On fillings of 2N-gons with rhombi

Ilda P.F. da Silva
CMAF/INIC, Av. Prof. Gama Pinto, No 2, 1699 Lisboa Codex, Portugal

Received 22 July 1991

Abstract
In the first part of this paper we give a generalization of a result of Ringel [11] on simple arrangements of pseudolines. In terms of fillings with rhombi of an N-zonogon, we obtain a way of generating every filling from a given one by successively performing the same local transformation.
In the second part we interpret, via oriented matroids, fillings of N-zonogons with rhombi as families of vectors in \( \mathbb{Z}^n \).
While for results in Section 1 the finiteness of the filling is essential, the aim of Section 2 is to strengthen the possibility already pointed out by Dress [7] of obtaining de Bruijn's results [4] on Penrose tilings from a purely combinatorial point of view.

1. Recalling some results

A filling with rhombi of a convex polygon \( P \) is a family \( \mathcal{F} \) of rhombi such that
(i) if \( R, R' \in \mathcal{F} \) then either \( R \cap R' = \emptyset \) or \( R \cap R' \) is a face (vertex or edge) of both \( R \) and \( R' \),
(ii) \( \bigcup_{R \in \mathcal{F}} R = P \),
(iii) the length of the edge of any rhombus \( F \in \mathcal{F} \) is 1.
Given a filling \( \mathcal{F} \) of a convex polygon \( P \), for every rhombus \( R \in \mathcal{F} \), with one edge contained in an edge \( e \) of \( P \), the zone of \( R, Z_R \), is the 'strip' of rhombi constructed from \( R \) in the following way.
Put \( R_0 = R, e_0 = e \cap R \) and \( e_1 \) the edge of \( R \) parallel to \( e_0, Z_1 = R_1 \). For \( i > 1 \) let \( R_i \) be the rhombus of \( \mathcal{F} \) which verifies \( K_i \cap K_{i-1} = e_{i-1} \), define \( e_i \) as the edge of \( K_i \) opposite to \( e_{i-1} \) and put \( Z_i = Z_{i-1} \cup R_i \).
Since \( P \) is a convex polygon, this procedure terminates at some stage \( n, R_n \) being a rhombus with one edge contained in an edge \( e' \) of \( P \) parallel to \( P \).
From the concept of zone (see [8]) we easily obtain a characterization of what convex polygons can be filled with rhombi: a convex polygon \( P \) can be filled with
rhombi if and only if every edge \( e \) of \( P \) has an opposite edge \( e \) parallel and equal in length. The length of any edge of \( P \) must be an integer.

A polygon \( P \) satisfying these conditions, with \( 2N \) edges, is called an \( N \)-zonogon and is described by an ordered list \( P = (v_1, \ldots, v_N; l_1, \ldots, l_N) \), where \( v_i \in \mathbb{R}^2 \), \( \| v_i \| = 1 \), are unit vectors, with the direction of the edges ordered in such a way that the angle between \( v_1 \) and \( v_i \) increases with \( i \), and \( l_i \in \mathbb{N} \) represents the length of the pair of edges of \( P \) with direction \( v_i \).

The cyclic filling of an \( N \)-zonogon \( P \) is obtained in the following way.
Fix a point \( 0 \in \mathbb{R}^2 \). The following formula gives a general expression for the set of vertices of a unique (see Section 2) filling of \( P \) with parallelograms:

\[
V_{i,j} = 0 + \sum_{k=j}^{j+i-1} l_k v_k, \quad i = 1, \ldots, N \text{ and } j = 1, \ldots, N-i+1.
\]

Filling each parallelogram with unit rhombi, we obtain the cyclic filling of \( P \).

The following properties of zones lead to a natural correspondence between fillings of an \( N \)-zonogon with rhombi and arrangements of pseudolines: (z1) every rhombus of \( \mathcal{F} \) is contained in one zone; (z2) two different zones \( Z, Z' \) connecting the same pair of opposite edges verify \( Z \cap Z' = \partial Z \cap \partial Z' \); (z3) two zones \( Z \) and \( Z' \) connecting different pairs of opposite edges of \( P \) have exactly one rhombus in common.

