Results on the Convergence of Braitenberg Vehicle 3a

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Abstract

Braitenberg vehicles are well known models of animal behaviour used as steering mechanisms in mobile robotics and Artificial Life. Because of their simplicity, they are mainly used for teaching robotics, whilst the lack of a quantitative theory turns its use for research purposes troublesome. This paper contributes to our formal understanding of Braitenberg vehicle 3a by presenting the convergence properties of its trajectories under parabolic shaped stimuli. We show previously unreported features of the motion of Braitenberg vehicle 3a like; their conditional stability, their oscillatory behaviour and the existence of periodic trajectories. The mathematical model used provides a theoretical relation between the environment, the internal control mechanism of the vehicle and some morphological parameters, a link already found in experimental works. This work provides theoretical support for experimental research using Braitenberg vehicle 3a, and paves the way for further research in biology, robotics and Artificial Life.

keywords  Adaptive Behaviour, Braitenberg Vehicles, Source Seeking, Tropotaxis, Dynamical systems.
1 Introduction

Braitenberg vehicles [4] qualitatively model animal behaviour at the steering level and they have long been used on an empirical basis in robotics and Artificial Life [18] [23]. While, according to the original taxonomy, vehicles 2b and 3a model general taxis behaviour, vehicles 2a and 3b model negative taxis. Specifically, vehicle 3a models tropotaxis, the behaviour of an agent with two sensors moving towards high intensity values of some stimulus [6], which can be used to implement target acquisition for instance. Despite the taxis model proposed by Braitenberg being simple, intuitive and efficient, it is only a qualitative model that requires simulations to analyse its behaviour. Therefore, no general conclusions can be derived or proved from the model as presented in [4]. This does not prevent Braitenberg vehicles to be used for research or as a teaching tool. Since they can be quickly understood qualitatively, without strong mathematical background, the vehicles represent the perfect tool to teach robotics [5], even at the school level [21].

Multiple instances of Braitenberg vehicles, or their principles, can be found in the robotics literature, in applications ranging from phototaxis to chemotaxis. Phonotaxis behaviour is implemented in a robotic rat [2] that models the peripheral auditory system in mammals. Even though their main contribution consists on simplifying sound source localisation through the pinnae and the cochlea model, they successfully implement the central auditory system, and also include a Braitenberg vehicle 3a to control the robot motion. Another example of phonotaxis behaviour is presented in [20], where they imitate the auditory system of a lizard. In fact, their implementation of the lizard ear model is good enough to work with a high success rate over a wide range of frequencies using both, a Braitenberg vehicle 2b and a bang-bang controller. Interestingly, the performance of both controllers was similar even though they did not have a model of the vehicle to tune the behaviour of the robot. In a series of works [24] [8] [17] female cricket phonotaxis model is implemented using spiking neural networks connected according to the principles of Braitenberg vehicles. This neural model of motion control is comparable to a combination of vehicles 2a and 3b, since excitatory units display a direct connection between sensors and motors, while inhibitory ones are crossed. The authors prove their robots perform very well even under quite adverse outdoor conditions.

Chemotaxis can also be found among the implementations of robotic Braitenberg vehicles. An experimental analysis of vehicles 3a and 3b for odour source localisation is presented in [10], where the connection between sensors and motors is linear, but sensor readings are normalised. They present one of the first working robots for chemical source localisation, which turns to be a very complex task. Phototaxis, the motion towards a light source, is used in [3] as a mechanism for target acquisition using vehicle 3a. Obstacle avoidance is achieved through infrared based Braitenberg vehicle 2b, which produces a very smooth obstacle avoidance behaviour. Even though it is not possible to review all of them, there are many successful robotic applications of Braitenberg vehicles for target seeking, wandering, sound source localisation or obstacle avoidance.

