Optimum Combining of Rician-Faded Signals: Analysis in the Presence of Interference and Noise

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Abstract—This paper analyzes the performance of optimum combining systems in the presence of both co-channel interference and thermal noise, addressing the case where the desired-user undergoes Rician fading. Exact expressions are derived for the moment generating function of the SINR which apply for arbitrary numbers of antennas and interferers. Based on these, we obtain expressions for the symbol error probability with $M$-PSK. We also derive exact closed-form expressions for the moments of the SINR, and show that they are directly related to the corresponding moments of a Rayleigh system via a simple scaling parameter. Numerical results are presented to validate the analysis, and to examine the impact of Rician fading.

I. INTRODUCTION

Adaptive antenna arrays with linear diversity combining provide an effective means of increasing the performance of wireless communications systems. Of the various combining strategies which have been proposed, optimum combining (OC) [1] is the most effective solution for multi-user systems which operate in the presence of both co-channel interference and thermal noise. These systems are designed to exploit channel knowledge to maximize the signal-to-interference-plus-noise ratio (SINR) for every channel use.

For OC systems where both the desired-user and interferer channels undergo Rayleigh fading, the performance has been well-studied [2]–[10]; with results now available for the symbol error probability (SEP), the SINR distribution, and the outage probability. In practical scenarios however, the channels often include line-of-sight (LoS) paths, in which case Rician fading is appropriate. Prominent examples supported by physical measurements, include microcellular mobile and indoor radio applications [11]. Rician paths are also expected to arise in ad-hoc networking applications (especially for dense networks), which are currently receiving considerable interest.

However, despite their practical significance, and in contrast to the Rayleigh fading case, there are currently very few OC performance results which apply for Rician fading, and none which consider error probability. The few results which do apply for these channels are presented in [9], [11]–[13], all of which focus on characterizing the SINR probability density function (p.d.f.) or outage probability, and restrict attention to interference-limited scenarios. For these scenarios, the thermal noise is neglected, and the number of interferers is restricted to be greater than or equal to the number of antenna elements. This simplifies the analysis since the resulting signal-to-interference-ratio (SIR) follows a noncentral multivariate $F$ distribution, thereby allowing classical multivariate statistics results from [14] to be directly leveraged.

In this paper we analyze the error probability of OC systems in Rician fading. We consider the general practical scenario where the received signals are subject to both co-channel interference and thermal noise. Our results allow for arbitrary numbers of interferers and receive antennas. We consider the case where the desired-user is subject to Rician fading and the interferers undergo Rayleigh fading (denoted Rician-Rayleigh). The analysis differs significantly from the interference-limited cases in [9], [11]–[13], since existing results from multivariate statistics cannot be directly applied.

We make the following key contributions:

- A new accurate approximation is presented for the SEP of OC with coherent $M$-PSK modulation. This approximation is a closed-form expression in terms of the moment generating function (m.g.f.) of the SINR.
- An exact expression is derived for the m.g.f. of the SINR. The expression applies for arbitrary numbers of antennas and interferers, and permits fast efficient evaluation of the exact and approximate SEP.
- Exact closed-form expressions are presented for the moments of the SINR. Remarkably, we find that all moments in this case are directly related to the corresponding moments of a Rayleigh-Rayleigh system via a simple scaling parameter.

Numerical results are presented to validate the analysis, and to examine the impact of Rician fading. These show that the presence of LoS yields a performance improvement.

