Cellular Binomial Ideals. Primary Decomposition of Binomial Ideals

IGNACIO OJEDA MARTÍNEZ DE CASTILLA†
AND RAMÓN PEIDRA SÁNCHEZ‡

Departamento de Álgebra, Universidad de Sevilla, Spain

Eisenbud and Sturmfels’ theoretical study assures that it is possible to find a primary decomposition of binomial ideals into binomial ideals over an algebraically closed field. In this paper we complete the algorithms in Eisenbud and Sturmfels (1996, Duke Math. J., 84, 1–45) by filling in the steps for which the authors say they have not been very precise.

Introduction

It is known that algorithms exist which compute primary decompositions of polynomial ideals (Gianni et al., 1988; Eisenbud et al., 1992; Becker and Weispfenning, 1993; and more recently Shimoyama and Yokoyama, 1996).

However, in case the ideal is binomial, binomiality of its primary components is not assured, that is, the above algorithms do not necessarily compute a decomposition into binomial components even if such a decomposition exists. The older algorithms immediately leave the category of binomial ideals. For example, Algorithm ZPDF in Gianni et al. (1988) and NORMPOS in Becker and Weispfenning (1993) make changes of coordinates and the algorithms in Eisenbud et al. (1992) use syzygy computations and Jacobian ideals. On the other hand, the binomial ideal \((x^4y^2 - z^6, x^3y^2 - z^5, x^2 - yz)\) has a binomial primary decomposition in \(\mathbb{Q}[x, y, z]\) (see Example B.2), but the algorithm by Shimoyama and Yokoyama (1996) (implemented in Singular Greuel et al., 1998) does not yield a binomial primary decomposition.

Eisenbud and Sturmfels (1996) show that, over an algebraically closed field, binomial primary decompositions of binomial ideals exist. However, these authors do not complete their algorithms in several steps in which it is necessary to know a sufficiently large integer which verifies certain properties, thus giving rise to some theoretical problems.

In this paper we give a solution to these problems and we fill all the gaps in the algorithms in Eisenbud and Sturmfels (1996).

In Appendix A, we present the algorithms for decomposing binomial ideals that emerge from the general theory.

We start with a binomial ideal \(I\) in \(S = k[x_1, \ldots, x_n]\) where \(k\) is an algebraically closed field.

†Partially supported by Universidad de Sevilla. E-mail: iojeda@csica.es
‡Partially supported by Junta de Andalucía. Ayuda a grupos FQM 218. E-mail: piedra@algebra.us.es
field. In the second section we find explicitly a cellular decomposition of $I$ (an ideal is cellular if every variable $x_i$ is either a nonzerodivisor modulo $I$ or is nilpotent modulo $I$).

Once we have a procedure (see Algorithm 2) to write a binomial ideal as an intersection of cellular binomial ideals, we can give an effective and improved version (cf. Theorem 3.2) of Theorem 7.1′ in Eisenbud and Sturmfels (1996) when $\text{char}(k) = 0$.

In positive characteristic, it is also necessary to compute a cellular decomposition, but unfortunately this is not enough. In this case, we have to make a new decomposition. Starting from a cellular binomial ideal $I$ in $S$, we give an algorithm (Algorithm 4) that writes $I$ as a finite intersection of unmixed cellular binomial ideals. The key is in Theorem 4.5 that finds, for nice choices of a binomial $b$ (cf. Algorithm 4, step 7), an integer $e$ such that the quotient ideal $(I : b^{[e]})$ is monomial modulo $I$. In Theorem 5.2 in Eisenbud and Sturmfels (1996) this property was only assured for a sufficiently divisible integer.

This last decomposition allows us (cf. Theorem 4.9) to complete a binomial primary decomposition of a binomial ideal. As before, this involves the choice of an integer which was assumed to be sufficiently large in Eisenbud and Sturmfels (1996, cf. Theorem 7.1′). We explain how to obtain such a integer effectively.

1. Lattice Ideals

In this section we recall briefly some definitions and results in Eisenbud and Sturmfels (1996). They are necessary to understand our constructions and will be used frequently in the following sections.

Throughout this paper $k$ denotes any field, $S := k[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables over $k$ and $x^\alpha$ denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$.

We will start defining a class of binomial ideals called lattice ideals. They are just a generalization of toric ideals (Fulton, 1993; Sturmfels, 1995) and they are exactly semigroup (commutative, cancelative and finitely generated) ideals (Vigneron, 1998). Given a lattice ideal it is very easy to find its associated primes and a primary decomposition into lattice ideals. We will obtain abundant properties from these structures in the general case and this is the main reason why lattice ideals are interesting for us.

**Definition 1.1.** A partial character on $\mathbb{Z}^n$ is a homomorphism $\rho$ from a sublattice $L_\rho$ of $\mathbb{Z}^n$ to the multiplicative group $k \setminus \{0\}$.

**Definition 1.2.** Given a partial character $(\rho, L_\rho)$ on $\mathbb{Z}^n$, we define the ideal in $S$

$$I_\rho(\rho) := (\{x^{\alpha_+} - \rho(\alpha)x^{\alpha_-} \mid \alpha \in L_\rho\})$$

called lattice ideal, where $\alpha_+$ and $\alpha_-$ denote the positive and negative part of $\alpha$, respectively.

The lattice ideals are studied in greater depth in other papers (Eisenbud and Sturmfels, 1996; Hosten and Shapiro, 1998; Ojeda, 1998b). Unfortunately, not all binomial ideals are lattice ideals. The following theorem provides a necessary and sufficient condition for a binomial ideal to be a lattice ideal.

**Theorem 1.3.** A binomial ideal $I \subset S$ is a lattice ideal if and only if $I \neq (1)$ and $I = (I : (x_1 \cdots x_n)^\infty)$, in other words, every variable is a nonzerodivisor modulo $I$. 


There are different methods to compute \((I : (x_1 \cdots x_n)^\infty)\), by elimination theory (Becker and Weispfenning, 1993) or using the methods presented by Hosten and Shapiro (1998) and by Vigneron (1998).

A lattice ideal \(I_+(\rho)\) is uniquely determined by the partial character \((\rho, L_\rho)\). It will be seen that it is enough to study this partial character to obtain abundant information of the lattice ideal \(I_+(\rho)\).

Now, let us describe how one computes the radical, the associated primes and a minimal primary decomposition of a lattice ideal using the method developed by Eisenbud and Sturmfels (1996).

Definition 1.4. If \(L\) is a sublattice of \(\mathbb{Z}^n\), then the saturation of \(L\) is the lattice
\[
\text{Sat}(L) := \{ \alpha \in \mathbb{Z}^n \mid d\alpha \in L \text{ for some } d \in \mathbb{Z} \setminus \{0\} \}.
\]
We say that \(L\) is saturated if \(L = \text{Sat}(L)\).

Note that the group \(\text{Sat}(L)/L\) is finite and \(\text{Sat}(L) \cong \mathbb{Z}^t\), where \(t = \text{rank}(L)\).

Proposition 1.5. The lattice ideal \(I_+(\rho)\) is prime if and only if \(L_\rho\) is saturated.

Proof. Consequence of Theorem 2.1(c) in Eisenbud and Sturmfels (1996). □

Definition 1.6. If \(p\) is a prime number, we define \(\text{Sat}_p(L)\) and \(\text{Sat}'_p(L)\) to be the largest sublattices of \(\text{Sat}(L)\) containing \(L\) such that \(\text{Sat}_p(L)/L\) has order a power of \(p\) and \(\text{Sat}'_p(L)/L\) has order relatively prime to \(p\). If \(p = 0\), we adopt the convention that \(\text{Sat}_p(L) = L\) and \(\text{Sat}'_p(L) = \text{Sat}(L)\).

