COVERING SETS FOR LIMITED-MAGNITUDE ERRORS

ZHIXIONG CHEN, IGOR E. SHPARLINSKI, AND ARNE WINTERHOF

Abstract. For a set \( M = \{-\mu, -\mu + 1, \ldots, \lambda\} \setminus \{0\} \) with non-negative integers \( \lambda, \mu < q \) not both 0, a subset \( S \) of the residue class ring \( \mathbb{Z}_q \) modulo an integer \( q \geq 1 \) is called a \((\lambda, \mu; q)\)-covering set if

\[ MS = \{ms \mod q : m \in M, s \in S\} = \mathbb{Z}_q. \]

Small covering sets play an important role in codes correcting limited-magnitude errors. We give an explicit construction of a \((\lambda, \mu; q)\)-covering set \( S \) which is of the size \( q^{1+o(1)} \max\{\lambda, \mu\}^{-1/2} \) for almost all integers \( q \geq 1 \) and of optimal size \( p \max\{\lambda, \mu\}^{-1} \) if \( q = p \) is prime. Furthermore, using a bound on the fourth moment of character sums of Cochrane and Shi we prove the bound

\[ \omega_{\lambda, \mu}(q) \leq q^{1+o(1)} \max\{\lambda, \mu\}^{-1/2}, \]

for any integer \( q \geq 1 \), however the proof of this bound is not constructive.

1. Introduction

Codes correcting limited magnitude errors have been introduced in flash memory devices, which are widely used nowadays, see for example the recent survey [8].

Let \( q \) be a positive integer and \( M = \{-\mu, -\mu + 1, \ldots, \lambda\} \setminus \{0\} \) for non-negative integers \( \lambda, \mu < q \) with \( \lambda + \mu > 0 \). Following Kløve and Schwartz [14], for a set \( S \subseteq \mathbb{Z}_q \) in the residue class ring \( \mathbb{Z}_q \) modulo an integer \( q \), we consider the product set

\[ MS = \{ms \mod q : m \in M, s \in S\} \]

and define

\[ \nu_{\lambda, \mu}(q, r) = \max_{S \subseteq \mathbb{Z}_q} \{|\mathcal{M}S| : |S| = r\}, \quad r \geq 1, \]

where \( |\mathcal{Z}| \) denotes the cardinality of a set \( \mathcal{Z} \). We have the trivial bound

\[ \max\{r, \lambda + \mu\} \leq \nu_{\lambda, \mu}(q, r) \leq \min\{(\lambda + \mu)r, q\}. \]

2010 Mathematics Subject Classification. 05B40, 11D79, 94B65.

Key words and phrases. covering sets, limited-magnitude errors, residue class rings, character sums.

1
Very recently, Kløve and Schwartz [14] have introduced the notion of \((\lambda, \mu; q)\)-covering sets. Namely, a subset \(S \subseteq \mathbb{Z}_q\) is called a \((\lambda, \mu; q)\)-covering set of \(\mathbb{Z}_q\) if \(MS = \mathbb{Z}_q\). As stated in [14], the problem of covering for certain parameters has applications such as rewriting schemes, see also [10]. The task is to find \((\lambda, \mu; q)\)-covering sets of size as small as possible. Define

\[ \omega_{\lambda, \mu}(q) = \min\{r \in \mathbb{N} : \nu_{\lambda, \mu}(q, r) = q\}. \]

Clearly, we have the lower bound

\[ \omega_{\lambda, \mu}(q) \geq \left\lceil \frac{q}{\lambda + \mu} \right\rceil, \quad \lambda + \mu < q. \]  

We prove the general upper bound

\[ \omega_{\lambda, \mu}(q) = O\left( \frac{q(\log q)^{r(q)}}{\max\{\lambda, \mu\}^{1/2}} \right), \]

where \(r(q)\) is the number of prime divisors of \(q\). In many cases our constructive method provides stronger bounds. In particular, if \(q = p\) is a prime we get

\[ \omega_{\lambda, \mu}(p) \leq 2 \left[ \frac{p}{\max\{\lambda, \mu\}} \right] - 1, \]

which is consistent with the lower bound (1).

Note that we can always assume that \(\lambda + \mu < q - 1\) as otherwise \(S = \{0, 1\}\) is trivially a \((\lambda, \mu; q)\)-covering set of smallest possible cardinality.

Although \(r(q)\) is typically quite small, for some \(q\) the bound (2) can be trivial. However, using a bound on the fourth moment of character sums of Cochrane and Shi [7] we prove the general bound

\[ \omega_{\lambda, \mu}(q) \leq q^{1 + o(1)} \max\{\lambda, \mu\}^{-1/2}, \]

however, the proof is not constructive.

