we get (51).

\[ \square \]

2.5 Unification Inequalities

**Proposition 2.5.** The following inequalities hold:
\[ 0 \leq D_{K_0T}(P||Q) \leq D_{K_0h}(P||Q) \leq \frac{1}{2} D_{T \Psi}(P||Q) \leq \frac{1}{2} D_{K_0h}(P||Q) \leq \frac{1}{2} D_{T \Psi}(P||Q) \leq \frac{1}{2} D_{K_0T}(P||Q). \] (56)

**Proof.** In view of (49), (50) and (51), we have
\[ 0 \leq D_{K_0T}(P||Q) \leq D_{K_0h}(P||Q) \leq \frac{1}{2} D_{T \Psi}(P||Q) \leq \frac{1}{2} D_{K_0h}(P||Q) \leq \frac{1}{2} D_{T \Psi}(P||Q). \] (57)
In view of Proposition 2.3 we have
\[ 0 \leq D_{K_0 T}(P||Q) \leq D_{K_{oh}}(P||Q) \leq \frac{1}{2} D_{\Psi T}(P||Q) \leq \frac{1}{128} D^*(P||Q). \] (58)

We observe that the inequalities (57) and (58) are the same except in the last term in the right side.

Below we shall give a relationship with \( D_{\Psi M T}(P||Q) \) and \( D^*(P||Q) \).

In view of (57), we can write
\[ -\frac{1}{8} K_0(P||Q) \leq -T(P||Q). \]

This gives
\[ 2\Psi_M(P||Q) - 4K_0(P||Q) \leq 2\Psi_M(P||Q) - 32T(P||Q). \] (59)

Now (37) and (59) together give
\[ D^*(P||Q) \leq 32 \left( \frac{1}{16} \Psi_M(P||Q) - T(P||Q) \right), \]
i.e.,
\[ \frac{1}{32} D^*(P||Q) \leq D_{\Psi M T}(P||Q). \] (60)

Finally, (58), (59) and (60) together completes the proof.

3 Exponential Divergence

Let consider the following general measure
\[ K_t(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{2(t+1)}}{p_i q_i^{2(t+1)}}, \quad t = 0, 1, 2, 3, ... \] (61)

When \( t = 0 \), we have the same measure as given in (39) and when \( t = 1 \), we have \( K_1(P||Q) = D^*(P||Q) \). When \( 2t + 1 = k \), it reduces to one studied by Jain and Srivastava [4].

Now we shall prove the convexity of the measure (61).
We can write

\[ K_t(P||Q) = \sum_{i=1}^{n} q_i f_t \left( \frac{p_i}{q_i} \right), \]

where

\[ f_t(x) = \frac{(x - 1)^{2(t+1)}}{x^{2t+1}}, \quad x > 0 \]

Calculating the first and second order derivative of \( f_t(x) \), we get

\[ f'_t(x) = \frac{(x - 1)^{2t+1}(2tx + 3x + 2t + 1)}{2x^{2t+1}} \]

and

\[ f''_t(x) = \frac{(x - 1)^{2t}(2t + 1) [(2t + 3)(x^2 + 1) + 2(2t + 1)x]}{x^{2t+1}}, \quad \forall x > 0, \ t \in \mathbb{N}. \quad (62) \]

Thus we have \( f''_t(x) \geq 0 \) for all \( x > 0 \), and hence, \( f_t(x) \) is convex for all \( x > 0 \) and \( t \in \mathbb{N} \). Also, we have \( f_t(1) = 0 \). In view of this we can say that the measure \( K_t(P||Q) \), \( t \in \mathbb{N} \) given by (61) is nonnegative and convex in the pair of probability distributions \((P, Q) \in \Gamma_n \times \Gamma_n\).

For \( t = 0, 1, 2, 3, \ldots \) and \( x > 0 \) we have the following sequence of convex functions

\[ f_0(x) = \frac{(x - 1)^2}{x^{1/2}}, \]
\[ f_1(x) = f_{D^*}(x) = \frac{(x - 1)^4}{x^{3/2}}, \]
\[ f_2(x) = \frac{(x - 1)^6}{x^{5/2}}, \]
\[ f_3(x) = \frac{(x - 1)^8}{x^{7/2}}, \]
\[ \vdots \]

Let write a linear combination of convex functions,

\[ f_{E_K}(x) = c_0 f_0(x) + c_1 f_1(x) + c_2 f_2(x) + \]
i.e.,

\[ f_{E_K}(x) = c_0 \frac{(x - 1)^2}{x^{1/2}} + c_1 \frac{(x - 1)^4}{x^{3/2}} + c_2 \frac{(x - 1)^6}{x^{5/2}} + c_3 \frac{(x - 1)^8}{x^{7/2}} + \ldots, \]
where $c_0$, $c_1$, $c_2$, $c_3$, ... are the constants.