Theorem 1.7. Assume \(k\) algebraically closed and \(\text{char}(k) = p \geq 0\). Let \((\rho, L_\rho)\) be a partial character on \(\mathbb{Z}^n\). Write \(g\) for the order of \(\text{Sat}'_p(L_\rho)/L_\rho\). There are \(g\) distinct characters \(\rho_1, \ldots, \rho_g\) of \(\text{Sat}'_p(L_\rho)\) extending \(\rho\) and for each \(j\) a unique character \(\rho'_j\) of \(\text{Sat}(L_\rho)\) extending \(\rho_j\). There is a unique partial character \(\rho'\) of \(\text{Sat}_p(L_\rho)\) extending \(\rho\). The radical, associated primes and minimal primary decomposition of \(I_+(\rho) \subseteq S\) are:
\[
\sqrt{I_+(\rho)} = I_+(\rho'),
\]
\[
\text{Ass}(S/I_+(\rho)) = \{ I_+(\rho'_j) \mid j = 1, \ldots, g \}
\]
and
\[
I_+(\rho) = \bigcap_{j=1}^{g} I_+(\rho_j)
\]
where \(I_+(\rho_j)\) is \(I_+(\rho'_j)\)-primary. In particular, if \(p = 0\), then \(I_+(\rho)\) is a radical ideal. The associated primes \(I_+(\rho'_j)\) of \(I_+(\rho)\) are all minimal and have the same codimension \(\text{rank}(L_\rho)\).

Proof. Corollary 2.2 and 2.5 in Eisenbud and Sturmfels (1996). □

In the light of the theorem above, it suffices to know the extensions of \(\rho\) to \(\text{Sat}_p(L_\rho)\),
Sat\((L_{\rho})\) and Sat\(_p\)'\(L_{\rho}\) to find the radical, associated primes and the minimal primary decomposition of a lattice ideal, respectively.

The next results say that every prime binomial ideal is uniquely defined by a partial character and a subset of the variable set.

**Theorem 1.8.** Let \(P\) be a binomial ideal in \(S\). Set \(\{y_1, \ldots, y_s\} := \{x_1, \ldots, x_n\} \cap P\) and let \(\{z_1, \ldots, z_t\} := \{x_1, \ldots, x_n\} \setminus P\). The ideal \(P\) is prime if and only if
\[
P = I_+(\sigma) + (y_1, \ldots, y_s)
\]
for a saturated partial character \((\sigma, L_\sigma)\) on \(Z^t\) corresponding to \(z_1, \ldots, z_t\).

**Proof.** Corollary 2.6 in Eisenbud and Sturmfels (1996).

Let \(I\) be a binomial ideal in \(S\). If \(\text{char}(k) = p > 0\) and \(q = p^h\) is a power of \(p\), then we write \(I[q]\) for the ideal generated by the \(q\)th powers of elements of \(I\).

**Proposition 1.9.** Assume \(k\) algebraically closed and \(\text{char}(k) = p > 0\). Let \(I = I_+(\rho')\) be a lattice ideal in \(S\). If \(I_+(\rho')\) is the radical of \(I_+(\rho)\) then
\[
(I_+(\rho'))^{[q]} \subseteq I_+(\rho),
\]
where \(q\) is the order of the group \(\text{Sat}_p(L_{\rho})/L_{\rho}\).

**Proof.** If \(x^{\alpha} = \rho'(\alpha)x^\rho \in I_+(\rho')\), then \(\alpha \in \text{Sat}_p(L_{\rho})\), that is, there exists a power \(q'\) of \(p\) such that \(q'\alpha \in L_{\rho}\). Since \(q'\) divides \(q\) we have \(q\alpha \in L_{\rho}\) and, by extension of partial characters, we have \(\rho(q\alpha) = \rho(q\alpha)\). Putting this together, one can deduce that
\[
x^{q\alpha} = \rho(q\alpha)x^{q\alpha} = x^{q\alpha} - (\rho(q\alpha))x^{q\alpha} \in I_+(\rho).
\]
Therefore \((I_+(\rho'))^{[q]} \subseteq I_+(\rho)\).

**2. Cellular Binomial Ideals and Cellular Decomposition**

**2.1. Cellular Binomial Ideals**

In this section, we are going to study another class of binomial ideals which generalize the concept of lattice and primary ideals.

**Definition 2.1.** We define an ideal \(I\) of \(S\) to be cellular if \(I \neq (1)\) and, for some \(\delta \subseteq \{1, \ldots, n\}\), we have that

1. \(I = (I : (\prod_{i \in \delta} x_i)^\infty)\).
2. For every \(i \notin \delta\) there exists an integer \(d_i \in \mathbb{Z}_+\) such that the ideal \((\{x_i^{d_i}\}_{i \in \delta})\) is contained in \(I\).

In other words, an ideal \(I\) is cellular if every variable is either a nonzerodivisor modulo \(I\) or is nilpotent modulo \(I\). Note that every primary ideal is cellular.
Given any binomial ideal $I \subset S$, we can manufacture cellular binomial ideals from $I$ as follows. For each vector of positive integers $d = (d_1, \ldots, d_n)$ and each subset $\delta$ of $\{1, \ldots, n\}$, we set

$$I_\delta^{(d)} := \left( I + \left( \{x_i^{d_i} \}_{i \notin \delta} \right) : \left( \prod_{i \in \delta} x_i \right)^\infty \right).$$

From Definition 2.1 it can be deduced that the binomial ideal $I$ is cellular if and only if $I = I_\delta^{(d)}$ for some $\delta \subseteq \{1, \ldots, n\}$ and $d \in \mathbb{Z}^n_+$. Let $I$ be a cellular binomial ideal in $S$. We write $\delta \subseteq \{1, \ldots, n\}$ for the set of indices $i$ such that $x_i$ is a nonzerodivisor modulo $I$ and $k[\delta]$ for the polynomial subring of $S$ in the variables $\{x_i\}_{i \in \delta}$. We write $M(I) := \{x_i\}_{i \in \delta}$ for the ideal generated by the variables which are zerodivisors modulo $I$. If $d = (d_1, \ldots, d_n)$ is a vector of positive integers, then we write $M^d(I)$ for the ideal $\{x_i^{d_i}\}_{i \in \delta}$.

With the notation above, if $I$ is a cellular binomial ideal with respect to $d \in \mathbb{Z}^n_+$ and $\delta \subseteq \{1, \ldots, n\}$, then

$$I = I_\delta^{(d)} := \left( I + \left( \{x_i^{d_i} \}_{i \notin \delta} \right) : \left( \prod_{i \in \delta} x_i \right)^\infty \right).$$

We write $\mathbb{Z}^d$ for $\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \mid \alpha_i = 0 \text{ if } i \notin \delta\}$, with $\delta \subseteq \{1, \ldots, n\}$.

When $d = (1, \ldots, 1)$ we will write $I_\delta$ instead of $I_\delta^{(1)}$. We are going to see that $I_\delta$ verifies very interesting properties. In fact, in the following proposition, the cellular binomial ideals $I_\delta$ will allow us to compute the radicals of cellular binomial ideals.

**Proposition 2.2.** Let $I = I_\delta^{(d)}$ be a cellular binomial ideal in $S$. There exists a partial character $(\rho, L_\rho)$ on $\mathbb{Z}^d$ such that

(a) $I \cap k[\delta] = I_\delta(\rho)$.
(b) $I_\delta = I_+(\rho) + M(I)$.
(c) $\sqrt{I_\delta} = \sqrt{I_+(\rho)} + M(I)$.
(d) $\sqrt{I} = \sqrt{I_\delta}$.