II. SYSTEM DESCRIPTION

We consider a multi-user system where the receiver of interest optimally-combines the output from $N_d$ receive antennas. The desired signal is corrupted by $N_f$ interferers and thermal noise. After matched filtering and sampling at the symbol rate, the array output vector at time $k$ can be written as [5]:

\[ z(k) = \sqrt{E_D}c_D b_0(k) + z_{IN}(k) , \]

with the interference plus noise term

\[ z_{IN}(k) = \sum_{j=1}^{N_f} \sqrt{E_I}c_{I,j} b_j(k) + n(k) , \]

where $E_D$ and $E_I$ are the mean (over fading) energies of the desired and interfering signals, respectively (assuming all
interferers have the same power); \( c_D = [c_{D,1}, \ldots, c_{D,N_A}]^T \) and \( c_{I,j} = [c_{I,j,1}, \ldots, c_{I,j,N_A}]^T \) are the desired and \( j \)th interference normalized propagation vectors, respectively; \( b_0(k) \) and \( b_j(k) \) are the desired and interfering data samples, respectively, and \( \mathbf{n}(k) \) represents the additive noise.

We model \( c_D \) and \( c_{I,j} \) as uncorrelated complex Gaussian vectors. The desired-user channel vector \( c_D \) exhibits Rician fading and is distributed according to

\[
e_D \sim \mathcal{CN}(0, \mathbf{R}^{-1} e_D),
\]

where \( \mathbf{R} \) denotes the short-term covariance matrix of \( z_{IN}(k) \), conditioned on all interference propagation vectors, as follows

\[
\mathbf{R} = N_0\mathbf{I}_{N_A} + E_D \mathbf{C}_I \mathbf{C}_I^T
\]

where \( \mathbf{C}_I = [c_{I,1}, c_{I,2}, \ldots, c_{I,N_I}] \). It is important to remark that \( \mathbf{R} \) and, consequently \( \gamma \), vary at the fading rate, which is assumed to be much slower than the symbol rate.

### III. Symbol Error Probability

#### A. Exact SEP

Here we present expressions for M-PSK. Similar expressions can be generated for M-QAM.

Using a standard approach [15], the SEP with coherent M-PSK can be expressed as

\[
P_E = \frac{1}{\pi} \int_0^\theta M_{\gamma} \left( \frac{\cos\theta}{\sin^2\theta} \right) d\theta
\]

where \( \theta \triangleq \pi(M-1)/M \), \( \gamma \triangleq \sin^2(\pi/M) \), and \( M_{\gamma}() \) is the m.g.f. of the SINR \( \gamma \) given by

\[
M_{\gamma}(s) = \mathbb{E}_{\gamma} [e^{s\gamma}] .
\]

Note that for OC, from (5), the expectation in (8) involves averaging over both the distribution of the user channel vector \( c_D \) and the interference channel matrix \( \mathbf{C}_I \).

#### B. Approximate SEP

Here we present a new approximation to (7) which does not require integration over \( \theta \). We start by using (7) and (8) to write

\[
P_E = \mathbb{E}_{\gamma} \left[ \frac{1}{\pi} \int_0^\theta \exp \left( -\frac{\cos\theta}{\sin^2\theta} \right) d\theta \right] + \frac{1}{\pi} \int_\frac{-\theta}{2}^\frac{\theta}{2} \exp \left( -\frac{\cos\theta}{\sin^2\theta} \right) d\theta .
\]

Now, using [16, eq. (3)] and [16, eq. (14)] we can approximate the left-hand integral as

\[
\frac{1}{\pi} \int_0^\theta \exp \left( -\frac{\cos\theta}{\sin^2\theta} \right) d\theta \approx \frac{1}{12} e^{-\cos\theta} + \frac{1}{4} e^{-4\cos\theta}/3 .
\]

The right-hand integral can also be well approximated by the area of a trapezoid with parallel sides of length \( e^{-\cos\theta} \) and \( e^{-\cos\theta}/\sin^2(\theta) \) and height \( \Theta - \pi/2 \); i.e.

\[
\frac{1}{\pi} \int_\frac{-\theta}{2}^\frac{\theta}{2} \exp \left( -\frac{\cos\theta}{\sin^2\theta} \right) d\theta \approx \frac{1}{2\pi} \left( e^{-\cos\theta} + e^{-\cos\theta}/\sin^2(\theta) \right) (\Theta - \pi/2) .
\]

Now substituting (11) and (10) into (9), we obtain the approximate SEP expression

\[
P_E \approx \left( \frac{\Theta - \pi/2}{2\pi} \right) M_{\gamma} \left( e^{-\cos\theta} + e^{-\cos\theta}/\sin^2(\theta) \right) \left( \frac{\Theta - \pi/2}{2\pi} \right).
\]

Our numerical results will show that this approximation is accurate.