For simplicity let us consider,

$$
c_0 = \frac{1}{0!}, \quad c_1 = \frac{1}{1!}, \quad c_2 = \frac{1}{2!}, \quad c_3 = \frac{1}{3!}, \ldots
$$

Thus we have

$$
f_{E_K}(x) = \frac{1}{0!} \frac{(x - 1)^2}{x^{1/2}} + \frac{1}{1!} \frac{(x - 1)^4}{x^{3/2}} + \frac{1}{2!} \frac{(x - 1)^6}{x^{5/2}} + \frac{1}{3!} \frac{(x - 1)^8}{x^{7/2}} + \ldots
$$

This gives us

$$
f_{E_K}(x) = \frac{(x - 1)^2}{x^{1/2}} \exp \left( \frac{(x - 1)^2}{x} \right).
$$

As a consequence of (63), we have the following divergence measure

$$
E_K(P||Q) = \sum_{i=1}^{n} q_i f_{E_K} \left( \frac{p_i}{q_i} \right)
= \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{\sqrt{p_i q_i}} \exp \left( \frac{(p_i - q_i)^2}{p_i q_i} \right), \quad (P, Q) \in \Gamma_n \times \Gamma_n,
$$

The measure (65) has been presented by Jain and Srivastava [4]. We shall call it exponential divergence.

As a consequence of the expression (63), it is obvious that

$$
K_0(P||Q) + D^*(P||Q) \leq E_K(P||Q),
$$

where $D^*(P||Q) = K_1(P||Q)$.

From the inequalities (8) and (42), we have

$$
\frac{1}{4} \Delta(P||Q) \leq I(P||Q) \leq h(P||Q) \leq \frac{1}{8} J(P||Q) \leq T(P||Q) \leq

\leq \frac{1}{8} K_0(P||Q) \leq \frac{1}{16} \Psi(P||Q) \leq \frac{1}{16} \Psi_M(P||Q).
$$

Here below we shall relate the measure $\Psi_M(P||Q)$ with exponential divergence $E_K(P||Q)$.  

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In view of (37), we have
\[ \frac{1}{2} \Psi_M (P \| Q) = K_0 (P \| Q) + \frac{1}{4} D^* (P \| Q) \leq K_0 (P \| Q) + D^* (P \| Q). \] (68)

Now (68) together with (66) gives
\[ \frac{1}{2} \Psi_M (P \| Q) \leq E_K (P \| Q). \] (69)

Inequalities (67) and (69) together give
\[ \frac{1}{4} \Delta (P \| Q) \leq I (P \| Q) \leq h (P \| Q) \leq \frac{1}{8} J (P \| Q) \leq T (P \| Q) \leq \] \[ \leq \frac{1}{8} K_0 (P \| Q) \leq \frac{1}{16} \Psi (P \| Q) \leq \frac{1}{16} \Psi_M (P \| Q) \leq \frac{1}{8} E_K (P \| Q). \] (70)

Also we have
\[ D_{K_0 T} (P \| Q) \leq D_{K_0 h} (P \| Q) \leq \frac{1}{2} D_{\Psi T} (P \| Q) \leq \frac{1}{128} D^* (P \| Q) \leq \frac{1}{128} E_K (P \| Q). \] (71)

Moreover in view of (48) and (69) the following inequalities among the new measures hold:
\[ F_N (P \| Q) \leq \frac{1}{192} D^* (P \| Q) \leq \frac{1}{96} \Psi_M (P \| Q) \leq \frac{1}{48} E_K (P \| Q). \] (72)

4 Bounds on New Divergence Measures

In this section we shall give bounds on the measures studied in sections 2 and 3. These bounds are based on the theorem given below.