We also consider some questions which appear in the case of very small values of \(\lambda\) and \(\mu\). For instance, Kløve and Schwartz [14, Corollary 3] have given a description of the integers \(q\) which admit an explicit formula for \(\omega_{2,1}(q)\). This description involves the property of the multiplicative order of 2 modulo all prime divisors of \(q\). We show that classical number theoretic tools allow to obtain an asymptotic formula for the number of such integers \(q \leq Q\) (this question has been investigated numerically in [14]).

Finally, we discuss the approach of [14] to estimating \(\omega_{\lambda, \mu}(p)\) (for a prime \(p\)) via the number of residues of a sequence of consecutive powers of a given primitive root modulo \(p\) in a short interval. We show that several recently obtained results due to Bourgain [4, 5] indicate that this approach has no chance to succeed.
Throughout this work, the implied constants in the symbols ‘$O$’, and ‘$\ll$’ are absolute. We recall that the notations $U = O(V)$ and $U \ll V$ are both equivalent to the assertion that the inequality $|U| \leq cV$ holds for some constant $c > 0$.

As usual, for an integer $q$, we use $\varphi(q)$ to denote the Euler function of $q$ and $\tau(q)$ the number of integer positive divisors of $q$.

Since $\omega$ has already got another meaning in this work, which stems from notation of [14], then, as before, we use $r(q)$ for the number of distinct prime divisors of $q$.

We also use $\mathbb{Z}_q^*$ to denote the set of invertible elements of $\mathbb{Z}_q$.

The letter $p$, with or without subscripts, always denotes a prime number.

2. Construction

Here we give explicit constructions of $(\lambda, \mu; q)$-covering sets

**Theorem 1.** For any integer $q \geq 1$ and non-negative integers $\lambda, \mu$ with $\lambda + \mu \geq 1$ in time $\#S(\log q)^{O(1)}$ one can construct a $(\lambda, \mu; q)$-covering set $S \subseteq \mathbb{Z}_q$ with

$$\#S = O \left( \frac{q(\log q)^{r(q)}}{\max\{\lambda, \mu\}^{1/2}} \right).$$

**Proof.** Note that $S$ is a $(\lambda, \mu; q)$-covering set whenever so is $-S = \{q - s : s \in S\}$. Hence, we may restrict ourselves to the case $\lambda \geq \mu$ and note that $\{1, 2, \ldots, \lambda\} \subseteq M$.

First we consider the case that $q = p^\ell$ with a prime $p$.

If $\lambda < p$, we put $H = \lceil p/\lambda \rceil - 1$ and

$$S = \{s_0 + s_1p : s_0 \in \{\pm j^{-1} \mod p : 1 \leq j \leq H\} \cup \{0\}, 0 \leq s_1 < p^{\ell - 1}\}.$$

Clearly

(3) $\#S = (2H + 1)p^{\ell - 1} < 3p^\ell / \lambda$.

Let $a = a_0 + a_1p \mod p^\ell$ be any integer with $0 \leq a_0 < p$ and $0 \leq a_1 < p^{\ell - 1}$. We have to show that

$$a_0 \equiv ms_0 \mod p$$

for some $m \in \{1, 2, \ldots, \lambda\}$ and $s_0 \in \{\pm j^{-1} \mod p : 1 \leq j \leq H\} \cup \{0\}$. Then taking $s_0 + s_1p \in S$ with

$$s_1 \equiv \left( \frac{a_0 - ms_0}{p} + a_1 \right) m^{-1} \mod p^{\ell - 1},$$
we derive
\[ a \equiv a_0 + a_1 p \equiv m(s_0 + s_1 p) \mod p^\ell \in \mathcal{MS}. \]
If \( a_0 = 0 \), we take \( s_0 = 0 \), \( m = 1 \), and \( s_1 = a_1 \). If \( a_0 \neq 0 \), there are at least two elements \( r_1 a_0, r_2 a_0 \) with \( 0 \leq r_1, r_2 \leq H \) such that
\[ 0 < (r_1 a_0 - r_2 a_0 \mod p) \leq \frac{p}{H+1} \leq \lambda \]
by the pigeon-hole principle. We take \( m = (r_1 - r_2) a_0 \mod p \in \mathcal{M} \) and \( s_0 = (r_1 - r_2)^{-1} \mod p \in \{ \pm j^{-1} \mod p : 1 \leq j \leq H \} \), and get
\[ a_0 \equiv m s_0 \mod p. \]
Hence, \( \mathcal{MS} = \mathbb{Z}_{p^\ell} \).