**Proof.** (a) Since $I \cap k[\delta] \neq (1)$ is binomial and the variables $x_i$ with $i \in \delta$ are nonzerodivisors modulo $I$, we can assure, by Theorem 1.3, that $I \cap k[\delta]$ is a lattice ideal.

(b) The proof is immediate by part (a).

(c) Since $I_\delta = I_+(\rho) + M(I)$, it follows that $\sqrt{I_\delta} = \sqrt{I_+(\rho)} + M(I)$. According to the property verified by the radical of a sum,

$$\sqrt{I_\delta} = \sqrt{I_+(\rho) + M(I)} = \sqrt{I_+(\rho)} + M(I) \supseteq \sqrt{I_+(\rho)} + M(I).$$

On the other hand, let $f \in \sqrt{I_\delta}$. We can write $f = \sum_{i \notin \delta} g_i x_i + h$ with $h \in k[\delta]$. Since the first summand is in $M(I) \subseteq \sqrt{I_\delta}$, we have $h \in \sqrt{I_\delta}$, then a positive integer $e$ exists such that $h^e \in I_\delta = I_+(\rho) + M(I)$. Since $h \in k[\delta]$, it follows that $h \in \sqrt{I_+(\rho)}$, hence $f = \sum_{i \notin \delta} g_i x_i + h \in \sqrt{I_+(\rho)} + M(I)$.

(d) By a similar argument as above, we obtain $\sqrt{I_+(\rho) + M^d(I)} = \sqrt{I_\delta}$. Since $I_+(\rho) + M^d(I) \subseteq I \subseteq I_\delta$, it follows that $\sqrt{I} = \sqrt{I_\delta}$. \( \square \)

Note that if $P$ is a prime binomial ideal then it is also cellular and the set of variables $\{y_1, \ldots, y_s\}$ in Theorem 1.8 generates the ideal $M(P)$. We are going to prove that the
associated primes of a cellular binomial ideal are cellular with respect to the same \( \delta \subseteq \{1, \ldots, n\} \).

**Proposition 2.3.** If \( I = I_{\delta}^{(d)} \) is a cellular binomial ideal and \( P \) is an associated prime, then \( M(P) = M(I) \).

**Proof.** Since \( I \) is cellular, \( x_i \in M(I) \) implies \( x_i^d_i \in I \subseteq P \), therefore \( x_i \in M(P) \). Conversely, let \( x_i \in M(P) \), since \( P \) is an associated prime of \( I \), we have that \( P = \text{Ann}_{S/I}(f) \) with \( f \in S \setminus I \). Thus \( x_i \cdot f \in I \), that is, \( x_i \) is a zerodivisor modulo \( I \), and necessarily \( x_i \in M(I) \), otherwise \( I \) could not be cellular. \( \Box \)

Obviously, every unmixed ideal has no embedded associated primes. In the next result we will prove that every cellular binomial ideal without embedded primes is an unmixed ideal.

**Proposition 2.4.** If \( I = I_{\delta}^{(d)} \) is a cellular ideal without embedded associated primes, then \( I \) is an unmixed ideal.

**Proof.** By Proposition 2.2(c)–(d), there is a partial character \( (\rho, L_\rho) \) on \( \mathbb{Z}^d \) such that \( \sqrt{I} = \sqrt{I_+(\rho)} + M(I) \). Thus the minimal associated primes of \( I \) have, by Theorem 1.7, codimension equals \( \text{rank}(L_\rho) \), so the minimal primes of \( I \) have the same dimension. Since \( I \) has no embedded primes, we can assure that the associated primes of \( I \) have the same dimension, hence we deduce that \( I \) is an unmixed ideal. \( \Box \)

**Proposition 2.5.** Assume \( k \) algebraically closed. Let \( I = I_{\delta}^{(d)} \) be an unmixed cellular binomial ideal, let \( P \) be a minimal prime of \( I \) and let \( Q \) be the \( P \)-primary component. If \( I \cap k[\delta] = I_+(\rho) \), then \( Q \cap k[\delta] = I_+(\rho_j) \) where \( \rho_j \) is an extension of \( \rho \) to \( \text{Sat}'_P(L_\rho) \).

**Proof.** Since \( P \) is an associated prime of \( I \), we know that there exists an extension \( \rho_j \) of \( \rho \) to \( \text{Sat}(L_\rho) \) such that \( P = I_+(\rho_j) + M(I) \). Since \( I \subseteq Q \subseteq P \), we can assure that

\[
I \cap k[\delta] = I_+(\rho) \subseteq Q \cap k[\delta] \subseteq P \cap k[\delta] = I_+(\rho_j).
\]

\( Q \cap k[\delta] \) is a primary ideal. By Theorem 1.7, \( Q \cap k[\delta] \) is the \( I_+(\rho_j) \)-primary component of \( I_+(\rho) \), then \( Q \cap k[\delta] = I_+(\rho_j) \) where \( \rho_j \) is the extension of \( \rho \) to \( \text{Sat}'_P(L_\rho) \) whose saturation is \( \rho_j \). \( \Box \)

### 2.2. Cellular Decomposition of Binomial Ideals

An algorithmic procedure can be deduced from the proof of Theorem 7.1 in Eisenbud and Sturmfels (1996) which allows us to write a binomial ideal as the intersection of cellular binomial ideals. Before presenting this algorithm it is necessary to recall the following elementary result.

**Proposition 2.6.** Let \( I \) be an ideal in a Noetherian ring \( R \). If \( g \in R \) and \( (I : g) = (I : g^\infty) \), then \( I = (I : g) \cap (I + (g)) \).

**Theorem 2.7.** Let \( I \) be a proper binomial ideal in \( S \). If \( I \) is not cellular then a monomial
\[
m \in S \text{ can be computed effectively such that } I = (I : m) \cap (I + (m)), \text{ with } (I : m) \text{ and } I + (m) \text{ binomial ideals strictly containing } I.\]

**Proof.** If \( I \) is a binomial ideal that is not cellular, then there exists at least one variable \( x_i \) which is zerodivisor modulo \( I \) but it is not nilpotent modulo \( I \). On the other hand, since \( S \) is a Noetherian ring, for a positive integer \( s \) sufficiently large \( (I : (x_i)^s) = (I : ((x_i)^s)_{\infty}) = (I : (x_i)^{\infty}) \). Taking \( m := x_i^s \), we have, by Proposition 2.6, that \( I = (I : m) \cap (I + (m)) \). The ideal \( I + (m) \) is binomial and \( I \) is strictly contained in it, due to \( x_i \) is not nilpotent modulo \( I \). On the other hand, \( (I : m) \) is binomial by Corollary 1.7 in Eisenbud and Sturmfels (1996), and \( I \) is strictly contained in \( (I : m) \) because \( m \) is zerodivisor modulo \( I \). \( \square \)

This last proposition is the key of Algorithm 2: if \( I \) is not a cellular ideal then we can find two new proper ideals strictly containing \( I \). If these ideals are cellular then we are done. Otherwise, we can repeat the same argument with these new ideals, getting strictly increasing chains of binomial ideals. Since \( S \) is a Noetherian ring each one of these chains has to be stationary. So, in the end, we obtain a (redundant) cellular decomposition of \( I \).