We sketch in Fig. 1 the procedure of associating with every filling \( \mathcal{F} \) of an \( N \)-zonogon \( P \) an arrangement \( \mathcal{A}(\mathcal{F}) \) of pseudolines in the real projective plane \( \mathbb{P}\mathbb{R}(2) \).

To the frontier of \( P \) corresponds \( l_\infty \), and to every zone a pseudoline. Pseudolines corresponding to zones connecting the same pair of opposite edges meet in the same point of \( l_\infty \).

Let \( P = (v_1, \ldots, v_N; l_1, l_2, \ldots, l_N) \) be an \( N \)-zonogon and \( \mathcal{F} \) a filling of \( P \). The corresponding arrangement of pseudolines \( \mathcal{A}(\mathcal{F}) \) verifies the two conditions that follow:

(A1) \( \mathcal{A} \) has \( N \) points in the line at infinity, \( P_1, \ldots, P_N \). Each \( P_i \) being the meeting with \( l_\infty \) of the \( l_i \) pseudolines corresponding to zones connecting edges with direction \( v_i \).
(A2) Except for the $N$ points in the line at infinity, every point of $\mathcal{A}(\mathcal{F})$ is the meeting of exactly two pseudolines.

We shall call an arrangement of pseudolines satisfying (A1) and (A2) an *almost simple* arrangement of pseudolines with $N$ points at infinity. We denote such an arrangement by $\mathcal{A}(L_1 \cup \cdots \cup L_N \cup I_\infty)$, $L_i$ being the set of pseudolines meeting $l_\infty$ in the same point $P_i$. We consider, without loss of generality, that the order $P_1 < \cdots < P_N$ is, up to a cyclic permutation, one of the orders by which you meet consecutively the points when travelling along $l_\infty$.

2. Main results

Next theorem is a generalization and strengthening to almost simple arrangements of pseudolines of a result of Ringel [11] for simple arrangement of pseudolines.

**Theorem 2.1.** Let $\mathcal{A} = \mathcal{A}(L_1 \cup \cdots \cup L_N \cup I_\infty)$ and $\mathcal{A}' = \mathcal{A}'(L_1 \cup \cdots \cup L_N \cup I_\infty)$ be two almost simple arrangements of pseudolines with the same $N$ points at infinity, each one with the same multiplicity in $\mathcal{A}$ and $\mathcal{A}'$.

We can obtain from $\mathcal{A}$ an arrangement $\mathcal{A}_1$ isomorphic\(^1\) to $\mathcal{A}'$ by a sequence of switchings of one triangle. Switching a triangle is the operation that consists in changing slightly the relative position of only three pseudolines from one of the positions in Fig. 2 to the other. Moreover, none of the triangles switched intersects $l_\infty$.

**Proof** (Sketch). The proof is constructive. Proceed as follows.

Draw a pseudoline $l_0$, meeting $l_\infty$ in a point $P_0$ between $P_1$ and $P_N$ and such that one of the half-spaces determined by $l_0$ and $l_\infty$ does not contain any point of $\mathcal{A}(L_1 \cup \cdots \cup L_N)$ or $\mathcal{A}'(L_1 \cup \cdots \cup L_N)$.

Let $l_{i_1} < \cdots < l_{i_{k_1}} < \cdots < l_{i_N} < \cdots < l_{Nk_1}$, $(l_{i_1} < \cdots < l_{i_{k_1}} < \cdots < l_{N_1} < \cdots < l_{Nk_1})$ be the total order of pseudolines of $\mathcal{A}$ ($\mathcal{A}'$), induced by the order by which you meet them when travelling along $l_0$ starting at $P_0$.

![Fig. 2.](image)

\(^1\) Two arrangements $\mathcal{A}$ and $\mathcal{A}'$ of $n$ pseudolines in $\mathcal{P}(2)$ are isomorphic if they determine isomorphic cell complexes on $\mathcal{P}(2)$, or, using the terminology of oriented matroids, if they determine the same class of orientations of the same matroid of rank 3.
Choose three pseudolines in $\mathcal{A}$, $l_i, l_j, l_k$, such that the corresponding pseudolines in $\mathcal{A}'$, $l'_i, l'_j, l'_k$, determine a triangle with the same orientation as the triangle defined by $l_i, l_j, l_k$ (i.e. none is a switching of the other).