Several evolutionary based works used Braitenberg vehicles for different purposes. The work in [11], for instance, presents an interesting study of the effect of sensor placement and range to achieve tasks of increasing complexity. They prove there is a relation between the number of sensors, their range and the complexity of the task, a very interesting result as it links behaviour and morphology. Unfortunately they need to include the controller within the evolutionary strategy slowing down the evolu-
Evolution of Braitenberg vehicle 2b using genetic algorithms and agent generative grammars is presented in [12], where the evolved vehicles outperform free random evolution of agent morphology and control. Through their experiments, the authors prove that random evolution can actually find features of Braitenberg vehicles, while the evolution of vehicle 2b produces a variant of vehicle 3a. Many other interesting works use evolutionary techniques to design Braitenberg vehicle controllers in robots [13] or simulated agents [7].

Braitenberg vehicles also inspired the creation of artificial characters like the well known “boids” [19], which show complex collective behaviours, like flocking or schooling, even though they are implemented using simple mechanisms. In fact this paper represents one of the first, and probably the best known, research work in agent formation. A set of artificial fishes, with behaviours also inspired by [4], are modelled and presented in [22]. Their work goes much further than the simple generation of obstacle avoidance, target reaching or wandering, since they propose a learning mechanism to acquire and associate simulated muscular motion with steering commands. However, they use simple behaviours at the steering level that can be still found among the Braitenberg vehicles. Many other works exist that model steering behaviours in simulated agents, but Braitenberg vehicles can be considered a de facto standard procedure to simulate agents performing foraging, target acquisition and many other sensor driven tasks.

As we saw, Braitenberg vehicles are present in a wide range of areas and applications, even though their understanding is limited to intuition and experimental or simulated results. They successfully model sensory driven motion and, therefore, provide a better understanding of animal motion that can be applied in robotics or further investigated in Artificial Life or biology. However, only recently a formal mathematical model to further understand and exploit the possibilities of Braitenberg vehicles was developed [15]. This paper contributes to our understanding of Braitenberg vehicle 3a by presenting a mathematical analysis of their behaviour in the vicinity of parabolic stimulus. Instead of relying on experimental data, we perform a top-down analysis of the general non linear differential equations describing their behaviour, such that the results are valid for any application of vehicle 3b. New and unexpected results that actually challenge our empirical understanding of Braitenberg vehicle 3a are presented. This formal analysis is required to ensure applicability in real world scenarios of the controllers presented by Braitenberg. So far, Braitenberg vehicles have an impact only in a restricted scientific community, even though their potential is of high interest for a wider audience like roboticists or control engineers.

The rest of the paper is organised as follows. Section 2 first reviews the assumptions and the mathematical model derived for Braitenberg vehicle 3a already presented in [15]. Then it presents some analytic solutions of the non linear dynamical system describing vehicle 3a under a parabolic, non circularly symmetric, stimulus or potential field. This is a case general enough as many stimuli can be approximated by a parabolic function near the source. Results for circular symmetric stimuli are presented at the end of this section. Section 3 contains a set of selected simulations that illustrate the theoretical results obtained. Finally, Section 4 presents a summary of the conclusions and future steps on the development to a theoretical framework for Braitenberg vehicles.
2 Vehicle 3a Model and Resulting Trajectories

A simple model of animal taxis is proposed for dual-drive wheeled vehicles in [4] as a thought experiment. Figure 1 shows the vehicle named 3a in the proximity of a light source. The wheels of the vehicle abstract the locomotive subsystems of animals to focus on the steering level [19]. In fact, similar steering models have been used to understand human motion [1]. As shown in the figure, the vehicle has two sensors connected to the ipsilateral wheel in a decreasing way. An intuitive analysis shows that the vehicle will turn towards, and approach, the light source. As it gets closer to the source, or maximum, it will slow down, and eventually, once it is close enough it will stop in front of it. Since intuitively the motion converges to the stimulus, no further formalisation or analysis was performed even though Brairtenberg vehicles were used in real robots and simulations of artificial agents. Therefore, all previous works assumed vehicle convergence to the stimulus under any circumstance.