Theorem 4.1. Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be differentiable convex and normalized i.e., \( f(1) = 0 \). If \( P, Q \in \Gamma_n \),
\[ 0 \leq C_f (P \| Q) \leq W_{C_f} (P \| Q), \] (73)
where
\[ W_{C_f} (P \| Q) = \sum_{i=1}^{n} (p_i - q_i) f' \left( \frac{p_i}{q_i} \right), \] (74)
In addition, if we have $0 < r \leq \frac{p_i}{q_i} \leq R < \infty$, $\forall i \in \{1, 2, \ldots, n\}$, for some $r$ and $R$ with $0 < r \leq 1 \leq R < \infty$, then the followings hold:

$$0 \leq C_f(P||Q) \leq W_{C_f}(P||Q) \leq Z_{C_f}(r, R),$$  \hspace{1cm} (75)

and

$$0 \leq C_f(P||Q) \leq Y_{C_f}(r, R) \leq Z_{C_f}(r, R),$$  \hspace{1cm} (76)

where

$$Z_{C_f}(r, R) = \frac{1}{4}(R - r)\left[f'(R) - f'(r)\right];$$  \hspace{1cm} (77)

and

$$Y_{C_f}(r, R) = \frac{(R - 1)f(r) + (1 - r)f(R)}{R - r}.$$  \hspace{1cm} (78)

More details on the above theorem refer to Taneja [10].

Here below we shall give examples of the inequality (73). This we have done for the measures $E_K(P||Q)$, $K_t(P||Q)$, $F_N(P||Q)$, and $\Psi_M(P||Q)$. The measures $D^*(P||Q)$ and $K_0(P||Q)$ are the particular cases of $K_t(P||Q)$. The applications of the inequalities (74) and (75) can be obtained on similar lines.

**Example 4.1.** Let us consider the measure $K_t(P||Q)$ given by (61). Then applying the inequality (73), we have the following bound:

$$0 \leq K_t(P||Q) \leq W_{K_t}(P||Q), \quad t = 0, 1, 2, \ldots$$ \hspace{1cm} (79)

where

$$W_{K_t}(P||Q) = \sum_{i=1}^{n} \left(\frac{p_i - q_i}{q_i}\right)^{2(t+1)} \left(\frac{q_i}{p_i}\right)^{\frac{2t+3}{2}} \left(\frac{(2t+3)p_i + (2t+1)q_i}{2}\right), \quad t = 0, 1, 2, \ldots$$

In particular we have

$$0 \leq K_0(P||Q) \leq \sum_{i=1}^{n} \left(\frac{p_i - q_i}{q_i}\right)^{2} \left(\frac{q_i}{p_i}\right)^{3/2} \left(\frac{3p_i + q_i}{2}\right),$$  \hspace{1cm} (80)

and

$$0 \leq D^*(P||Q) \leq \sum_{i=1}^{n} \left(\frac{p_i - q_i}{q_i}\right)^{4} \left(\frac{q_i}{p_i}\right)^{5/2} \left(\frac{5p_i + 3q_i}{2}\right).$$  \hspace{1cm} (81)
Inequalities (80) and (81) are obtained by considering $t = 0$ and $t = 1$ in (78) respectively. The measures $K_0(P||Q)$ and $D^*(P||Q)$ are the same as given by (39) and (9) respectively.

**Example 4.2.** Let us consider the measure $E_K(P||Q)$ given by (65). Then applying the inequality (73), we have the following bound:

$$0 \leq E_K(P||Q) \leq W_{E_K}(P||Q),$$

where

$$W_{E_K}(P||Q) = \sum_{i=1}^{n} \left( \frac{(p_i - q_i)^2 [q_i(p_i - q_i)^2 + 2p_i^3 + p_i q_i^2 + q_i^3]}{2p_i^{5/2} q_i^{3/2}} \right) \exp \left( \frac{(p_i - q_i)^2}{p_i q_i} \right).$$

**Example 4.3.** Let us consider the measure $\Psi_M(P||Q)$ given by (38). Then applying the inequality (73), we have the following bound:

$$0 \leq \Psi_M(P||Q) \leq E_{\Psi_M}(P||Q),$$

where

$$E_{\Psi_M}(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^4(p_i + q_i)(5p_i^2 + 3q_i^2)}{2p_i^{5/2} q_i^{3/2}}.$$  

**Example 4.4.** Let us consider the measure $F_N(P||Q)$ given by (44). Then applying the inequality (73), we have the following bound:

$$0 \leq F_N(P||Q) \leq W_{F_N}(P||Q),$$

where

$$E_{F_N}(P||Q) = \sum_{i=1}^{n} \frac{(\sqrt{p_i} - \sqrt{q_i})^4 (\sqrt{p_i} + \sqrt{q_i}) (p_i + 2q_i + \sqrt{p_i q_i})}{(p_i + q_i)^2 \sqrt{p_i}}.$$  

As we mentioned above we can also obtain bounds applying the inequalities (74) and (75) by simple calculations. The results obtained shall be in terms of $r$ and $R$. 

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References