If \( p^i \leq \lambda < p^{i+1} \) for some \( 1 \leq i < \ell \), we take
\[ S = \{ s_0 + s_1 p^i : s_0 \in \{ p^i : i = 0, \ldots, j-1 \} \cup \{0\}, 0 \leq s_1 < p^{\ell-j} \}. \]
We show that any \( a \equiv a_0 + a_1 p^i \mod p^\ell \) with \( 0 \leq a_0 < p^i \) and \( 0 \leq a_1 < p^{\ell-j} \) can be written as \( a \equiv m s \mod p^\ell \) with \( 1 \leq m \leq \lambda \) and \( s \in S \).
If \( a_0 = 0 \), we take \( m = 1 \) and \( s = a \in S \). If \( \gcd(a_0, p^i) = p^i \) for some \( 0 \leq i < j \), we take \( m = a_0/p^i < p^i \leq \lambda \) and \( s = p^i + s_1 p^i \) with
\[ s_1 \equiv \left( \frac{a_0}{p^i} \right)^{-1} a_1 \mod p^{\ell-j}. \]
Hence, we have \( \mathcal{MS} = \mathbb{Z}_{p^\ell} \) and
\begin{equation}
\#S = (j+1)p^{\ell-j} < (j+1)p^\ell/\lambda^{j/(j+1)} \leq (j+1)p^\ell/\lambda^{1/2}.
\end{equation}

Now we assume that \( q = p_1^{\ell_1} \cdots p_r^{\ell_r} \) is the prime decomposition of \( q \) with different primes \( p_1, \ldots, p_r \) and \( p_1^{\ell_1} > p_2^{\ell_2} > \ldots > p_r^{\ell_r} \). We inductively construct a covering set \( S \subseteq \mathbb{Z}_q \) of size
\[ \#S = O \left( q \lambda^{-1/2} (\log q)^r \right). \]
For \( r = 1 \) this result follows from (3) and (4) and we assume \( r \geq 2 \).
We put \( \bar{q} = p_1^{\ell_1} \cdots p_{r-1}^{\ell_{r-1}} \) and \( p^\ell = p_r^{\ell_r} \). Note that \( \gcd(\bar{q}, p) = 1 \) and \( \bar{q} > q^{1/2} > p^\ell \).
If \( \lambda < \bar{q} \), let \( S_0 \) be a \( (\lambda, 0; \bar{q}) \)-covering set of size
\[ \#S_0 = O \left( \bar{q} \lambda^{-1/2} (\log \bar{q})^{r-1} \right) \]
which exists by induction. Now we put
\[ S_1 = \{ p^i s_0 : i = 0, \ldots, \lfloor \log \bar{q}/\log p \rfloor, s_0 \in S_0 \}. \]
Let $m \in \{1, \ldots, \lambda\}$, $s_0 \in S_0$, be a solution of $ms_0 \equiv a_0 \mod \tilde{q}$ and $p^i$ be the largest power of $p$ which divides $m$, that is, $i \leq \lfloor \log \tilde{q}/\log p \rfloor$. Then $m_1 = m/p^i$ and $s_1 = p^is_0 \in S_1$ is another solution of

$$m_1s_1 \equiv a_0 \mod \tilde{q} \quad \text{with} \quad \gcd(m_1, p) = 1.$$ 

Put

$$s_2 = m_1^{-1}\left(a_1 + \frac{a_0 - m_1s_1}{\tilde{q}}\right) \mod p^\ell$$

and verify that

$$m_1(s_1 + s_2\tilde{q}) \equiv a_0 + a_1\tilde{q} \mod q.$$ 

Consequently, $S = \{0 \leq s < q : s \mod p^\ell \in S_1\}$ is a $(\lambda, 0; q)$-covering set of size

$$\#S = O\left(\tilde{q}\lambda^{1/2}(\log \tilde{q})^{r-1}\cdot p^\ell \log(\tilde{q})\right) = O\left(q\lambda^{-1/2}(\log q)^r\right).$$ 

If $\tilde{q} \leq \lambda < q$, we write $a \equiv a_0 + a_1\tilde{q} \mod q$ with $0 \leq a_0 < \tilde{q}$ and $0 \leq a_1 < p^\ell$. If $a_0 = 0$, we take $m = 1$ and $s = a$. Otherwise let $p^i$ be the largest power of $p$ dividing $a_0$ and take $m = a_0/p^i < \tilde{q} \leq \lambda$ and $s = p^i + s_1\tilde{q}$ with $s_1 \equiv (a_0/p^i)^{-1}a_1 \mod p^\ell$. Hence,

$$S = \{s_0 + s_1\tilde{q} : s_0 \in \{p^i : i = 0, \ldots, \lfloor \log \tilde{q}/\log p \rfloor\} \cup \{0\}, 0 \leq s_1 < p^\ell\}$$

is a $(\lambda, 0; q)$-covering set of size

$$\#S \leq p^\ell(\lfloor \log \tilde{q}/\log p \rfloor + 2) = O\left(q \log \tilde{q}/\tilde{q}^{1/2} \cdot p^\ell/\lambda^{1/2}\right) = O\left(q \log q/\lambda^{1/2}\right)$$

and the result follows since any $(\lambda, 0; q)$-covering set is a $(\lambda, \mu; q)$-covering set.