The cellular decomposition obtained above by algorithm is different from the one given by Eisenbud and Sturmfels (1996, Formula (6.4)); it does not depend on a (fixed) vector of positive integers (each component depends on a different one) and in many cases this decomposition has less components than Eisenbud and Sturmfels’ one (see Examples B.1 and B.2).

Given a cellular decomposition of a binomial ideal, one can find a vector of positive integers such that Formula (6.4) in Eisenbud and Sturmfels (1996) holds. The following result solves, in some sense, Problem 6.3 in Eisenbud and Sturmfels (1996).

**Theorem 2.8.** Let \( I \) be a proper binomial ideal in \( S \) and let \( I = \cap_{j=1}^t I_{d_j}^{(d)} \) be a cellular decomposition of \( I \). Then
\[
I = \bigcap_{d \subseteq \{1, \ldots, n\}} I_{d}^{(d)}
\]
is a cellular decomposition of \( I \), with \( d = (\max_j d_1^{(d)}, \ldots, \max_j d_n^{(d)}) \).

**Proof.** We know that the ideals \( I_{d}^{(d)} \) are cellular, binomial and \( I \subseteq I_{d}^{(d)} \) for every \( d \subseteq \{1, \ldots, n\} \) and \( d \in \mathbb{Z}^n \), therefore \( I \subseteq \bigcap_{d \subseteq \{1, \ldots, n\}} I_{d}^{(d)} \). So, it is enough to prove the opposite inclusion for \( d = (\max_j d_1^{(d)}, \ldots, \max_j d_n^{(d)}) \). If \( d = d_j \) for some \( j = 1, \ldots, t \), then \( I_{d}^{(d)} \subseteq I_{d_j}^{(d_j)} \), because \( d_i \geq d_i^{(d)} \) for every \( i = 1, \ldots, n \). This implies
\[
\bigcap_{d \subseteq \{1, \ldots, n\}} I_{d}^{(d)} \subseteq \bigcap_{j=1}^t I_{d_j}^{(d_j)} = I. \quad \square
\]

**Remark.** By Theorem 6.1 in Eisenbud and Sturmfels (1996), it is known that, over an algebraically closed field, the associated primes of a binomial ideal are binomial. Unfortunately this result is not true in general, for example, consider the binomial ideal
$I = (x^3 - 1)$ in $\mathbb{Q}[x]$. For this reason, it will be necessary to suppose $k$ algebraically closed in the following sections.

3. Primary Decomposition of Cellular Binomial Ideals in Zero Characteristic

Definition 3.1. We write $\text{Hull}(I)$ for the intersection of the minimal primary components of an ideal $I$.

In the following theorem, an improved version of Theorem 7.1′(b) in Eisenbud and Sturmfels (1996) is given.

Theorem 3.2. Let $I = I_0^{(d)}$ be a cellular binomial ideal in $S$. If $\text{char}(k) = 0$, then

$$I = \bigcap_{P \in \text{Ass}(S/I)} \text{Hull}(I + (P \cap k[\delta]))$$

is a minimal primary decomposition into binomial ideals.

Proof. Let $P$ be an associated prime of $I$ and $Q$ be a $P$-primary component. By Theorem 7.1 in Eisenbud and Sturmfels (1996), we may assume that $Q$ is binomial, therefore $Q \cap k[\delta]$ is a lattice ideal whose radical is $P \cap k[\delta]$. Since the characteristic of $k$ is zero, $Q \cap k[\delta]$ is radical by Theorem 1.7, so

$$P \cap k[\delta] = Q \cap k[\delta] \subseteq Q. \quad (3.1)$$

By Proposition 2.3, $M^d(P) = M^d(I) \subseteq I$, for every $P \in \text{Ass}(S/I)$. Thus

$$\sqrt{I + (P \cap k[\delta])} = \sqrt{I + M^d(P) + (P \cap k[\delta])} = \sqrt{I + P} = P. \quad (3.2)$$

Let $I = \bigcap_{P \in \text{Ass}(S/I)} Q_P$ be a minimal primary decomposition of $I$ and let $P = \sqrt{Q_P}$. From (3.1) and (3.2) it follows, by localization in $P$, that

$$I \subseteq \text{Hull}(I + (P \cap k[\delta])) \subseteq \text{Hull}(Q_P) = Q_P.$$  

The ideals $\text{Hull}(I + (P \cap k[\delta]))$ are primary and binomial, by Corollary 6.5 in Eisenbud and Sturmfels (1996), for every $P \in \text{Ass}(S/I)$. Putting this together, it can be easily deduced that

$$I = \bigcap_{P \in \text{Ass}(S/I)} \text{Hull}(I + (P \cap k[\delta]))$$

is a minimal primary decomposition into binomial ideals. □

In the second section we have shown that every binomial ideal can be written as an intersection of cellular binomial ideals. Therefore, we can use Theorem 3.2 to compute a primary decomposition into binomial ideals of binomial ideals in zero characteristic.
4. Primary Decomposition of Cellular Binomial Ideals in Positive Characteristic

4.1. Quotient of Cellular Binomial Ideals by Binomials

It is known that the quotients of binomial ideals by binomials are generally not binomial. In the fifth section of Eisenbud and Sturmfels (1996) a sufficient condition is given for certain quotients by binomials to be binomial. Before seeing this result it is necessary to introduce the concept of quasi-powers.

In the following, we suppose that $<$ is a fixed monomial order on $S$.

**Definition 4.1.** If $m < n$ are terms so that $b := m - n$ is a binomial, and if $e$ is a positive integer, then we set $b^{[e]} := m^e - n^e$ and call it the $e$th quasipower of $b$.

**Theorem 4.2.** Let $I$ be a binomial ideal in $S$. Suppose $b := x^α - ax^β$ is a binomial and $f \in S$ such that $bf \in I$ but $x^α$ is a nonzerodivisor modulo $I$. Let $f_1 + \cdots + f_s$ be the normal form of $f$ modulo $I$ with respect to $<$, where $f_1, \ldots, f_s$ are terms. If $d$ is a sufficiently divisible positive integer, then we have the following.

(a) The binomials $b^{[d]}f_j$ lie in $I$ for $j = 1, \ldots, s$.
(b) $(I : b^{[d]})$ is generated by monomials modulo $I$, and is thus a binomial ideal.
(c) Let $p = \text{char}(k)$. If $p = 0$, let $q = 1$, while if $p > 0$, let $q$ be the largest power of $p$ that divides $d$. If $e$ is a divisor of $d$ that is divisible by $q$, then $(I : (b^{[d]}/b^{[e]}))$ is a binomial ideal.

**Proof.** Theorem 5.2 in Eisenbud and Sturmfels (1996). \(\square\)

In this subsection, we will show that this integer, $d$, can be computed when $I$ is a cellular binomial ideal.

**Definition 4.3.** Let $I = I_δ^{[d]}$ be a cellular binomial ideal in $S$. If $b \in k[δ]$ is a binomial which is zero divisor modulo $I$, then we define

$$V_b := \{m \in U \mid \exists e, b^{[e]} \in (I : m)\},$$

where $U$ denotes the set of standard monomials modulo $I$ in the variables $\{x_i\}_{i \in S}$.