Note that both the exact SEP (7) and the approximate SEP (12) require the m.g.f. of the SINR \( \gamma \). Although not shown, the same is true for M-QAM. This m.g.f. has been evaluated for Rayleigh fading scenarios [10], and in the following section we evaluate it for the more general practical Rician scenario considered in this paper.

### IV. SINR Results for Rician-Rayleigh Fading

#### A. Moment Generating Function of the SINR

**Theorem 1:** The SINR m.g.f. of an OC system with Rician-faded users and Rayleigh-faded interferers is given by

\[
M_{\gamma}(s) = K_1 \sum_{k=1}^{N_{\max}} (-1)^{k+1} \det(\mathbf{X}_k) \beta_k(s)
\]

where

\[
K_1 = \frac{(-1)^{N_A}(N_0/E_1)^\tau}{\Gamma_{N_{\min}}(N_{\max})} \sum_{i=1}^{N_{\max}} \Gamma(N_{\max} - i + 1)
\]

Also, \( \beta_k() \) is defined as

\[
\beta_k(s) \triangleq \int_0^\infty x^{N_{\max} - k} (N_0/E_1 + x) e^{-x} h_1(s, x) dx - \sum_{i=1}^{\tau} \zeta_i(k + 1) h_t(s, 0)
\]

\(^1\)Note that this is related to the standard complex multivariate gamma function \( \Gamma_N(M) \) (as defined in [14]) via \( \Gamma_N(M) = \pi^{-N(N-1)/2} \Gamma_N(M) \).
\[ \zeta(t) \triangleq (N_0/E_1)(N_1 + t - k)! + (N_1 + t - k + 1)! \]  
(17)

\[ h(t, x) \triangleq \frac{1}{\mathcal{F}_1(t; N; x)} \frac{a_{N_k}^N}{(b_0 E_0 / E_1 - N_0 / E_0 - b_0)^t} \]  
(18)

where \( \mathcal{F}_1(\cdot) \) is the scalar confluent hypergeometric function, and \( X_k \) corresponds to the \( N_{\min} \times N_{\min} \) Hankel matrix \( X \) with elements

\[ \{X\}_{i,j} = \zeta N(i + j), \quad i, j = 1, \ldots, N_{\min} \]  
(19)

but with the first column and \( k \)th row removed.

**Proof:** See Appendix I.

We can see that the SINR m.g.f. depends on the Rician channel component only via the power scaling parameters \( a \) and \( b \), and is independent of the mean vector \( m \). This reveals that the structure of \( m \) has no impact on the statistics of the SINR, and therefore does not affect the system performance.

Note that (13)–(19) contains only standard functions, and therefore \( \mathcal{M}_s(s) \) can be easily and efficiently evaluated numerically with software packages such as Maple and Mathematica (note that the hypergeometric function in (18) can be expressed as a finite sum involving only exponential and polynomial terms).

### B. Moments of the SINR

The following theorem presents a closed-form expression for the moments of the SINR.