We will briefly review the modelling assumptions leading to the motion equations for vehicle 3a (see [15] for further details). First, we will assume that the stimulus field can be modelled as a non negative smooth scalar function \( S(x) \) of the position of the vehicle in its workspace \( x = (x,y) \in \mathcal{W} \subseteq \mathbb{R}^2 \), and \( S(x) \) is \( C^\infty \). If there is a single source or maximum, without loss of generality, we can set the origin of a coordinate system such that \( S(0) \geq S(x) \forall x \in \mathcal{W} \). This means the gradient of \( S(x) \) vanishes at the origin while the Hessian matrix is negative definite, i.e. \( \nabla S(0) = 0 \) and \( y^T D^2 S(0)y < 0 \forall y \in \mathbb{R}^2 \). Probably the best known example of Brairtenberg vehicles is the one implementing phototaxis through light sensors, with a light source placed at some height \( h_0 \) above the ground. It can be seen that, according to the inverse-square law, the light intensity will fulfil the above conditions. Furthermore, if the emission pattern of the light source is isotropic, the stimulus \( S(x) \) will be such that
\[
S(x) \propto \frac{1}{h_0^2 + x^2 + y^2}.
\]

As the connection between the sensor readings ‘\( s \)’ and the wheel velocities \( v_{L/R} \) is direct and decreasing, we can model it as a smooth function \( F(s) \), such that: i) \( F : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\} \); i.e. the vehicle does not move backwards and \( F(s) = 0 \) only at the source maximum; and ii) \( F'(s) < 0 \), since the connection is decreasing the derivative of \( F(s) \) must be negative. This function is usually selected to be just a linear function or an Artificial Neural Network with the sensor readings as inputs and the outputs connected to the motor outputs. Approximating the stimulus around the midpoint between the sensors of the vehicle \( (x) \) as a Taylor series, we obtain the
general motion equations for vehicle 3a (see [15]):

\[
\begin{align*}
\dot{x} &= F(S(x)) \cos \theta \\
\dot{y} &= F(S(x)) \sin \theta \\
\dot{\theta} &= -\frac{\delta}{d} \nabla F(S(x)) \cdot \hat{e}_p
\end{align*}
\]

where \( \delta \) is the distance between the sensors, \( d \) is the wheelbase of the vehicle and \( \hat{e}_p = [-\sin \theta, \cos \theta] \). This is a general model that can be used to analyse the behaviour of the vehicle or to design \( F(s) \) for a known stimulus. Equations (1), (2) and (3) are valid for any smooth stimulus with any number of extrema, and they actually indicate that, while the value of \( F(s) \) directly affects the linear velocity, its derivative is used to control the steering rate. This is an interesting new view of Braitenberg vehicles that does not rely on pure intuition.

2.1 Behaviour under Parabolic Stimuli

If the stimulus with a maximum at the origin can be modelled as a smooth function, it can also be approximated as a Taylor series like \( S(x) \approx S(0) - x^T \Sigma x \). On the other hand, if the stimulus has elliptic symmetry, we can generalise it as a function \( S(\rho) \) where \( \rho = x^T \Sigma x \geq 0 \) for all \( x \neq 0 \) and \( \frac{dS}{d\rho} \leq 0 \) for all \( \rho \). Without loss of generality, we can assume \( \Sigma \) is a diagonal matrix, which means the principal axes of the stimulus function coincide with the axes of the Cartesian coordinates. However, the elements of the diagonal do not need to be equal, therefore representing a non-symmetric emission pattern of a stimulus source. In the above mentioned case of a light source, it can be seen this is an appropriate approach, and \( \rho_0 = x^T \Sigma x \) corresponds to a curve with constant light intensity. Under these condition the motion equations of the vehicle can be stated as:

\[
\begin{align*}
\dot{x} &= F(S(\rho)) \cos \theta \\
\dot{y} &= F(S(\rho)) \sin \theta \\
\dot{\theta} &= -2\frac{\delta}{d} F'(S(\rho)) S'(\rho) x^T \Sigma \hat{e}_p
\end{align*}
\]

where \( F'(s) \) and \( S'(\rho) \) represent the derivatives of the function w.r.t. \( s \) and \( \rho \) respectively.

We assumed the compound function \( F(S(\rho)) \) will only vanish at the origin, \( \rho = 0 \), therefore equations (4) and (5) vanish simultaneously, while equation (6) vanishes too since \( \rho = 0 \iff x = 0 \). This means the motion equations have an equilibrium point at the origin for parabolically shaped stimuli, which according to the intuitive understanding of Braitenberg vehicles should be stable, i.e. vehicles starting near the source will end up approaching it.