Clearly, the inductive construction works in polynomial time per every element of $S$, which yields the desired complexity bound. □

Using that

$$\sum_{q \leq Q} 2^{2r(q)} \leq \sum_{q \leq Q} \tau^2(q) = (1 + o(1))Q(\log Q)^3,$$

as $Q \to \infty$, see [17, Chapter 1, Theorem 5.4], we see that for any $\varepsilon > 0$ the inequality

$$r(q) < \varepsilon \frac{\log Q}{\log \log Q}$$

fails for at most

$$Q \exp\left(-\varepsilon \log 4 + o(1)\right) \frac{\log Q}{\log \log Q} \ll Q \exp\left(-\varepsilon \frac{\log Q}{\log \log Q}\right).$$
positive integers $q \leq Q$. Indeed to derive this from (5) we simply discard all
\[ q \leq Q \exp \left( -\varepsilon \frac{\log Q}{\log \log Q} \right) \]
and note that for
\[ Q \exp \left( -\varepsilon \frac{\log Q}{\log \log Q} \right) < q \leq Q \]
we have $\log q = (1 + o(1)) \log Q$. For the remaining values of $q$, satisfying (6), the size of the set $S$ of Theorem 2 is
\[ \#S = O \left( \frac{q^{1+\varepsilon}}{\max\{\lambda, \mu\}^{1/2}} \right) . \]

We now note that if $q = p$ is a prime, then we always have $\lambda < p$ so the bound (3) applies and we obtain the following stronger result:

**Theorem 2.** For any prime $p$ and non-negative integers $\lambda, \mu$ with $\lambda + \mu \geq 1$ in time $\#S(\log p)^{O(1)}$ one can construct a $(\lambda, \mu; p)$-covering set $S \subseteq \mathbb{Z}_p$ with
\[ \#S \leq 2 \left\lceil \frac{p}{\max\{\lambda, \mu\}} \right\rceil - 1 . \]

### 3. Upper Bound

As we have mentioned, Theorem 1 applies to the majority of positive integers $q$, however there is a set of integers $q$ for which it gives only a trivial estimate. We now use a different approach to give a non-constructive bound on $\omega_{\lambda, \mu}(q)$ which applies to any $q$.

We start with recalling the following well-known estimates on the divisor and Euler functions
\[ \tau(q) = q^{o(1)} \quad \text{and} \quad \varphi(q) = q^{1+o(1)} , \]
as $q \to \infty$, see [17, Chapter 1, Theorems 5.1 and 5.2].

We also need the following well-known consequence of the sieve of Eratosthenes.

**Lemma 3.** For any integers $q, U \geq 1$,
\[ \sum_{\substack{u=1 \leq U \\text{gcd}(u, q) = 1}} 1 = \frac{\varphi(q)}{q} U + O(2^{r(q)}) . \]
Proof. Using the Mœbius function \( \mu(d) \) over the divisors of \( q \) to detect the co-primality condition and interchanging the order of summation, we obtain the Legendre formula

\[
\sum_{\substack{u=1 \\ \gcd(u,q)=1}}^{U} 1 = \sum_{d \mid q} \mu(d) \left\lfloor \frac{U}{d} \right\rfloor = U \sum_{d \mid q} \mu(d) d + O \left( \sum_{d \mid q} |\mu(d)| d \right)
\]

from which the result follows immediately. \( \square \)

Let \( \mathcal{X} \) be the set of all multiplicative characters \( \chi \) modulo \( q \) and let \( \mathcal{X}^* \) be the set of non-principal characters \( \chi \neq \chi_0 \). We now recall the bound of Cochrane and Shi [7] on the fourth moment of character sums, which we present in the following slightly less precise form, which follows from [7, Theorem 1] and (7).

Lemma 4. For arbitrary integers \( U \geq 1 \), and \( V \), the bound

\[
\sum_{\chi \in \mathcal{X}} \left| \sum_{u=V+1}^{V+U} \chi(u) \right|^4 \leq q^{1+o(1)} U^2
\]

holds.

We now derive an extension of the result of Garaev and Garcia [9, Theorem 2], which is our main technical tool.

Lemma 5. Let \( \varepsilon > 0 \) be fixed and \( q \) be a sufficiently large positive integer. For any intervals \( \mathcal{I} = [K+1, K+M] \) and \( \mathcal{J} = [L+1, L+N] \) with \( \mathcal{I}, \mathcal{J} \subseteq [1, q-1] \), and \( M, N \geq q^\varepsilon \), for all but at most \( q^{2+o(1)} M^{-1} N^{-1} \) elements \( a \in \mathbb{Z}_q^* \), the congruence

(8) \( a \equiv mn \mod q, \quad m \in \mathcal{I}, \ n \in \mathcal{J}, \)

has a solution.