**Proposition 4.4.** The monomial set $V_b$ defined above is not empty.

**Proof.** Since $b$ is zero divisor modulo $I$, we can assure that there exists an associated prime $P = I_+(σ) + M(I)$ of $I$, such that $b$ lies in $P$. So $b = x^γ(x^α+ − σ(α)x^α−)$, for $α \in L_σ$. By Theorem 8.1 in Eisenbud and Sturmfels (1996), we have that there is a monomial $m \in U$ and a partial character $(τ, L_τ)$ such that $σ$ is a saturation of $τ$ and $(I : m) \cap k[δ] = I_+(τ)$. Since $α \in L_σ$ and $σ$ is a saturation of $τ$, there exists a positive integer $e$ such that $eα \in L_τ$ and $τ(eα) = σ(eα)$, this implies that $b^{[e]} = x^{eγ}(x^{eα+} − σ(eα)x^{eα−}) \in (I : m) \cap k[δ]$. Therefore $m \in V_b$. \(\square\)

For every $m \in V_b$ we consider the least positive integer $e_m$ such that $b^{[e_m]} \in (I : m)$. We define an ideal, $J_b$, and an integer, $e_b$, as follows

$$J_b := \{m \mid m \in V_b\} \quad \text{and} \quad e_b := \text{lcm}\{e_m \mid m \in V_b\}.$$
It could be possible that \( J_b = (1) \), this happens when \( b \) lies in a minimal prime of \( I \), but there is no problem if we consider 1 as monomial in \( S \).

**Theorem 4.5.** Let \( I = I_{\delta}^{(d)} \) be a cellular binomial ideal in \( S \). If \( b \in k[\delta] \) is a binomial which is zerodivisor modulo \( I \), then the following holds:

\[
\begin{align*}
(a) & \ I + J_b \subseteq (I : b^{[s]}). \\
(b) & \ (I : b^{[t]}) \subseteq I + J_b, \text{ for every } t \in \mathbb{Z}_+.
\end{align*}
\]

**Proof.** (a) It suffices to show that \( J_b \subseteq (I : b^{[s]}) \). If \( m \in \mathcal{V}_b \), then there exists an integer \( e_m \) such that \( mb^{[s]} \in I \), that is, \( m \in (I : b^{[s]}) \). Since \( e_m \) divides \( e_b \) we have that \( (I : b^{[s]}) \subseteq (I : b^{[s]}) \), hence \( m \in (I : b^{[s]}) \). Therefore \( J_b \subseteq (I : b^{[s]}) \) and \( I + J_b \subseteq (I : b^{[s]}) \).

(b) Suppose that \( t \) is a fixed positive integer. If \( f \in (I : b^{[t]}) \), then \( fb^{[t]} \in I \). Since \( b^{[t]} \in k[\delta] \), it follows, by Theorem 4.2, that if \( f_1 + \cdots + f_s \) is the normal form of \( f \) modulo \( I \) with respect to \( < \), where \( f_1, \ldots, f_s \) are terms, then for a sufficiently large integer \( e' \), \( f_j b^{[t]} \in I \), for every \( j = 1, \ldots, s \). Since each \( f_j \) is a term in \( S \setminus I \) we can write \( f_j = c_j m_{j1} m_{j2} \) with \( c_j \in k \), \( m_{j1} \in k[\delta] \) and \( m_{j2} \in \mathcal{U} \). If \( m_{j2} b^{[t]} \notin I \), then \( m_{j1} \) is zerodivisor modulo \( I \), in contradiction with the hypothesis \( I \) cellular. Therefore \( m_{j2} b^{[t]} \in I \), that is, \( b^{[t]} \in (I : m_{j2}) \). It follows, by definition of \( \mathcal{V}_b \), that \( m_{j2} \in \mathcal{V}_b \), thus \( f_j \in J_b \), for every \( j = 1, \ldots, s \). Finally, we know that there is \( f' \in I \) such that \( f = f' + \sum_j f_j \), the first summand is in \( I \) and we have just proved that the second one is in \( J_b \), consequently \( f \in I + J_b \). \( \square \)

**Corollary 4.6.** Let \( I = I_{\delta}^{(d)} \) be a cellular binomial ideal in \( S \). If \( b \in k[\delta] \) is a binomial which is zerodivisor modulo \( I \), then

\[
I + J_b = (I : b^{[s]}) = (I : (b^{[s]})^\infty).
\]

**Proof.** The first equality is an immediate consequence of Theorem 4.5. The second one follows from Theorem 4.5 and Proposition 7.3 in Eisenbud and Sturmfels (1996). \( \square \)

**Corollary 4.7.** Let \( p = \text{char}(k) \), let \( I = I_{\delta}^{(d)} \) be a cellular binomial ideal in \( S \) and let \( b \in k[\delta] \) be a binomial which is zerodivisor modulo \( I \). If \( p = 0 \), let \( q = 1 \), while if \( p > 0 \), let \( q \) be the largest power of \( p \) that divides \( e_b \). If \( e \) is a divisor of \( e_b \) that is divisible by \( q \), then \( (I : (b^{[s]} b^{[e]})^q) \) is a binomial ideal.

**Proof.** The proof is an immediate consequence of Theorems 4.2 and 4.5. \( \square \)

### 4.2. The unmixed decomposition

Our aim in this section will be to find an effective method to compute a decomposition of a cellular binomial ideal into unmixed cellular binomial ideals, that is, cellular binomial ideals without embedded primary components (cf. Proposition 2.4).

**Theorem 4.8.** Let \( I = I_{\delta}^{(d)} \) be a cellular binomial ideal in \( S \). If \( I \) has an embedded prime, then a binomial \( b \) and a positive integer \( e \) can be computed effectively such that for \( g = b^e \) the following holds

\[
0.
1. \( I = (I : g) \cap ((I + (g)) : (\prod_{i \in \delta} x_i)^\infty) \).

2. \((I : g)\) and \((I + (g)) : (\prod_{i \in \delta} x_i)^\infty\) are cellular binomial ideals strictly containing \( I \).

**Proof.** Let \( I \cap k[\delta] = I_+ (\rho) \). If \( P \) is an embedded associated prime of \( I \), then

(i) \( P = I_+ (\sigma) + M(I) \), where \( \sigma \) is a saturated partial character on \( \mathbb{Z}^\delta \).

(ii) There exists a minimal prime \( P \) of \( I \) such that \( P = I_+ (\rho_j) + M(I) \subseteq I_+ (\sigma) + M(I) = P \), where \( \rho_j \) is an extension of \( \rho \) to \( \text{Sat}(L_\rho) \).

From (ii) it can be deduced that there is \( \alpha \in L_\sigma \setminus \text{Sat}(L_\rho) \). Taking \( b := x^\alpha - \sigma(\alpha)x^\alpha \), we have that \( b \in k[\delta] \) is zero divisor modulo \( I \), therefore, by the results in Section 4.1, we can compute the integer \( e_b \). Let \( g := b^{e_b} \). Since, by Corollary 4.6, \((I : g) = (I : g^\infty)\), by Proposition 2.6, it follows that \( I = (I : g) \cap (I + (g)) \).

Since \( \alpha \notin \text{Sat}(L_\rho) \) no multiple of \( \alpha \) lies in \( L_\rho \), therefore \( g \notin P \). Besides this, since \( g \in P \), then it is a zero divisor modulo \( I \). From both statements, it follows that \((I : g)\) and \((I + g)\) are proper ideals strictly containing \( I \).

The ideal \((I + g)\) is binomial and, by Corollary 4.6, \((I : g)\) is also binomial. The last one is cellular with respect to \( \delta : M^d(I) \subseteq I \subseteq (I : g) \), and since \( I = (I : (\prod_{i \in \delta} x_i)^\infty) \), it follows that \((I : g) : (\prod_{i \in \delta} x_i)^\infty) = ((I : (\prod_{i \in \delta} x_i)^\infty) : g) = (I : g) \).