2.1.1 Analytic Solutions to the Dynamic Equations

Given our assumptions, we can prove that the axes contain solution trajectories. If we consider \( \Sigma \), a diagonal matrix, as \( diag(\Sigma) = (\sigma_1, \sigma_2) \), the condition for equation (6) to be zero is \( -x \sigma_1 \sin \theta + y \sigma_2 \cos \theta = 0 \). This condition is fulfilled if \( x = 0 \) and \( \cos \theta = 0 \) simultaneously, or \( y = 0 \) and \( \sin \theta = 0 \) simultaneously. Therefore, if the
point \( x = 0, \cos \theta = 0 \) belongs to the trajectory, as the solution is unique, it is given by \( x(t) = 0, \theta(t) = \mp \pi/2 \) and \( \dot{y} = \pm F(S(\sigma_2 y^2)) \), with the appropriate initial conditions. This means that a vehicle starting at the \( y \) axis with the right heading will move along that axis. Moreover, as we will see, the stability of the trajectory will depend on the position and heading of the vehicle. While intuition tells a vehicle heading the source will move following a straight line, the mathematical model says this is only the case for the principal axis if the stimulus has elliptic symmetry.

As a case study, let us analyse the situation of the vehicle starting on the \( x \) Cartesian axis, i.e. \( y_0 = 0 \). The trajectory will be a straight line only if \( \sin \theta_0 = 0 \), which makes equations (5) and (6) vanish simultaneously. The vehicle motion is described by the solution to \( \dot{x} = \pm F(S(\sigma_1 x^2)) \), where the sign depends on whether \( \theta_0 = 0 \) or \( \theta_0 = \pi \), giving two possible vehicle trajectories. It is worth reminding that \( F(S(x)) \geq 0 \) and that the identity only occurs for \( x = 0 \), which means the origin is an equilibrium point. However, the linear stability test does not work for the equation at hand as \( F(S(x)) \) has an extremum at the origin. Moreover, the non linear differential equation describing the motion of vehicle 3a is conditionally stable as shown in figure 2. According to figure 2(a), when the initial heading of the vehicle is \( \theta_0 = 0 \), the equilibrium point at the origin is an attractor for initial conditions \( x_0 < 0 \), and a repeller for \( x_0 > 0 \). The interpretation of this result is straightforward; when the vehicle is heading the source from a negative position it will move in a straight line approaching the origin. However, if the source lies exactly behind the vehicle, as the stimulus perceived in both sensors is identical, the vehicle will move forward without turning, leaving the origin exactly at its back. In a real situation, where noise has an effect, there is some chance that the vehicle will get out of this perfectly symmetric case. The outcome of the symmetry breaking will be presented in the next section when trajectories close to the straight line are analysed. A similar analysis can be performed for the initial condition \( \theta_0 = \pi \), however the flow vectors point in the opposite direction as shown in figure 2(b).

![Figure 2: Stability of the motion equations along the \( x \) axis.](image)

As a result we found eight analytic solutions to the motion equations of the Braitenberg vehicle 3a. When the vehicle initial condition lies on the principal axis of the stimulus function and the heading points towards the source, or in the opposite direction, the trajectories are straight lines towards, or moving away from the source. These are represented in figure 3(a) and correspond to the principal axes of the stimulus. As we will see in the next section we can extract useful information from the obtained trajectories. On the other hand, contrary to what one could expect, if the vehicle heads perfectly the source somewhere outside the main axis, its trajectory will not be a straight line unless the stimulus is circularly symmetric.
2.1.2 Trajectories close to the Analytic Solutions