**Theorem 2:** The \( \ell \)th moment of the SINR of an OC system with Rician-faded users and Rayleigh-faded interferers is given by

\[ \mu_{\ell}^{\text{Ric-Ray}} = \alpha_{\ell}^{\text{Ric}} \mu_{\ell}^{\text{Ray-Ray}} \]  
(20)

where \( \mu_{\ell}^{\text{Ray-Ray}} \) is the \( \ell \)th moment for an OC Rayleigh-Rayleigh system (\( a = 0, b = 1 \)), given by

\[ \mu_{\ell}^{\text{Ray-Ray}} = K_1 \sum_{k=0}^{N_{\min}} (-1)^k \det(X_k) \xi_k \]  
(21)

where

\[ \xi_k = \ell!(E_0/E_1)^\ell e^{N_0/E_1} \sum_{t=0}^{N_1-k} \binom{N_1-k}{t} \frac{\Gamma(t - \ell + 1, N_0/E_1)}{-E_1/N_0} - (E_0/N_0)^\ell \]  
(22)

and \( K_1, X_k, \) and \( \zeta(\cdot) \) are defined as in Theorem 1. Also, \( \alpha_{\ell}^{\text{Ric}} \) is a constant given by

\[ \alpha_{\ell}^{\text{Ric}} = b^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} \frac{(aN_k/b)^k}{(N_k)} \]  
(23)

and \( (N_k)_k = N_A(N_A + 1) \cdots (N_A + k - 1) \) is the Pochhammer symbol.

**Proof:** See [17].

Given the intricacies of the derivation of \( \mathcal{M}_s(s) \) in the proof of Theorem 1, it is remarkable to see that such a simple relationship exists between \( \mu_{\ell}^{\text{Ric-Ray}} \) and \( \mu_{\ell}^{\text{Ray-Ray}} \). To highlight the implications of this relationship, we now make the following remarks. Let us consider the common power normalization model with \( a = K/(K+1) \) and \( b = 1/(K+1) \), where \( K \) is the Rician \( K \)-factor.

**Remark 1:** The Rician component has no effect on the average SINR, since \( \alpha_{\ell}^{\text{Ric}} = 1 \).

**Remark 2:** The second moment of the SINR decreases monotonically with the Rician \( K \)-factor according to

\[ \mu_{2}^{\text{Ric-Ray}} = \left(1 - \frac{K}{K+1}\right)^2 \frac{1}{N_A+1} \mu_{2}^{\text{Ray-Ray}}. \]  
(24)

Combined with Remark 1, this also shows that the variance of the SINR decreases monotonically with the Rician \( K \)-factor.

**Remark 3:** The ratio of SINR moments \( \alpha_{\ell}^{\text{Ric}} = \mu_{\ell}^{\text{Ric-Ray}}/\mu_{\ell}^{\text{Ray-Ray}} \) depends only on the Rician \( K \)-factor and the number of antennas \( N_A \), for all \( \ell \); and is independent of the number of interferers \( N_I \), the SINR \( E_0/N_0 \), and the (normalized) signal-to-interference ratio (SIR) \( E_1/N_0 \).

**Remark 4:** For fixed numbers of interferers \( N_I \) and antennas \( N_A \), the SINR moments are bounded as a function of the Rician-\( K \) factor by

\[ \left(\prod_{i=0}^{\ell-1} \frac{N_A}{N_A+i}\right) \frac{\mu_{\ell}^{\text{Ray-Ray}}}{\mu_{\ell}^{\text{Ric-Ray}}} \leq \frac{\mu_{\ell}^{\text{Ric-Ray}}}{\mu_{\ell}^{\text{Ray-Ray}}} \leq \frac{\mu_{\ell}^{\text{Ric-Ray}}}{\mu_{\ell}^{\text{Ray-Ray}}} \]  
(25)

for all \( \ell \), where the left-hand equality is approached as \( K \to \infty \) (i.e., when the desired user’s channel becomes nonfading), and the right-hand equality is met when \( K \to 0 \).

**Remark 5:** The SINR distributions for Rician-Rayleigh and Rayleigh-Rayleigh optimum combining systems are asymptotically equal as \( N_A \to \infty \). This can be proved from (23) by showing that all of the moments are asymptotically equal as follows:

\[ \lim_{N_A \to \infty} \frac{\mu_{\ell}^{\text{Ric-Ray}}}{\mu_{\ell}^{\text{Ray-Ray}}} = \lim_{N_A \to \infty} \alpha_{\ell}^{\text{Ric}} = b^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} (a/b)^k \lim_{N_A \to \infty} \frac{N_A}{N_A} \cdots \frac{N_A}{N_A+k-1} = (b+a)^\ell = 1 \]  
(26)

for all \( \ell \).