Because intersections and quotients of ideals commute, then

\[
(I : (\prod_{i \in \delta} x_i)^\infty) = (I : g) : (\prod_{i \in \delta} x_i)^\infty) \cap (I + g) : (\prod_{i \in \delta} x_i)^\infty),
\]

since \( I \) and \((I : g)\) are cellular with respect to \( \delta \), we have that

\[
I = (I : g) \cap (I + g) : (\prod_{i \in \delta} x_i)^\infty)
\]

Note that the cellular binomial ideal \((I + g) : (\prod_{i \in \delta} x_i)^\infty)\) is proper, otherwise \( I = (I : g) \).

In order to get an unmixed decomposition, we have to use a similar argument as the one applied to find a cellular decomposition. If the cellular binomial ideals \((I : g)\) and \((I + g) : (\prod_{i \in \delta} x_i)^\infty)\) are both unmixed we are done, otherwise we can find cellular binomial ideals strictly containing them. Since \( S \) is Noetherian this procedure has to be finite. So, in the end, we obtain a (not necessarily minimal) set of unmixed cellular binomial ideals whose intersection is equal to \( I \). \( \square \)

4.3. PRIMARY DECOMPOSITION OF UNMIXED CELLULAR BINOMIAL IDEALS IN POSITIVE CHARACTERISTIC

Throughout this section we assume \( \text{char}(k) = p > 0 \).

**Theorem 4.9.** Let \( I = I_{(d)}^S \) be an unmixed cellular binomial ideal in \( S \). If \( I \cap k[\delta] = I_+ (\rho) \) and \( q \) is the order of the group \( \text{Sat}_p(L_\rho) \), then

\[
I = \bigcap_{P \in \text{Ass}(S/I)} \text{Hull}(I + (P \cap k[\delta])^{[q]})
\]

is a minimal primary decomposition into binomial ideals.
Proof. By hypothesis, the associated primes of $I$ are minimal, then the primary components of $I$ are uniquely defined. Let $P$ be an associated prime of $I$ and let $Q$ be the $P$-primary component. By Proposition 2.5 we have that $Q \cap k[\delta] = I_+(\rho_j)$ where $\rho_j$ is an extension of $\rho$ to $\text{Sat}'_P(L_\rho)$. Since $\sqrt{Q \cap k[\delta]} = I_+(\rho'_j) = P \cap k[\delta]$, where $\rho'_j$ is the only extension of $\rho_j$ to $\text{Sat}(L_\rho)$, then we have, by Proposition 1.9,

$$(P \cap k[\delta])^{[\alpha]} \subseteq Q \cap k[\delta] \subseteq Q.$$ 

As in the proof of Formula (3.2), it is easy to see that $\sqrt{I + (P \cap k[\delta])^{[\alpha]}} = P$. And, by the same argument as in Theorem 3.2, it follows that

$$I = \bigcap_{P \in \text{Ass}(S/I)} \text{Hull}(I + (P \cap k[\delta])^{[\alpha]})$$

is a minimal primary decomposition into binomial ideals. □

In the Section 2 we have shown that every binomial ideal can be written as an intersection of cellular binomial ideals and, in Section 4.2, we have seen that every cellular binomial ideal can be written as an intersection of unmixed cellular binomial ideals. Therefore, we can use these constructions and Theorem 4.9 to compute a primary decomposition into binomial ideals of binomial ideals in positive characteristic.

Acknowledgements

We are grateful to the referees for their useful comments. We also thank Pilar Pisón-Casares for helpful conversations, Irena Swanson for her constructive remarks and Pepe Rodríguez-García for polishing up our English.

References


Appendix A. Algorithms

Here we present our algorithms for computing a primary decomposition of a binomial ideal. The following figure summarizes the way to get a primary decomposition of a binomial ideal.

![Flowchart of Cellular Ideals](Image)

**Algorithm 1.** Cellular binomial ideal.

**Input:** A binomial ideal $I$ in $S = k[x_1, \ldots, x_n]$.

**Output:** The decision (“YES”, “NO”), whether $I$ is cellular. In the affirmative case, the algorithm finds the largest subset $\delta$ for which $I$ is cellular. In the negative case, the algorithm finds a variable which is a zerodivisor modulo $I$ but is not nilpotent modulo $I$.

1. If $I = (1)$ then output “NO”.
2. Otherwise, set $\delta = \emptyset$.
3. For $i = 1, \ldots, n$:
   3.1 Compute $(I : (x_i)^\infty)$.
   3.2 If $(I : (x_i)^\infty) = (1)$, then $\delta = \delta \cup \{i\}$.
4. Set $\delta := \{1, \ldots, n\} \setminus \overline{\delta}$.
5. Compute $J := (I : (\prod_{i \in \delta} x_i)^\infty)$.
6. If \( J = I \) then output “YES, \( I \) is cellular with respect to” \( \delta \).
7. Otherwise, for \( i \in \delta : \)
   7.1 Compute \( J := (I : (x_i)) \).
   7.2 If \( J \neq I \) then output “NO, the variable” \( x_i \) “is a zero divisor modulo \( I \) but is not nilpotent modulo \( I \)”.

Comments. The third step of Algorithm 1 follows basically from the next result: \( x_i^{d_i} \in I \) for some \( d_i > 0 \) if and only if \((I : (x_i)^\infty) = (1)\). Then we can select the indices which cannot belong to \( \delta \subseteq \{1, \ldots, n\} \). Once we have defined \( \delta \) we check if the first condition in Definition 2.1 holds.

Algorithm 2. Cellular decomposition.

**Input:** A binomial ideal \( I \neq (1) \) in \( S = k[x_1, \ldots, x_n] \).

**Output:** A cellular decomposition of \( I \).

1. If \( I \) is cellular then output \( I \).
2. Otherwise, choose one variable \( x_i \) which is a zero divisor modulo \( I \) but is not nilpotent modulo \( I \).
3. Compute an integer \( s \) such that \((I : x_i^s) = (I : (x_i^s)^\infty)\).
4. Compute the ideals \((I : x_i^s)\) and \( I + (x_i^s) \).
5. Compute cellular decompositions of the binomial ideals \((I : x_i^s)\) and \( I + (x_i^s) \) by calling recursively Algorithm 2.

Comments. The correctness follows from Theorem 2.7. There are different methods to compute the integer in Step 3. For example, from Corollary 6.36 in Becker and Weispfenning (1993) an algorithm is deduced to compute this integer.

Algorithm 3. Quotient of cellular binomial ideal by binomials.

**Input:** A binomial ideal \( I \) in \( S \) which is cellular with respect to \( \delta \) and a binomial \( b := x^\alpha - ax^\beta \in k[\delta] \) which is zerodivisor modulo \( I \).

**Output:** The positive integer \( e_b \) and the binomial ideal \((I : b^{[e_b]})(I : b^{[e_b]})\).

1. Set \( U' := \emptyset, D := \emptyset \) and \( J = (0) \).
2. Compute a Gröbner basis of \( I \).
3. Let \( U \) be the set of standard monomials modulo \( I \) in the variables \( \{x_i\}_{i \notin \delta} \).
4. For each \( m \in U \) :
   4.1. Compute the partial character \((\tau, L_\tau)\) that satisfies \( I_\tau(\tau) = (I : m) \cap k[\delta] \).
   4.2. If \( \alpha - \beta \in \text{Sat}(L_\tau) \), then
      4.2.1. Redefine \( U' := U' \cup \{m\} \).
      4.2.2. Compute the order, \( e_m \), of \( \alpha - \beta \) in \( \text{Sat}(L_\tau) / L_\tau \).
   4.3. Redefine \( D := D \cup \{e_m\} \).
5. Define \( e' := \text{lcm}\{e_m \mid e_m \in D\} \). Set \( D := \emptyset \).
6. For each \( m \in U' \) :
   6.1 If \( b^{[e']} \in (I : m) \), then
6.1.1 Redefine $J := J + (m)$.