Finding analytic solutions to non linear dynamical systems is very useful as it allows to analyse the solution trajectories in their vicinity. That means we can get also information on how the vehicle behaves when it moves close to the main axis of the stimulus. The tool used under these circumstances is linearisation of the dynamical system around the trajectory, which for equilibrium points turns into the linear stability test. Therefore, we need to compute the Jacobian matrix of the equations (1), (2) and (3), which can be stated as:

\[
J = \begin{bmatrix}
\hat{e} \nabla F(S(x))^T & F(S(x))\hat{e}_p^T \\
\frac{\delta}{\delta \nabla x} F(S(x))^T \hat{e}_p & \frac{\delta}{\delta \partial S} F(S(x))
\end{bmatrix}
\] (7)

where \(\nabla F(S(x))\hat{e}^T\) is the \(2 \times 2\) matrix resulting from the product of the gradient of \(F(S(x))\) and the heading unit vector of the vehicle \(\hat{e} = [\cos \theta, \sin \theta]^T\), \(\frac{\delta}{\delta \nabla x} F(S(x))^T \hat{e}_p\) is a \(1 \times 2\) matrix formed by second order cross derivatives of \(F(S(x))\), and \(\partial_s F(S(x))\) is the directional derivative of \(F(S(x))\) along the vehicle heading. To apply the linear stability test or to investigate trajectories close to the ones found, this matrix has to be evaluated at the analytic solutions and the linear system \(\dot{x} = A(t)\Delta x\), for \(\Delta x = x - x_l\), solved to obtain the behaviour of nearby trajectories. Specifically, the straight line trajectory can be stated as; \(y_l(t) = 0\), \(\theta_l(t) = 0\) (or \(\theta_l(t) = \pi\)) and \(x_l(t)\) is the solution of \(\dot{x} = F(S_{\sigma_1}x^2)\). If we substitute the trajectory in expression (7), taking the case \(\theta_l(t) = 0\) we obtain:

\[
A(t) = \begin{bmatrix}
2F'S'\sigma_1x & 0 & 0 \\
0 & 0 & F(S_{\sigma_1}x^2) \\
0 & -2F'S'\sigma_2 & 2F'S'\sigma_1x
\end{bmatrix}
\] (8)

where \(F' = F'(S_{\sigma_1}x^2)\) and \(S' = S'(\sigma_1x^2)\) are the derivatives of \(F(s)\) and \(S(\rho)\) w.r.t. their arguments along the trajectory \(x_l(t)\) respectively. This linear time dependent dynamical system describes the behaviour of the trajectories close to the analytic solutions obtained earlier. We can compute the eigenvalues of the above matrix to check whether trajectories close to that solution converge to it or diverge. The eigenvalues \(\lambda_i\) for the case at hand are
\[ \lambda_1 = 2\sigma_1 x F' S' \]

\[ \lambda_{2,3} = \frac{\delta}{d} \sigma_1 x F' S' \mp \sqrt{\left( \frac{\delta}{d} \sigma_1 x F' S' \right)^2 - 2\frac{\delta}{d} F' S' F \sigma_2} \]

where \( F = F(S(\sigma_1 x^2)) \) is evaluated along \( x_1(t) \).

According to our assumptions \( S'(\rho) < 0 \), \( F'(s) < 0 \) and \( \sigma_1 > 0 \) and \( \sigma_2 > 0 \). Having this in mind, we can see that the sign of the first eigenvalue \( \lambda_1 = 2F'S\sigma_1 x \) depends on the value of \( x \). Moreover, since \( F'S\sigma_1 > 0 \) the trajectories of the linearised system will diverge from the straight line trajectory when \( x > 0 \). We saw that the vehicle with the initial conditions \( \theta_0 = 0, y_0 = 0 \) and \( x_0 > 0 \) will move away from the source following the principal stimulus axis. However, if \( \theta_0 \) or \( y_0 \) are not exactly zero, it will start turning to head the source and will diverge from the stimulus principal axis. It is worth noting that the second and third eigenvalues will produce the same effect as they share this term, while the part inside the square root will always have an absolute value smaller than the first term. Therefore, all the eigenvalues have the same sign and the vehicle will move away from the straight line trajectory. On the other hand, when \( x < 0 \) all the eigenvalues are negative, hence any trajectory starting close to the analytic solution will get closer to it. A complete analysis of the trajectories is presented in figure 3(b). In short, all the trajectories approximately heading the source will converge to one of the solutions, while the ones with the source on the back will move away from the axis. Therefore, an agent close to a stimulus maximum will always turn towards it and, eventually, reach it. This result nicely matches the intuitive understanding of Braitenberg vehicle 3a.