Choose $l'_1 \in (L_1 \cup \cdots \cup L_N) \setminus \{l_i, l_j, l_k\}$ and let $l'_1$ be the corresponding pseudoline in $\mathcal{A}'$. Draw a copy $l'_1$ of $l_1$ in $\mathcal{A}'$ such that $\mathcal{A}(l'_1, l'_j, l'_k, l'_1, l'_\infty) \cong \mathcal{A}(l_i, l_j, l_k, l_1, l_\infty)$, and $l'_1$ does not pass through any point of $\mathcal{A}$ except $l_1 \cap l_\infty$. Consider the ‘lens’ $l'_1l'_1$ (half-space determined by $l'_1$ and $l'_1$, which does not contain $l_\infty$; see [8]). Either there is no point of $\mathcal{A}'$ inside the lens or there is a triangle contained in the lens with one edge in $l'_1$ (see [8]). Switch this triangle. Continue switching triangles ‘inside’ the lens till you obtain an arrangement $\mathcal{A}'$ where the lens $l'_1l'_1$ does not contain any point. The corresponding switchings, when performed in $\mathcal{A}$, transform $\mathcal{A}$ into an arrangement $\mathcal{A}'$ whose restriction to $\{l_i, l_j, l_k, l_1, l_\infty\}$ is isomorphic to the restriction of $\mathcal{A}'$ to $\{l'_1, l'_j, l'_k, l'_1, l'_\infty\}$.

This procedure is continued by adding a copy $l''_2$ of a line $l_2$ of $(L_1 \cup \cdots \cup L_N) \setminus \{l_i, l_j, l_k, l_1, l_\infty\}$ to $\mathcal{A}'$ and performing the sequence of switchings needed to obtain an arrangement where the lens $l'_2l''_2$ has no points of $\mathcal{A}'$. The corresponding switchings when performed in $\mathcal{A}'$ transform $\mathcal{A}'$ into an arrangement $\mathcal{A}_2$ whose restriction to $\{l_i, l_j, l_k, l_1, l_2, l_\infty\}$ is isomorphic to the restriction of $\mathcal{A}'$ to $\{l'_1, l'_j, l'_k, l'_1, l'_2, l'_\infty\}$.

Continue this procedure by adding at each time a copy of a pseudoline of $\mathcal{A}$ not yet considered. Since $\mathcal{A}$ (and $\mathcal{A}'$) have a finite number of pseudolines, this procedure ends at some arrangement $\mathcal{A}_k'$, isomorphic to $\mathcal{A}'$.

This theorem has two consequences in terms of fillings of an $N$-zonogon: first, it gives a way of obtaining every filling from a given one; second, it shows that every almost simple arrangement of pseudolines corresponds to a filling of some $N$-zonogon.

Corollary 2.2. Let $\mathcal{F}$ and $\mathcal{F}''$ be two fillings of a convex $N$-zonogon $P$. Then $\mathcal{F}''$ can be obtained from $\mathcal{F}$ by a sequence of inversions of the ‘projection of one cube’ (The next section explains why we call this operation inversion of the projection of a cube.) which consists in changing the relative positions of three rhombi filling a hexagon from one of the positions in Fig. 3 to the other.

Corollary 2.2 is an immediate consequence of Theorem 2.1 once we note that, if $\mathcal{F}$ is a filling and $\mathcal{A}(\mathcal{F})$ the corresponding arrangement of pseudolines, there is a one-to-one correspondence between regions in $\mathcal{A}(\mathcal{F})$ and vertices of $\mathcal{F}$, points of

![Fig. 3.](image-url)
\(A(\mathcal{F})\) that are not in 1, and rhombus in 1. These one-to-one correspondences maintain adjacency; therefore, triangles in \(A(\mathcal{F})\) are in one-to-one correspondence with hexagons filled with three rhombi in 1 and switchings of one triangle in \(A(\mathcal{F})\) corresponds in 1 to inverting the position of the three rhombi inside the hexagon.

**Remark.** As an application of this corollary, we can answer and generalize to every filling the following question about the 'jeu des calissons' (see [6]), which consists in filling a regular hexagon of edge length \(n\) with rhombi with angles \(\pi/3\) and \(2\pi/3\). Up to a translation, we use only calissons in three positions. How many in each position? From Corollary 2.2 since any filling can be obtained from the cyclic filling by successively reversing positions of three rhombi in a hexagon, we can say that we use as much rhombi in each position as in the cyclic filling in Fig. 4. Clearly, \(n^2\) in each position.