6.1.2 Redefine $D := D \cup \{e_m\}$, with $e_m$ defined as in Step 4.2.2.

7. Define $e_b := \text{lcm}\{e_m \mid e_m \in D\}$.

8. Compute a Gröbner basis $\mathcal{G}$ of $I + J$.

9. Output $\mathcal{G}$ and $e_b$.

Comments. The correctness of this algorithm follows from the results in Section 4.1. Recall that $m \in \mathcal{V}_b$ if there is a positive integer $e$ such that $b^e \in (I : m)$. Since $b^e \in k[\delta]$, it suffices to find $e$ such that $b^e \in (I : m) \cap k[\delta] = I_+$. If $\alpha - \beta \notin \text{Sat}(L_\tau)$ then no multiple of $\alpha - \beta \in L_\tau$, therefore $b^e \notin (I : m)$ for every $e$. In another case, if we define $e_m$ as in Step 4.2.2, then we can assure that $e_m(\alpha - \beta) \in L_\tau$, but it is not always true that $b^{e_m} \in (I : m)$, for example when $a^{e_m} \neq \tau(e_m(\alpha - \beta))$. If we define $e'$, as in Step 5, it suffices to check if $b^{e'} \in (I : m)$, otherwise, there is no integer $e''$ such that $b^{e''} \in (I : m)$ by the second equality in Corollary 4.6.

Algorithm 4. Unmixed decomposition.

**Input:** A binomial ideal $I$ in $S$ which is cellular with respect to $\delta$.

**Output:** A decomposition of $I$ into unmixed cellular binomial ideals.

1. Compute the associated primes of $I$.
2. If they are minimal then output $I$.
3. Otherwise, compute the partial character $(\rho, L_\rho)$ such that $I \cap k[\delta] = I_+(\rho)$.
4. Choose an embedded prime $P$ of $I$.
5. Compute the partial character $(\sigma, L_\sigma)$ such that $P = I_+(\sigma) + M(I)$.
6. Compute a minimal prime $P_j = I_+(\rho_j'') + M(I)$ of $I$ such that $P_j \subset P$.
7. Choose $\alpha \in L_\sigma \setminus \text{Sat}(L_\rho)$ and define $b := x^{\alpha+} - \sigma(x^{\alpha-})$.
8. Compute $e_b$ and $g := b^{e_b}$.
9. Compute the ideals $(I : g)$ and $((I + (g)) : (\prod_{i} x_i)^\infty)$.
10. Compute a decomposition into unmixed cellular binomial ideals of the ideals $(I : g)$ and $((I + (g)) : (\prod_{i} x_i)^\infty)$ by calling recursively Algorithm 4.

Comments. The correctness of Algorithm 4 follows from Theorem 4.8. The associated primes of $I$ can be obtained by using Algorithm 9.5 in Eisenbud and Sturmfels (1996). The integer $e_b$ can be computed by Algorithm 3.

Recall that binomiality of the associated primes of binomial ideals is only assured when $k$ is algebraically closed (see the Remark before Section 3). Therefore, one finds serious problems when carrying out Algorithm 9.5 in Eisenbud and Sturmfels (1996) with the help of a computer algebra system.

Algorithm 5. Primary decomposition.

**Input:** A binomial ideal $I$ in $S$ which is cellular with respect to $\delta$ (and unmixed if $\text{char}(k) > 0$).

**Output:** Primary binomial ideals $Q_i$ whose intersection is irredundant and equals $I$.

1. Compute the partial character $(\rho, L_\rho)$ such that $I \cap k[\delta] = I_+(\rho)$.
2. Compute the associated primes $P_1, \ldots, P_i$ using Algorithm 9.5 in Eisenbud and Sturmfels (1996).
3. For each prime $P_i$:

3.1. If $\text{char}(k) = p > 0$, then

3.1.1 Let $q$ be the order of the group $\text{Sat}_p(L_\rho)/L_\rho$.

3.1.2 Let $R_i := I + (P_i \cap k[\delta])^{[d]}$.

3.2. If $\text{char}(k) = 0$, then let $R_i := I + P_i \cap k[\delta]$.

3.3. Compute $\text{Hull}(R_i)$. Output $Q_i := \text{Hull}(R_i)$.

Comments. The correctness of Algorithm 5 follows from the Theorems 3.2 and 4.9 in zero and positive characteristic, respectively. Note that, if $\sqrt{I} = P$ is prime, then $\text{Hull}(I)$ is the only $P$-primary component of $I$. In this case, Algorithm 9.6 in Eisenbud and Sturmfels (1996) provides a method to compute $\text{Hull}(I)$; however, it is necessary to know a sufficiently large integer again. Corollary 4.6 and Corollary 4.7 give us this integer. Since $\sqrt{R_i} = P_i$ is prime, then the primary ideals $\text{Hull}(R_i)$ can be computed by removing Steps 3.2 and 4.2 in Algorithm 9.6 in Eisenbud and Sturmfels (1996): “Select an integer $d$ which might be sufficiently divisible”, and adding a new step: “2bis. Compute $e_b$. Set $d = e_b$”.

Remark. At this moment, we have only implemented Algorithms 1 and 2 in Macaulay2, Grayson and Stillman (1993). The package is available by E-mail to the authors.

Appendix B. Examples

In the following examples the cellular decomposition has been done using our package. The other computations have been done theoretically using the algorithms in Appendix A and in Eisenbud and Sturmfels (1996).

Example B.1. Consider the binomial ideal $I = (x_1x_4^2 - x_2x_5, x_1^2x_3^3 - x_2^4x_4^2, x_2x_3^8 - x_3^3x_5^5)$ of $S = \mathbb{Q}[x_1, \ldots, x_5]$. Using Algorithm 1 we see that $I$ is not cellular. $I$ is not cellular because $x_1, x_2, x_4$ and $x_5$ are zerodivisors modulo $I$, but none of them is nilpotent modulo $I$.

If we use Algorithm 2 to find a cellular decomposition of $I$, we obtain that

$I_1 = (x_1^2x_4^2 - x_2x_5, x_1^4x_3^3 - x_2^4x_4^2, x_2^3x_4x_5^2 - x_1^3x_3^2x_5^2, x_2^2x_4^2x_5^4 - x_1x_3^3x_5^4, x_2x_3^8 - x_3^3x_5^5)$,

$I_2 = (x_1^2x_4^2 - x_2x_5, x_1^4x_3^3 - x_2^4x_4^2, x_2^3x_4x_5^2 - x_1^3x_3^2x_5^2, x_2^2x_4^2x_5^4 - x_1x_3^3x_5^4, x_2x_3^8 - x_3^3x_5^5)$,

is a cellular decomposition, that is, $I = I_1 \cap I_2$ with $I_1$ and $I_2$ cellular binomial ideals.

This computation has been done using Macaulay2 Grayson and Stillman (1993) running in a PC Pentium333 Mhz, 32 Mb RAM. It has spent 2.11 seconds.

It is easy to see that the ideals $I_1$ and $I_2$ are primary. Thus $I = I_1 \cap I_2$ is a minimal primary decomposition of $I$.

Example B.2. Consider the binomial ideal $I = (x^4y^2 - z^6, x^3y^2 - z^5, x^2 - yz)$ of $S = \mathbb{Q}[x, y, z]$.