Proof. Let \( \mathcal{I}^* \) and \( \mathcal{J}^* \) denote the set of integers \( m \in \mathcal{I} \) and \( n \in \mathcal{J} \), respectively with \( \gcd(m,q) = \gcd(n,q) = 1 \). Since \( 2^{r(q)} \leq r(q) \), we conclude from (7) and Lemma 3 that

(9) \( \# \mathcal{I}^* = (1+o(1)) \frac{\varphi(q)}{q} M \) \quad and \quad \( \# \mathcal{J}^* = (1+o(1)) \frac{\varphi(q)}{q} N. \)

Using the orthogonality of characters, we see that the number \( J(a) \) of solutions to (8) can be written as

\[
J(a) = \frac{1}{\varphi(q)} \sum_{m \in \mathcal{I}} \sum_{n \in \mathcal{J}} \sum_{\chi \in \mathcal{X}} \chi(mna^{-1}).
\]
Changing the order of summation and separating the contribution \(\#\mathcal{I}^* \#\mathcal{J}^*/\varphi(q)\) of the principal character, we obtain

\[
J(a) - \frac{\#\mathcal{I}^* \#\mathcal{J}^*}{\varphi(q)} = \frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}^*} \chi(a^{-1}) \sum_{m \in \mathcal{I}} \chi(m) \sum_{n \in \mathcal{J}} \chi(n).
\]

Hence

\[
\sum_{a \in \mathbb{Z}_q^*} \left( J(a) - \frac{\#\mathcal{I}^* \#\mathcal{J}^*}{\varphi(q)} \right)^2 = \frac{1}{\varphi^2(q)} \sum_{a \in \mathbb{Z}_q^*} \sum_{\chi_1, \chi_2 \in \mathcal{X}^*} \chi_1(a^{-1}) \chi_2(a^{-1}) \sum_{m_1 \in \mathcal{I}} \chi_1(m_1) \sum_{m_2 \in \mathcal{I}} \chi_2(m_2) \sum_{n_1 \in \mathcal{J}} \chi_1(n_1) \sum_{n_2 \in \mathcal{J}} \chi_2(n_2).
\]

By the orthogonality of characters again, we see that the inner sum vanishes, unless \(\chi_1 = \chi_2\) (in which case it is equal to \(\varphi(q)\)). Hence

\[
\sum_{a \in \mathbb{Z}_q^*} \left( J(a) - \frac{\#\mathcal{I}^* \#\mathcal{J}^*}{\varphi(q)} \right)^2 = \frac{1}{\varphi^2(q)} \sum_{\chi \in \mathcal{X}^*} \left( \sum_{m \in \mathcal{I}} \chi(m) \right)^2 \left( \sum_{n \in \mathcal{J}} \chi(n) \right)^2.
\]

Thus, from the Cauchy-Schwarz inequality, Lemma 4 and the bound (7) we obtain

\[
\sum_{a \in \mathbb{Z}_q^*} \left( J(a) - \frac{\#\mathcal{I}^* \#\mathcal{J}^*}{\varphi(q)} \right)^2 \leq N M q^{o(1)}.
\]

Hence, using (7) and (9), we see that \(J(a) = 0\) is possible for at most

\[
N M q^{o(1)} = \frac{\varphi(q)^2}{(\#\mathcal{I}^* \#\mathcal{J}^*)^2} = q^{2+o(1)} M^{-1} N^{-1}
\]

values of \(a \in \mathbb{Z}_q^*\).

Now we are able to prove the main result of this section.
Theorem 6. For any integer \( q \) and any positive integers \( \lambda, \mu < q \) with \( \lambda + \mu \geq 1 \) we have
\[
\omega_{\lambda,\mu}(q) \leq \frac{q^{1+o(1)}}{\max\{\lambda, \mu\}^{1/2}}.
\]

Proof. We define \( \omega^*_\lambda,\mu(q) \) in exactly the same way as \( \omega_{\lambda,\mu}(q) \) with respect to \( \mathbb{Z}_q^* \) instead of \( \mathbb{Z}_q \). Collecting the elements \( a \in \mathbb{Z}_q \) with the same value \( d = \gcd(a, q) \), we see that
\[
\omega_{\lambda,\mu}(q) \leq \sum_{d|q} \omega^*_{\lambda,\mu}(q/d).
\]

We now see from (7) that it is enough to show that for an arbitrary parameter \( \varepsilon > 0 \), we have
\[
(10) \quad \omega^*_{\lambda,\mu}(q) \leq \frac{q^{1+\varepsilon}}{\max\{\lambda, \mu\}^{1/2}}
\]
provided that \( \lambda, \mu < q \).