**Corollary 2.3.** Let \(A = A(L_1 \cup \cdots \cup L_N \cup l_\infty)\) be an almost simple arrangement of pseudolines. Then there is a filling \(\mathcal{F}\) of an \(N\)-zonogon \(P(v_1, \ldots, v_N; |L_1|, \ldots, |L_N|)\) such that \(A = A(\mathcal{F})\).

Let \(P\) be such a zonogon, \(\mathcal{F}_1\) a filling of \(P\) and \(A(\mathcal{F}_1)\) the corresponding arrangement of pseudolines.

\(A\) can be obtained from \(A(\mathcal{F}_1)\) by a sequence of switchings of triangles; therefore, \(\mathcal{F}\) is the filling obtained from \(\mathcal{F}_1\) by performing the corresponding inversions of hexagons.

**Remark A.** Ringel's result [11] on simple arrangements of pseudolines has been conjectured by Cordovil and Las Vergnas [3] to be valid for simple arrangements of pseudohyperplanes in \(\mathbb{P}(d)\). However, this conjecture meets with a conjecture of Las Vergnas [10] which is now open for about 15 years and which says that every arrangement of pseudohyperplanes determines a region which is a simplex (see also [2]).

3. Fillings, vectors and grids

The combinatorial structure of an arrangement of pseudolines in \(\mathbb{P}(2)\) is described by an oriented matroid of rank 3. From this point of view, fillings of zonogons with rhombi are particular cases of rank-3 affine matroids and can be coded by any one of the lists of signed sets defining the corresponding oriented matroid.
In this section we want to present a geometrical interpretation of maximal vectors of the oriented matroid associated with a filling of a zonogon.

In contrast to the previous section, whose results do not make sense for fillings of the plane, the aim of this section is to strengthen the possibility already pointed out by Dress [7] of obtaining de Bruijn’s [4] results on Penrose tilings from a purely combinatorial point of view.

3.1. Families of vectors corresponding to fillings

Let $\mathcal{A} = \mathcal{A}(L_1 \cup \cdots \cup L_N \cup \ell_\infty)$ be an almost simple arrangement of pseudolines in $\mathbb{R}^2$, and $\mathcal{F}$ a filling of an $N$-zonogon such that $\mathcal{A} = \mathcal{A}(\mathcal{F})$.

Choose a maximal cell $R$ of $\Delta(\mathcal{A})$. For every pseudoline $l$ distinguish the two half-spaces determined by $l$ and $\ell_{\infty}$ with a + or − sign in such a way that $R \in l^\circ$.

Let $\mathcal{L} \subseteq \{+, −\}^{L_1 \cup \cdots \cup L_N}$ be the family of signed vectors corresponding to maximal cells of $\Delta(\mathcal{A})$, i.e. $X \in \mathcal{L}$ if and only if there is a maximal cell $R'$ of $\Delta(\mathcal{A})$ such that $R' \in l^+$ if $X(l) = +$ and $R' \in l^−$ if $X(l) = −$.

$\mathcal{L}$ is in one-to-one correspondence with the vertices of $\mathcal{F}$. Next we associate with $\mathcal{L}$ a family $\mathcal{L}'$ of integral vectors $\mathcal{L}' \subseteq \mathbb{Z}^N$, which determines the filling as well.

Note that the choice of $R$ determines a partition of each $L_i$, $i = 1, \ldots, N$, in two totally ordered subsets $L_i^+$ and $L_i^−$, two pseudolines of $L_i$, $l_{ij}$ and $l_{ik}$ being in the same $L_i^\varepsilon$, $\varepsilon = +, −$, if $l_{ij} < l_{ik}$, or $l_{ik} < l_{ij}$, in which case $l_{ik} < l_{ij}$.

Given a signed vector $X \in \mathcal{L}$, for $i = 1, \ldots, N$, put $X_i^+ = \{l \in L_i : X(l) = +\}$. It is not difficult to prove that, for every $X \in \mathcal{L}$, either $X_i^+ \subseteq L_i^+$, or $X_i^− \subseteq L_i^−$; moreover, if $l \in X_i^+$ then every $l' \in L_i$ such that $l' < l$ verifies $l' \in X_i^+$.