Using Algorithm 2 we obtain (in 1.04 seconds) the following cellular decomposition, $I = I_1 \cap I_2 \cap I_3$, where

$I_1 = (y - z, x - z)$
\[ I_2 = (z^2, xz, x^2 - yz) \]
\[ I_3 = (x^2 - yz, xy^3z - z^5, xz^5 - z^6, z^7, y^7). \]

It is easy to see that \( I = I_1 \cap I_2 \cap I_3 \) is a minimal primary decomposition of \( I \).

**Example B.3.** Consider the cellular binomial ideal \( I = (x^6_i x_6 - x^3_i x_3^3 x_6, x^2_i x_4 x_6 - x^1_i x_4 x_6, x^2_i x_5 - x_4, x_4 x_5 x_6 - x_4 x_6, x^2_i x_6 - x_6, x^5_i) \) of \( S = \mathbb{k}[x_1, \ldots, x_6] \), with \( \mathbb{k} \) an algebraically closed field and \( \text{char}(k) = 3 \). \( I = I_8^{(d)} \) with \( \delta = \{1, 2, 3, 5\} \) and \( d = (0, 0, 0, 2, 0, 2) \).

**Associated Primes**

Using Algorithm 9.5 in Eisenbud and Sturmfels (1996), we see that the ideal \( I \) has one minimal and twelve embedded primes,

\[ P_1 := (x_4, x_6), \quad P_2 := (x_4, x_5 - 1, x_6), \quad P_3 := (x_4, x_5 + 1, x_6) \]
\[ P_4 := (x_1^2 - x_2 x_3, x_4, x_5 - 1, x_6), \quad P_5 := (x_1^2 - x_2 x_3, x_4, x_5 + 1, x_6), \quad P_6 := (x_1 - \zeta_i x_3, x_2 + \zeta_i^3 x_3, x_4, x_5 - 1, x_6), \quad i = 6, \ldots, 13, \]

where \( \zeta_i, i = 6, \ldots, 13 \) are the eighth roots of unity.

We have the two following chains of associated primes \( P_1 \subset P_2 \subset P_5 \) and \( P_1 \subset P_2 \subset P_3 \subset P_1, \quad i = 6, \ldots, 13 \).

**Unmixed Decomposition (Algorithm 4)**

Since \( I \) has embedded primes, it is necessary to compute the unmixed decomposition. Using Algorithm 4, we obtain

\[
\begin{array}{c}
\vdots \\
\downarrow \\
I_1 \\
\vdots \\
\downarrow \\
I_2 \\
\vdots \\
\downarrow \\
I_21 \\
\vdots \\
\downarrow \\
I_22 \\
\vdots \\
\downarrow \\
I_221 \\
\vdots \\
\downarrow \\
I_222 \\
\end{array}
\]

\[ I = I_1 \cap I_21 \cap I_221 \cap I_222 \] is an unmixed decomposition of \( I \), where

\[
I_1 := (x_4, x_6),
I_21 := (x^2_4, x_4 x_5^2 - x_4, x^6_6 - 1, x_6),
I_211 := (x^6_1 - x^3_2 x^3_3, x^2_4, x_4 x_5^2 - x_4, x_4 x_6, x^6_6 - 1, x^2_3 x_6 - x_6, x^2_6),
I_222 := (x^6_1 - x^3_2 x^3_3, x^2_4, x^12_4 - x^1_1, x^4_4 x_6 - x^1_4 x_4 x_6, x^2_4, x_4 x_5^2 - x_4, x_4 x_5 x_6 - x_4 x_6, x^2_6 - 1, x^2_3 x_6 - x_6, x^2_6).
\]

**Primary Decomposition (Algorithm 5)**
Now, we compute a primary decomposition of $I_1, I_{21}, I_{221}$ and $I_{222}$.

$I_1$. The ideal $I_1 = (x_4, x_6)$ is prime.

$I_{21}$. Since $I_4(p) := I_{21} \cap k[\delta] = (x_6^3 - 1)$, we have $L_p \cong \mathbb{Z}$ and $\text{Sat}_p(L_p) \cong \mathbb{Z}_2$. So, the order of the group $\text{Sat}_p(L_p)/L_p$ is 3. The (minimal) associated primes of $I_{21}$ are $P_2$ and $P_3$. Thus,

$$R_{21,2} := I_{21} + (P_2 \cap k[\delta])^{[3]} = (x_4^2, x_4x_5 - x_4, x_5^3 - 1, x_6)$$

and

$$R_{21,3} := I_{21} + (P_3 \cap k[\delta])^{[3]} = (x_4^2, x_4x_5 - x_4, x_5^3 + 1, x_6).$$

By Algorithm 9.4 in Eisenbud and Sturmfels (1996), one can see that $R_{21,2}$ and $R_{21,3}$ are primary.

$I_{221}$. Now, $I_4(p) := I_{221} \cap k[\delta] = (x_4^6 - x_2^3x_3, x_5^6 - 1)$, the order of the group $\text{Sat}_p(L_p)/L_p$ is 9 and the (minimal) associated primes of $I_{221}$ are $P_4$ and $P_5$. As before, we obtain two binomial ideals

$$R_{221,4} := I_{221} + (P_4 \cap k[\delta])^{[9]} = (x_4^6 - x_2^3x_3, x_4^2x_5 - x_4, x_4x_6, x_5^3 - 1, -x_6 + x_5x_6, x_6^3)$$

and

$$R_{221,5} := I_{221} + (P_5 \cap k[\delta])^{[9]} = (x_4^6 - x_2^3x_3, x_4^2x_5 + x_4, x_4x_6, x_5^3 + 1, -x_6 + x_5x_6, x_6^3)$$

which are primary.

$I_{222}$. Since $I_4(p) := I_{222} \cap k[\delta] = (x_4^6 - x_2^3x_3, x_5^6 - 1)$, we have $L_p \cong \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$ and $\text{Sat}_p(L_p) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$. So, the order of the group $\text{Sat}_p(L_p)/L_p$ is 27. The (minimal) associated primes of $I_{222}$ are $P_i$, $i = 6, \ldots, 13$. Thus,

$$R_{221,i} := I_{221} + (P_i \cap k[\delta])^{[27]} = (x_4^6 - x_2^3x_3, x_5^6 - 1, x_4^2x_5x_6 - x_4x_6, x_5^3x_6 - x_6, x_6^2, x_6^2, x_5^3x_6^2, x_5^3x_6^2, -x_6 + x_5x_6, x_6^3 - 1),$$

$$i = 6, \ldots, 13.$$ 

By Algorithm 9.4 in Eisenbud and Sturmfels (1996), we have that $R_{222,i}$, $i = 6, \ldots, 13$, are not primary. Therefore, it is necessary use Algorithm 9.6 (improved by Algorithm 3) that computes $\text{Hull}(R_{222,i})$, $i = 6, \ldots, 13$. So, in the end, we obtain the following primary ideals

$$Q_{222,i} := \text{Hull}(R_{222,i}) = (x_4^6 + x_2^3x_3, x_5^6 - 1, x_4^2x_5x_6 - x_4x_6, x_5^3x_6 - x_6, x_6^2, x_6^2, x_5^3x_6^2, x_5^3x_6^2, -x_6 + x_5x_6, x_6^3 - 1),$$

$$i = 6, \ldots, 13.$$ 

Putting this together, we have

$$I = I_1 \cap R_{21,2} \cap R_{21,3} \cap R_{221,4} \cap R_{222,5} \cap \left( \bigcap_{i=6,\ldots,13} Q_{222,i} \right)$$

is a minimal primary decomposition of $I$. 

Originaly Received 22 December 1998
Accepted 27 July 1999