However, a previously unreported situation, not completely matching intuition, can occur if the term inside the square root becomes negative, as the eigenvalues \( \lambda_{2,3} \) will be complex numbers. If this happens, oscillatory behaviour will be displayed by these vehicles, while the specific condition for that to occur is \( \frac{\delta}{d} F' S' \sigma_1^2 x^2 - 2F \sigma_2 < 0 \). We can see that the distance, for which oscillations around the principal stimulus axis occur, depends, among other factors, on the relation between \( \sigma_2 \) and \( \sigma_1 \), but also on the morphology of the vehicle and its internal connection, the function \( F(s) \). This implies that for a given morphology and internal control, an adequate threshold distance should be selected to stop the vehicle if we do not want the trajectory to oscillate. Similar results can be obtained if the trajectories are analysed around the other analytic solution, \( x_2(t) = 0, \theta(t) = \pi \) and \( y = F(S(\sigma_2 y^2)) \), but with exchanged roles of \( \sigma_1 \) and \( \sigma_2 \). To the best knowledge of the author, this represents the first time a theoretical relation between morphology and behaviour has been found in agent based controllers.

### 2.2 The Case of Circular Symmetry

If the stimulus function displays circular symmetry a deeper analysis can be performed. In this case \( S(x) \), and therefore \( F(S(x)) \), can be expressed as a function of the distance to the origin. Therefore, we will write \( F(S(r)) \) where \( r \) is the polar coordinate of the vehicle, and the gradient of \( F(S(r)) \) has only a radial component. This is the case, for instance, of a light source emitting equally in all directions. Under this assumption, if we transform the system of differential equations (1), (2) and (3) to polar coordinates \((r, \psi)\) and we define a new variable \( \eta = \psi - \theta \), the system of differential equations describing the motion of the Braitenberg vehicle 3a is:
\[ \dot{r} = F(r) \cos \eta \quad (9) \]

\[ \dot{\eta} = -\left[ \frac{F(r)}{r} - \frac{\delta}{d} \frac{\partial F(r)}{\partial r} \right] \sin \eta \quad (10) \]

where \( F(r) = F(S(r)) \) and \( \eta \) represents the vehicle heading relative to the polar coordinate \( \psi \). Due to the symmetry of the problem, only the relative angle is a relevant variable, and it can be seen that \( \eta = 0 \) corresponds to the vehicle with the source at its back, while for \( \eta = \pi \) the vehicle heads the origin, \( \theta \) and \( \psi \) are complementary angles. Formulating the motion equations as a function of the relative heading simplifies the analysis of the trajectories, but the results of this analysis need to be interpreted accordingly. We will define the discriminant function \( G(r) = \frac{F(r)}{r} - \frac{\delta}{d} \frac{\partial F(r)}{\partial r} \) which can be used to analyse the stability of these equations and, therefore, of the Braitenberg vehicle 3a.

One way to analyse the behaviour of the system, defined by equations (9) and (10), is to qualitatively draw the phase plot, as done in figure 4, where the directions of the arrows show how the variables change for different regions of the phase plane [9]. While the sign of the radial flow component depends only on \( \cos \eta \), the sign of the angular component can change with the sign of the discriminant function. Figure 4(a) shows the flow vectors for a stable situation \( (G(r) < 0) \) where the vehicle moves towards the point in polar coordinates \((0, \pi)\). This point corresponds to the vehicle heading the origin and moving towards it, the expected behaviour of Braitenberg vehicle 3a, always reported in experimental works. On the other hand, if the discriminant function is positive, \( G(r) > 0 \), the vehicle will end up moving away from the stimulus or potential minimum as shown in figure 4(b), heading the angular coordinate \( \eta = 0 \) which means \( \psi = \theta \). This last situation cannot be easily guessed from the intuitive understanding of Braitenberg vehicle 3a as intuition indicates a stable motion towards the source. This situation occurs, for instance, when the slope of the function \( F(s) \) is not strong enough, and it can be fixed by changing \( F(s) \) if the aim is to obtain a taxis behaviour. It is worth noting that the morphological parameter \( \delta/d \) has a significant effect on the convergence, as it is a factor in \( G(r) \) multiplying the slope of \( F(s) \).