With these remarks, defining, for every $i = 1, \ldots, N$, $n_i^− = |L_i^−|$ and $n_i^+ = |L_i^+|$ and $I_i^\pm$ as the interval $[−n_i^−, n_i^+]$ of $\mathbb{Z}$, we can identify every signed vector $X \in \mathcal{L}$ with the integer vector $X' \in I_1^\pm \times \cdots \times I_N^\pm$, $X' = (x_1, \ldots, x_N)$, where $x_i = |X_i^+|$ if $X_i^+ \subseteq L_i^+$, $x_i = −|X_i^−|$ if $X_i^− \subseteq L_i^−$.

From $\mathcal{L}'$ we can easily construct a filling $\mathcal{F}'$ of an $N$-zonogon $P(v_1, \ldots, v_N; |L_1|, \ldots, |L_N|)$ such that $\mathcal{A}(\mathcal{F}') = \mathcal{A}(L_1 \cup \cdots \cup L_N \cup \ell_\infty)$. For any choice of unit vectors $v_1, \ldots, v_N$, with the angle between $v_1$ and $v_i$ increasing with $i$, $\mathcal{F}'$ is the filling whose vertices are given by $V_X = \sum_{i=1}^N x_i v_i$, for every $X' \in \mathcal{L}'$.

Remark B. It is possible from general axiomatics for the set of maximal vectors of an oriented matroid (see [1, 5, 9]) to characterize the families $\mathcal{L}$ of signed vectors corresponding to vertices of a filling of an $N$-zonogon and, thus, obtain a characterization of $\mathcal{L}'$. However, we do not have a practical and direct characterization of $\mathcal{L}'$ (not even of $\mathcal{L}$; see [5]).

3.2. Fillings and grids

Suppose $\mathcal{A}(L_1 \cup \cdots \cup L_N \cup \ell_\infty)$ is a stretchable (see [8]) almost simple arrangement of pseudolines, i.e. $\mathcal{A}$ can be realized by lines in $\mathbb{R}^2$, each $L_i$ being a set of parallel lines.
Let $R$ be a region of $\mathbb{R}^2 \setminus \mathcal{A}$.

Choose for origin of coordinates a point $0 \in R$. Let $v_1, \ldots, v_N \in \mathbb{R}^2$ be unit vectors orthogonal to directions of lines in $L_1, \ldots, L_N$, respectively, ordered in such a way that the angle between $v_i$ and $v_{i+1}$ increases with $i$.

For every $i=1, \ldots, N$ and $j=1, \ldots, |L_i|$, there are constants $c_{ij} \in \mathbb{R}$ such that $l_{ij} = \{ x \in \mathbb{R}^2 : \langle x, v_i \rangle + c_{ij} = 0 \}$. Order each $L_i$ by increasing order of $c_{ij}$'s. Each region $R'$ of $\mathbb{R}^2 \setminus \mathcal{A}$ is then the set of points $(x, y) \in \mathbb{R}^2$ satisfying a system of strict inequalities of the following form:

$$\begin{bmatrix} c_{11} & \vdots & c_{1|L_1|} \\ \vdots & \ddots & \vdots \\ c_{2|L_2|} & \vdots & c_{21} \\ \vdots & \vdots & \vdots \\ c_{N1} & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ c_{N|L_N|} & \vdots & c_{N1} \end{bmatrix} + \begin{bmatrix} a_1 b_1 \\ \vdots \\ a_{|L_i|} b_1 \\ \vdots \\ a_2 b_2 \\ \vdots \\ a_{|L_i|} b_2 \\ \vdots \\ a_N b_N \\ \vdots \\ a_N b_N \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \not> 0. \tag{1}$$

where $(a_i, b_i) = v_i$ and $c_{11} < c_{12} < \cdots < c_{N|L_N|}$.