**Remark 6:** For large \( N_A \), the Rayleigh-Rayleigh SINR moments satisfy

\[ \mu_{\ell}^{\text{Ray-Ray}} = \left(\frac{E_D}{N_0}(N_A+N_1)\right)^\ell + \mathcal{O}(N_A^{\ell-1}), \]  
(27)

which is obtained by taking \( N_A \) large in (21), and applying the following determinant identity for Hankel...
V. NUMERICAL RESULTS

We now validate the preceding theoretical analysis and examine the effect of the various system/channel parameters. Throughout this section we define $\text{SIR} = \frac{E_D}{(N_I E_1)}$, and $\text{SNR} = \frac{E_D}{N_0}$. We also assume the common Rician fading power normalization strategy, with $a = K/(K + 1)$ and $b = 1/(K + 1)$, where $K$ corresponds to the Rician $K$-factor.

Recall that we have shown that the specific choice of $m$ has no effect on the system performance, and as such, for all simulation results the mean vector $m$ was randomly-generated.

Fig. 1 shows analytical and Monte-Carlo simulated SEP curves for a Rician-Rayleigh OC system, comparing BPSK and 8-PSK modulation, and different $N_I$. The “Analytical (Exact)” curves were generated by combining (13) and (7), and the “Analytical (Approx)” curves were generated by combining (13) and (12). In all cases the “Analytical (Exact)” SEP curves match precisely with the simulated curves, and the “Analytical (Approx)” curves are accurate. For both modulation schemes, we also observe that an error floor exists when the number of interferers exceeds the number of degrees of freedom of the array (i.e. $N_A - 1$), which aligns with well-established results for Rayleigh-Rayleigh systems (see, e.g., [5]).

Fig. 2 further investigates the impact of the Rician $K$-factor on the SEP performance, comparing systems with $N_A = 2$ and $N_A = 3$, for a range of SIRs. The curves were generated based on the exact analytical SEP results, obtained by combining (13) and (7). Note that each curve essentially shows how the SEP performance varies as the desired-user channel undergoes a transition from Rayleigh fading ($K = 0$) to a deterministic nonfading channel ($K \to \infty$). We see that, in all cases, the SEP decreases monotonically with $K$, and the rate of decrease is most significant for low $K$ values (e.g. $< 5$). We also see that $K$ has the most impact for high SIRs and that the relative effect of $K$ appears to be independent of $N_A$.

Fig. 3 shows analytical and Monte-Carlo simulated curves for the second moment of the SINR as a function of $K$; comparing different $N_I$. The “Analytical” curves were generated using (20). In all cases we see a precise agreement between the analysis and simulations. Also, as predicted from Section IV-B (see Remarks 2 and 4), we see that for all values of $N_I$ this moment decreases monotonically with $K$; eventually converging to a deterministic constant.

Fig. 4 shows the ratio of the SINR moments of a Rician-Rayleigh and Rayleigh-Rayleigh system $\alpha_{\text{Ric}}$ (defined in (23)) as a function of $K$, for the first, second, and third moments, and for $N_A = 2, 3$, and 4. Recall that this ratio is independent of the SNR, SIR, and $N_I$. As suggested in Section IV-B (see Remark 1), we see that the Rician-$K$ factor has no effect on the average SINR. We see that the second and third moments, however, decrease monotonically with $K$, this being most significant for the third moment. Moreover, for the second and third moments, we see that the effect of $K$ becomes less (and $\alpha_{\text{Ric}}$ becomes closer to 1) as $N_A$ increases. This behavior is also consistent with the analytic conclusions given in Section IV-B (Remark 5).