Without loss of generality we restrict ourselves to the case \( \lambda \geq \mu \) and choose
\[
\Delta = \sqrt{\lambda}.
\]

We can also assume that \( \lambda \geq q^{\varepsilon} \) as otherwise the bound is trivial. Hence
\[
(11) \quad \lambda \geq \Delta \geq q^{\varepsilon/2}.
\]

Set
\[
S_0 = \{1, \ldots, \lceil \lambda^{-1}\Delta q^{1+\varepsilon/2} \rceil \}.
\]
Taking into account (11) we infer from Lemma 5 that all but a set \( S_1 \) of
\[
\#S_1 \leq q^{1+o(1)}\Delta^{-1}
\]
residue classes \( a \in \mathbb{Z}_q^* \) can be represented as \( ms \equiv a \mod q \) with \( 1 \leq m \leq \lambda \) and \( s \in S_0 \).

Setting \( S = S_0 \cup S_1 \), after elementary calculations, we derive (10) and conclude the proof. \( \square \)

4. Some Special Cases

Klöve and Schwartz [14] have also studied \( \omega_{\lambda,\mu}(p) \) for primes \( p \) and very small values of \( \lambda + \mu \) and presented several explicit formulas.

First we observe that the expression that appears in the formula for \( \omega_{2,0}(q) \) with odd \( q \) is very similar to the expression that has been investigated in [16]. Thus several results and methods of [16] apply directly to this expression too.
The density of integers in [14, Corollary 3] can be evaluated via the classical Wirsing theorem [18] and a result of Chinen and Murata [6]. More precisely, let $\ell_p$ denote the multiplicative order of 2 modulo an odd prime $p$. Kløve and Schwartz [14, Corollary 3] show that for integers $q \equiv 2 \mod 4$ for which $\ell_p \equiv 0 \mod 4$ for every odd prime divisor $p \mod q$, we have $\omega_{2,1}(q) = (3q + 2)/8$ and in fact there is an explicit construction that achieves this value, see also [13, Corollary 3]. Note that $\ell_p \equiv 0 \mod 4$ implies that $p \equiv 1 \mod 4$, since we always have $\ell_p \mid p - 1$. So in fact we have $q \equiv 2 \mod 8$ and thus $(3q + 2)/8 \in \mathbb{Z}$.

Thus, it is interesting to investigate the number $N(Q)$ of such integers $q \leq Q$. We note that the calculations of Kløve and Schwartz [14, Example 3] show that $N(40002) = 1745$.

We now present an asymptotic formula for $N(Q)$.

We say that a function $f(n)$ defined on positive integers is multiplicative for $f(uv) = f(u)f(v)$ for any relatively prime integer $u, v \geq 1$.

We recall the classical theorem of Wirsing [18].

**Lemma 7.** Assume that a real-valued multiplicative function $f(n)$ satisfies the following conditions:

- $f(n) \geq 0$, $n = 1, 2, \ldots$;
- $f(p^\alpha) \leq ab^\alpha$, $\alpha = 2, 3, \ldots$, for some constants $a, b > 0$ with $b < 2$;
- there exists a constant $\tau > 0$ such that

$$\sum_{p \leq x} f(p) = (\tau + o(1)) \frac{x}{\log x}.$$

Then, for $x \to \infty$ we have

$$\sum_{n \leq x} f(n) = \left( \frac{1}{e^{\gamma\tau} \Gamma(\tau)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \sum_{\alpha = 0}^{\infty} \frac{f(p^\alpha)}{p^\alpha},$$

where $\gamma$ is the Euler constant, and

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$$

is the $\Gamma$-function.

Let $Q_4(x)$ denote the number of odd primes $p \leq x$ with $\ell_p \equiv 0 \mod 4$. Then by [6, Theorem 1.1] we obtain the following result.

**Lemma 8.** We have

$$Q_4(x) = \left( \frac{1}{3} + o(1) \right) \frac{x}{\log x}$$

as $Q \to \infty$. 


Now we establish an analogue of the Mertens formula.