A more complex situation occurs if the sign of \( G(r) \) changes for different values of \( r \), as shown in figures 4(c) and 4(d). For the flow corresponding to figure 4(c), any trajectory heading the origin within a range of \( \pi \) and starting on the left side of the phase plane will converge to the point \((0, \pi)\), the stimulus source. However, initial conditions on the right side with the origin on their back will be unstable, making the vehicle to move away from the source. Nothing general can be stated about the stability for other initial conditions. However, two unstable periodic solutions appear at the intersection of the line defined by \( G(r) = 0 \) and \( \theta = \pm \pi/2 \) making the vehicle to turn around the source in clockwise and counter-clockwise directions. In sum, trajectories heading the origin are prone to drive the vehicle to the source, while those with the origin on the back will probably make it go away. On the other hand, figure 4(c) shows a flow where these two areas of the phase plane are swapped with respect to the previous case. This generates an equilibrium point at the intersection of the line defined by \( G(r) = 0 \) and \( \theta = \pm \pi/2 \), with oscillatory trajectories close to it, i.e. the superposition of two oscillatory motions. Even though theoretically possible, to the best knowledge of the author, this behaviour has never been reported before as the result of a Braitenberg vehicle 3a implementation or simulation. The techniques used to compute the frequencies of motion and approximated trajectories for Braitenberg
vehicle 2b presented in [14] can also be used in this case. More complex situations can appear mixing areas with different stability conditions, but the ones analysed here are the most relevant.

3 Computer Simulations of the Model

This section illustrates the above obtained theoretical results with several computer simulations of the dynamic equations describing the motion of the vehicle. However, only the most interesting results will be presented, using the appropriate stimulus functions.

3.1 Oscillatory Behaviour

To illustrate the oscillatory behaviour for non symmetric stimulus sources, we integrated the motion equations under a stimulus field for $\sigma_1 = 1$ and different $\sigma_2$ values. On the one hand, it is worth noting that this is a general setting, since values of $\sigma_1 \neq 1$ can be codified inside the derivative of the function $S(\rho)$ with the proper scaling of $\sigma_2$. On the other hand, it is worth reminding that smooth stimuli coming from a source with a non-isotropic emission pattern can be always approximated close to the source as a parabolic function. In order to compare the results with the symmetric trajectory we also plot the simulation corresponding to $\sigma_2 = 1$.

Figure 5(a) shows the trajectories for $\sigma_2 = \{1, 5, 10\}$ with initial conditions close to the axis $y = 0$, specifically $(x_0, y_0, \theta_0) = (5, 0, 2, 0)$ for all trajectories. The initial pose of the vehicle is represented by the triangle. Since $\sigma_2 > \sigma_1$ oscillations appear around the $x$ axis, and they start earlier as $\sigma_2$ increases relative to $\sigma_1$. Because $\lambda_1$ is always negative and $\lambda_{2,3}$ have negative real part, the amplitude of oscillations decreases exponentially as a first order approximation. This exponential modulation could make the
Oscillatory behaviour for non-symmetric stimulus. The point where oscillations start can be selected as a condition to switch to another behaviour in a more complex scenario. The figure also shows that the frequency of the oscillations increases as the vehicle approaches the stimulus maximum. This is due to the eigenvalues being a function of the vehicle’s $x$ coordinate. To the best knowledge of the author, this oscillatory behaviour has not been reported on any experimental work using Braitenberg vehicle 3a, probably because the parameters on the controller function $F(s)$ are experimentally tuned to avoid this situation. However, having a solution for the trajectories helps tuning the value of $F(s)$ to attenuate the oscillations.

Figure 5(b) shows the trajectories of the vehicle with initial conditions close to the axis $x = 0$, $(x_0, y_0, \theta_0) = (0.2, 5, -\pi/2)$ using the same $\sigma_1$ and $\sigma_2$ values. As the figure shows, the trajectories quickly turn towards the $x$ axis with an increasing strength as the value $\sigma_2$ increases. Despite the figure scale does not allow to see it, as the trajectories approach the $x$ axis, they start to oscillate like in the previous case (except
for the circularly symmetric case of $\sigma_2 = 1$). Finally we performed the simulations from random starting positions shown in figure 6 to verify trajectories reach indeed the source for different initial poses.