VI. CONCLUSIONS

We considered the analysis of Rician-Rayleigh OC systems in the presence of interference and thermal noise. An exact expression was derived for the m.g.f. of the SINR, which was
used to examine the SEP. Exact insightful closed-form expressions were also derived for the moments of the SINR. Our results have demonstrated that Rician fading has a beneficial impact on the performance of OC systems.

APPENDIX I

PROOF OF THEOREM 1

Proof: We first derive the m.g.f. when conditioned on the interference channel gains $\mathbf{C}_i$ (equivalently $\mathbf{R}$); then average over the interference channel statistics. With $\mathbf{C}_0$ distributed according to (3), the conditional m.g.f. corresponds to the m.g.f. of a noncentral quadratic form in complex Gaussian random vectors, given in [18, App. B]. Using this result, along with (5), we can write

$$M_\gamma(s) = \mathbb{E}_{\mathbf{R}} \left[ \frac{\exp\left(-\frac{s}{2} \mathbf{m}^\dagger \left( \left(1 - \frac{N_0}{E_I s} \right) \mathbf{I}_{N_A} - \frac{1}{s} \tilde{\mathbf{R}} \right)^{-1} \mathbf{m} \right)}{\det \left( \mathbf{I}_{N_A} - \frac{1}{s} \left( \frac{N_0}{E_I} \mathbf{I}_{N_A} + \tilde{\mathbf{R}} \right)^{-1} \right)} \right]$$

(28)

where $\tilde{s} = b s E_D / E_I$ and $\tilde{\mathbf{R}} \triangleq \mathbf{C}_0 \mathbf{C}^\dagger_0$. The major challenge here is to perform the averaging over $\tilde{\mathbf{R}}$. Let us first define

$$\mathbf{Y} \triangleq \left( \left(1 - \frac{N_0}{E_I \tilde{s}} \right) \mathbf{I}_{N_A} - \frac{1}{\tilde{s}} \tilde{\mathbf{R}} \right)^{-1}.$$

(29)

Since $\tilde{\mathbf{R}}$ and $\mathbf{Y}$ are both Hermitian, they admit the following eigenvalue decompositions:

$$\tilde{\mathbf{R}} = \mathbf{U}_{\tilde{\mathbf{R}}} \mathbf{A}_{\tilde{\mathbf{R}}} \mathbf{U}_{\tilde{\mathbf{R}}}^\dagger, \quad \mathbf{A}_{\tilde{\mathbf{R}}} = \text{diag}(\lambda_1, \ldots, \lambda_{N_A}), \quad \mathbf{U}_{\tilde{\mathbf{R}}} \in \mathcal{U}(N_A)$$

$$\mathbf{Y} = \mathbf{U}_{\mathbf{Y}} \mathbf{A}_{\mathbf{Y}} \mathbf{U}_{\mathbf{Y}}^\dagger, \quad \mathbf{A}_{\mathbf{Y}} = \text{diag}(\lambda_1, \ldots, \lambda_{N_A}), \quad \mathbf{U}_{\mathbf{Y}} \in \mathcal{U}(N_A)$$

(30)

where $0 = \lambda_{N_A} = \cdots = \lambda_{N_{\text{null}} + 1} < \lambda_{N_{\text{null}}} \leq \ldots \leq \lambda_1 < \infty$ are the ordered eigenvalues of $\tilde{\mathbf{R}}$, and $\mathcal{U}(N_A)$ is the unitary manifold consisting of $N_A \times N_A$ unitary matrices with real diagonal elements. After some manipulations, it can be shown that the m.g.f. in (28) can be expressed as

$$M_\gamma(s) = \left( \frac{N_0}{E_I \tilde{s}} \right)^{N_A} \times \mathbb{E}_{\mathbf{A}_{\tilde{\mathbf{R}}}} \left[ \prod_{i=1}^{N_A} \left( \frac{N_0}{E_I \tilde{s}} + \lambda_i \right) \mathcal{I}(\mathbf{A}_{\tilde{\mathbf{R}}}) \right]$$