**Lemma 9.** There exists an absolute constant \( \eta \) such that

\[
\prod_{\substack{3 \leq p \leq x \\ \ell_p \equiv 0 \mod 4}} \left( 1 + \frac{1}{p-1} \right) = \eta (\log x)^{1/3} + O \left( (\log x)^{-2/3} \right).
\]

**Proof.** In view of the fact that

\[
\log \left( 1 + \frac{1}{p-1} \right) = \frac{1}{p} + O \left( \frac{1}{p^2} \right)
\]

it is equivalent to prove that there exists an absolute constant \( \kappa \) such that

\[
(12) \quad \sum_{\substack{3 \leq p \leq x \\ \ell_p \equiv 0 \mod 4}} \frac{1}{p} = \frac{1}{3} \log \log x + \kappa + O \left( \frac{1}{\log x} \right).
\]

Let us define the function \( \vartheta_4(x) \) by the identity

\[
\vartheta_4(x) = \sum_{\substack{3 \leq p \leq x \\ \ell_p \equiv 0 \mod 4}} \frac{\log p}{p}.
\]

Observe that by Lemma 8, we have

\[
\vartheta_4(x) = \frac{Q_4(x) \log x}{x} + \int_2^x \frac{\log t - 1}{t^2} Q_4(t) \, dt
\]

\[
= \frac{1}{3} \int_2^x \frac{\log t - 1}{t^2} \pi(t) \, dt + O(1),
\]

where, as usual, \( \pi(t) \) denotes the number of primes \( p \leq t \).

The same arguments also imply that

\[
\sum_{p \leq x} \frac{\log p}{p} = \int_2^x \frac{\log t - 1}{t^2} \pi(t) \, dt + O(1)
\]

and the Mertens theorem, see [17, Chapter 1, Theorem 3.1], yields

\[
\vartheta_4(x) = \sum_{\substack{3 \leq p \leq x \\ \ell_p \equiv 0 \mod 4}} \frac{\log p}{p} = \frac{1}{3} \log x + R(x)
\]
for some function $R(x)$ with $R(x) = O(1)$. We now derive

$$\sum_{3 \leq p \leq x, \ell_p \equiv 0 \mod 4} \frac{1}{p} = \frac{\vartheta_4(x)}{\log x} + \int_2^x \frac{\vartheta_4(t)}{t(t\log t)^2} dt$$

$$= \frac{1}{\log x} \left( \frac{1}{3} \log x + R(x) \right) + \int_2^x \frac{1}{t(t\log t)^2} \left( \frac{1}{3} \log t + R(t) \right) dt$$

$$= \frac{1}{3} \log \log x - \frac{1}{3} \log 2 + \frac{1}{3} + \int_2^\infty \frac{R(t)}{t(t\log t)^2} dt + O \left( \frac{1}{\log x} \right)$$

$$= \frac{1}{3} \log \log x - \frac{1}{3} \log 2 + \frac{1}{3} + \int_2^\infty \frac{R(t)}{t(t\log t)^2} dt + O \left( \frac{1}{\log x} \right)$$

(here the existence of the improper integral follows from $R(t) = O(1)$).

So we now obtain (12) with

$$\kappa = \frac{1 - \log \log 2}{3} + \int_2^\infty \frac{R(t)}{t(t\log t)^2} dt,$$

which concludes the proof.

We are now ready to establish an asymptotic formula for $N(Q)$.

**Theorem 10.** There is an absolute constant $\rho > 0$ such that We have

$$N(Q) = (\rho + o(1)) \frac{Q}{(\log Q)^{2/3}}$$

as $Q \to \infty$.

**Proof.** Let us define the multiplicative function $f(n)$ by its values on prime powers $p^\alpha$, $\alpha = 1, 2, \ldots$,

$$f(p^\alpha) = \begin{cases} 1, & \text{if } p \geq 3 \text{ and } \ell_p \equiv 0 \mod 4; \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$N(Q) = \sum_{n \leq (Q-2)/2} f(n).$$

Applying Lemma 7, where by Lemma 8 we can take $\tau = 1/3$, we derive

$$N(Q) = \left( \frac{1}{e^{\gamma/3} \Gamma(1/3)} + o(1) \right) \frac{Q}{2 \log Q} \prod_{p \leq (Q-2)/2} \sum_{\alpha=0}^\infty \frac{f(p^\alpha)}{p^\alpha}.$$

We note that

$$\sum_{\alpha=0}^\infty \frac{f(p^\alpha)}{p^\alpha} = \begin{cases} 1 + 1/(p - 1), & \text{if } p \geq 3 \text{ and } \ell_p \equiv 0 \mod 4; \\ 1, & \text{otherwise}. \end{cases}$$

Hence, using Lemma 9, we conclude the proof. □
It is certainly interesting to get a closed form expression for the constant $\rho$ in Theorem 10 or at least evaluate it numerically.