Let $\mathcal{L}$ be the family of signed vectors on $L_1 \cup \cdots \cup L_N$ defined in the preceding paragraph. There is a one-to-one correspondence between $\mathcal{L}$ and those systems of linear inequalities of type (1) that are solvable. A signed vector $X \in \{+, -, \}^{L_1 \cup \cdots \cup L_N}$ is a vector in $\mathcal{L}$ if and only if the system of linear inequalities of type (1) defined, for every $i=1, \ldots, N$, $j=1, \ldots, |L_i|$, by $\langle x, v_i \rangle + c_{ij} > 0$ if $X(l_{ij}) = +$, $\langle x, v_i \rangle + c_{ij} < 0$ if $X(l_{ij}) = -$ is a solvable system of inequalities.

$\mathcal{L}$ can, therefore, be interpreted as the family of orthants of $\mathbb{R}^{L_1 \cup \cdots \cup L_N}$ which are intersected by the affine plane

$$(c_{11}, \ldots, c_{N|L_N|}) + x(a_1, \ldots, a_1, \ldots, a_N, \ldots, a_N) + \beta(b_1, \ldots, b_1, \ldots, b_N, \ldots, b_N)$$

On the other hand, each solvable system of form (1) is clearly equivalent to the existence of $(x, y) \in \mathbb{R}^2$ verifying (2)

$$\begin{bmatrix} a_1 b_1 \\ \vdots \\ a_{|L_i|} b_N \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in I_1 \times \cdots \times I_N, \quad \text{for } i=1, \ldots, N. \tag{2}$$

$I_i$ being an open interval of $\mathbb{R}$ of the form $(c_{ij}, c_{ij+1})$, $j=1, \ldots, |L_i| - 1$, or $(-\infty, c_{i1})$ or yet $(c_{i|L_i|}, +\infty)$.

Let $H_{ij} = \{ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_i = c_{ij} \} \ i=1, \ldots, N, \ j=1, \ldots, |L_i|$. Denote by $G$ the 'irregular $N$-grid' $G = \mathbb{R}^N \setminus \bigcup_{i,j} H_{ij}$, whose regions are precisely of the form $I_1 \times \cdots \times I_N$.

The family of vectors of $\mathcal{L}$ corresponds to those regions of the grid $G$ which are intersected by the plane $\mathcal{A} = x(a_1, \ldots, a_N) + \beta(b_1, \ldots, b_N)$.
The correspondence is given as follows. For every $i = 1, \ldots, N$, let $j_i = \sup\{ j \in \mathbb{Z}: c_{ij} < 0 \}$. Identify every interval $I_i = (c_{ij}, c_{ij+1})$ with $j - j_i$, $I_i = (-\infty, c_{ii})$ with $-j_i$, $I_i = (c_{il}[L_i], +\infty)$ with $|L_i| - j_i$. Each region $I_1 \times \cdots \times I_N$ of $G$ is then represented by a vector $(z_1, \ldots, z_N) \in \mathbb{Z}^N$ and vectors of $\mathcal{L}''$ correspond to the regions intersected by the plane $\mathcal{P}$.

It is always possible to define a piecewise linear transformation $f: \mathbb{R}^N \to \mathbb{R}^N$ (linear in every region of $G$) which transforms $G$ into the regular grid $G$ defined by $\mathbb{R}^N \setminus \bigcup_{ij} H_{ij}$, where $H_{ij} = \{ x \in \mathbb{R}^N: x_i = j \}$, $i = 1, \ldots, N$, $j = -j_i, \ldots, |L_i| - j_i$. Particular attention deserves those strechable arrangements for which $f(\mathcal{P})$ is still a plane. This happens if and only if the arrangement is what we call equidistantstrechable, i.e., for every $i = 1, \ldots, N$ there exist $d_i, k_i \in \mathbb{R}$, $l_{ij} = \{ x \in \mathbb{R}^N: \langle x, v_i \rangle + k_i + j d_i = 0 \}$.

Remark C. We remark that de Bruijn [4] proved that Penrose tilings of the plane are obtained in the same way as the intersection of an affine plane of $\mathbb{R}^5$ with a regular pentagrid.

From a matroidal point of view, the following questions arise naturally:

**Question 1:** Characterize 'strechable' almost simple arrangement of pseudolines. The answer to this question implies an answer to: 'What simple arrangements of pseudolines are strechable?'. This is a known open problem.

**Question 2:** Characterize 'equidistantstrechable' almost simple arrangements of pseudolines.

### References