### 3.2 Different Stability Conditions

This section presents simulations of the symmetric stimulus for the stability conditions found. The value used for the ratio $\delta/d$ was 0.85 for all the simulations, and we always used different random starting poses but for the periodic motion around the source. Figures 7(a) and 7(b) show results for the cases where the discriminant function does not change its sign, and therefore the stability is the same for all $r$ values, meaning all trajectories converge to the source or move away from it. The initial pose of the vehicle is also drawn in the figures for each simulation. While the trajectories shown in figure 7(a) ($G(r) < 0$) are all stable, the ones in figure 7(b) ($G(r) > 0$) are unstable. In the stable case the trajectories pointing initially away from the origin reach the source after turning to head the source. On the other hand, trajectories in the unstable case can move first towards the stimulus source and diverge later. It is worth noting that even though one of the trajectories in figure 7(b) appears to head the origin (where the maximum of the stimulus occurs) it also diverges following an almost straight line with a different direction.

Figure 7(c) shows the first sign change case, where a $r_0$ exists such that $G(r) < 0$ for $r < r_0$ and $G(r) > 0$ for $r > r_0$. As already shown in figure 4(c), the stability of the trajectory depends on the initial conditions such that for $|\eta| > \pi/2$ and $r < r_0$ all the trajectories reach the source at the origin, for $|\eta| < \pi/2$ and $r > r_0$ all move away from it and any other trajectory will eventually reach any of these sets of the state space, therefore becoming stable or unstable. As it can be seen, from the 10 random initial conditions, only trajectories close to the origin ($r_0 = 3.34$ for the simulation) and pointing towards the source are stable. The case of trajectories oscillating around a fixed distance is presented in figure 7(d), where the distance is $r_0 = 4.3$. As shown in the figure, the initial pose of the vehicle is in the region with the flow pointing towards the source. However, once the trajectory reaches the area where $G(r) > 0$, it moves away from the source. The motion of the vehicle can be approximated as the sum of a circular trajectory around the origin and an oscillation around it as in [16]. Therefore
the solution in this case is either periodic or quasi-periodic depending on the ratio between the oscillation frequencies but it will never reach the source.

4 Conclusions and Further Work

This paper contributes to our theoretical understanding of Braitenberg vehicle 3a, a widely used qualitative model of animal steering control for goal seeking and target acquisition. Contrary to the intuitive understanding, we were able to theoretically predict the existence of unstable configurations, trajectories of the vehicle moving away from the stimulus, periodic trajectories around the stimulus and oscillatory behaviour when approaching the source. Moreover, we theoretically proved the link between morphological parameters, the sensor-motor connection and the resulting behaviour. While relations between morphology and control have been found in experimental works, instability of the behaviour of Braitenberg vehicle 3a was previously unreported. Neither it was the possibility of having periodic trajectories. Obviously the existing experimental works used this Braitenberg vehicle in a limited range of working conditions, the ones generating a proper taxis behaviour, also explained by the theoretical model. Even though we only provided the example of light stimulus, any other stimulus following the inverse-square law, or even the inverse distance law (like sound pressure) can be used to implement taxis behaviour. However, for real sensors the assumptions might not be adequate as some are not omnidirectional. In robotics, for instance, this issue is solved by adding sensors that cover the body of the robot and combining their
measures.

A limitation of the analysis can occur when the signal to noise ratio is too low and therefore the behaviour is mostly driven by noise. This is an important point in the future development of the theory of Braitenberg vehicles, as it would provide fully theoretical coverage of existing, simulated and real, experimental works. This lack of theoretical basis is the most probable reason why the impact of Braitenberg vehicles in different research areas is not as high as it should be. In sum, the existence of a formal model of Braitenberg vehicles paves the way to a further development in many research areas like robotics, where a formal biologically inspired controller can be used to steer robots; biology, as it provides a new deterministic model of tropotaxis behaviour that accounts for the animal heading direction; and Artificial Life, where the model allows computationally sound simulations of in steering related tasks.

References