5. Remarks

Kløve and Schwartz [14, Theorem 3] showed that if $g$ is a primitive root modulo a prime $p$ and the interval $\mathcal{M} = \{-\mu, -\mu + 1, \ldots, \lambda\}$ contains $\delta$ consecutive powers of $g$, then

$$\omega_{\lambda,\mu}(p) \leq \left\lceil \frac{p-1}{\delta} \right\rceil + 1.$$ 

Unfortunately, one expects that $\delta$ is rather small if, say $\lambda + \mu < p/2$ and thus this approach does not seem to be able to produce strong results. For example, by a result of Bourgain [4, Theorem B], for a primitive root $g$ modulo a prime $p$, the sequence of fractional parts

$$\{g^n/p\}, \quad n = 1, \ldots, \delta,$$

is uniformly distributed modulo 1, provided that $\delta > p^{C/\log \log p}$ for some absolute constant $C$. Thus, for any fixed $\varepsilon > 0$ and a sufficiently large $p$, we have

$$\delta \leq p^{C/\log \log p}$$

for any $\lambda$ and $\mu$ with $\lambda + \mu < (1 - \varepsilon)p$. Using [1, 2, 3] one can obtain similar results for arbitrary composite moduli.

Furthermore, if $\lambda + \mu$ is small (but not very small), namely if

$$p^\varepsilon \leq \max\{\lambda, \mu\} < (\sqrt{1/2} - \varepsilon)p^{1/2},$$

then using a different result of Bourgain [5, Theorem 1] we can get

$$\delta = o((\log p)^{\psi(p)})$$

where $\psi(p)$ is an arbitrary function with $\psi(p) \to \infty$ as $p \to \infty$. Indeed, if say $\lambda > 0$ then by [5, Theorem 1] the set

$$\{mg^n/p\}, \quad m = 1, \ldots, \lambda, \quad n = 1, \ldots, \delta,$$

is uniformly distributed modulo 1. On the other hand if $g^n \in \mathcal{M}$, $n = 1, \ldots, \delta$, then these elements are all at the distance at least $1/2 - (\sqrt{1/2} - \varepsilon)^2 = (\sqrt{2} - \varepsilon)\varepsilon$ from 1/2.

We call a set $\mathcal{S}$ of size $\#\mathcal{S} = N$ with

$$\nu_{\lambda,\mu}(q, N) = \#(\mathcal{M}\mathcal{S}) = \#\mathcal{M}\#\mathcal{S} = (\mu + \lambda)N$$

a $(\lambda, \mu; q)$-packing set of order $N$. In [11, 12, 13], the authors applied packing sets to define codes that correct single limited-magnitude errors. It is certainly interesting to find constructions of such sets and
in particular obtain non-trivial estimates on the introduced quantity in \([14]\)

\[ \vartheta_{\lambda,\mu}(q) = \max\{N : \nu_{\lambda,\mu}(q, N) = (\mu + \lambda)N\} . \]

We note that it is very easy to achieve an asymptotically optimal value of \(\nu_{\lambda,\mu}(q, N)\). Indeed, we may restrict ourselves to the case \(\lambda \geq \mu\). If \(N\lambda < q\), we simply take \(S = \{1, \ldots, N\}\) and using the classical asymptotic formula for the average value of the square of the divisor function, see (5), and the Cauchy-Schwarz inequality, we obtain

\[
(#M#S)^2 \leq #(MS) \sum_{k \leq N\lambda} \tau^2(k) \ll #(MS) #M#S (\log q)^3 .
\]

Hence, \(\nu_{\lambda,\mu}(q, N) \geq c(\lambda + \mu)N(\log q)^{-3}\) for some absolute constant \(c > 0\). In fact the result of Koukoulopoulos [15] yields an even tighter bound.

If \(N\lambda \geq q\), we consider only the subset \(\{1, \ldots, \lfloor q/N\rfloor\}\) of \(M\) and get similarly the bound \(\nu_{\lambda,\mu}(q) \geq cq(\log q)^{-3}\). However, investigating when \(\nu_{\lambda,\mu}(q, N) = (\mu + \lambda)N\) and thus estimating \(\vartheta_{\lambda,\mu}(q)\) seems to be more challenging.

**Acknowledgements**

Parts of this paper were written during a very pleasant visit of the first author to RICAM, Austrian Academy of Sciences in Linz. He wishes to thank for the hospitality.

During the preparation of this work, Z.X.C. was partially supported by the National Natural Science Foundation of China grant 61373140 and the Special Scientific Research Program in Fujian Province Universities of China under grant JK2013044; I.S. was partially supported by Australian Research Council grant DP130100237 and by Singapore National Research Foundation grant CRP2-2007-03.

**References**


[7] T. Cochrane and S. Shi, ‘The congruence $x_1x_2 \equiv x_3x_4 \pmod{m}$ and mean values of character sums’, *J. Number Theory*, **130** (2010), 767–785.


**School of Applied Mathematics, Putian University, Putian, Fujian 351100, P.R. China**

*E-mail address*: ptczx@126.com

**Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia**

*E-mail address*: igor.shparlinski@unsw.edu.au

**Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenberger Strasse 69, A-4040 Linz, Austria**

*E-mail address*: arne.winterhof@oeaw.ac.at