(31)

where

$$\mathcal{I}(\mathbf{A}_{\tilde{\mathbf{R}}}) = \int_{\mathcal{U}(N_A)} \exp \left(-\text{tr} \left( \mathbf{B} \mathbf{U}_{\tilde{\mathbf{R}}} \mathbf{A}_{\mathbf{Y}} \mathbf{U}_{\tilde{\mathbf{R}}}^\dagger \right) \right) [d\mathbf{U}_{\tilde{\mathbf{R}}}]$$

(32)

where $\mathbf{B} = \frac{a}{\pi} \mathbf{m} \mathbf{m}^\dagger$, and $[d\mathbf{U}_{\tilde{\mathbf{R}}}]$ is the normalized Haar invariant probability measure on $\mathcal{U}(N_A)$. To evaluate (32), we force both $\mathbf{B}$ and $\mathbf{A}_{\tilde{\mathbf{R}}}$ to be full-rank by perturbing all of the zero-valued eigenvalues. These eigenvalues will subsequently be driven back to zero. We can then use the identity [14, eq. (89)] and the splitting property of hypergeometric functions [14, eq. (92)] to perform the integration in (32) which yields

$$\mathcal{I}(\mathbf{A}_{\tilde{\mathbf{R}}}) = 0 \mathcal{F}_0 \left( \mathbf{B}, -\mathbf{A}_{\mathbf{Y}} \right).$$

(33)

Now, noting that the $N_A$ eigenvalues of $\mathbf{B}$ are given by $\{a||\mathbf{m}||^2/b, 0, 0, \ldots, 0\}$, we can invoke [19, Corr. 1] to write $0 \mathcal{F}_0 \left( \cdot \right)$ in a determinant form which accounts for the zero-valued eigenvalues of $\mathbf{B}$. Then, recalling that $||\mathbf{m}||^2 = N_A$, and after lengthy manipulations, we obtain

$$\mathcal{I}(\mathbf{A}_{\tilde{\mathbf{R}}}) = \frac{\det(\mathbf{F})}{\det \left( \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_{N_A} \\ \end{array} \right)}$$

(34)

where $\mathbf{F}$ is an $N_A \times N_A$ matrix with $(i, j)^{th}$ element

$$\left( \mathbf{F} \right)_{i,j} = \begin{cases} 1 & \text{if } i = j = 1; \\ 1 & \text{if } i = 1, \ldots, N_A; \\ \frac{N_A \tilde{s}^i}{w(\tilde{s}, \lambda_i)} & \text{if } i = 1, \ldots, N_A; \\ \frac{N_A \tilde{s}^j}{w(\tilde{s}, \lambda_j)} & \text{if } j = 2, \ldots, N_A \end{cases}$$

(35)
where
\[ w(\mathbf{s}, \lambda_i) = \mathbf{s} - \lambda_i - N_0 / E_i. \] (36)

We now take the limits \( \lambda_i \to 0 \), for \( i = N_{\text{min}} + 1, \ldots, N_A \). To this end, we can apply the general limiting result in [19, Corr. 1] to (34) and, after more algebraic manipulations, evaluate these limits to yield
\[ \mathbb{I}(\mathbf{A}) = w(\mathbf{s}, 0)^T \left( \prod_{i=1}^{N_{\text{min}}} w(\mathbf{s}, \lambda_i) \right) \frac{\det(\mathbf{J})}{\det \left( \lambda_i^{N_{\text{min}}-j} \right)_{i,j=1}^{N_{\text{min}}}}. \] (37)

where \( \mathbf{J} \) is an \( N_{\text{min}} \times N_{\text{min}} \) matrix with elements
\[ \left( \mathbf{J} \right)_{i,j} = \begin{cases} h_1(s, \lambda_i) - \sum_{t=1}^{s} h_t(s, 0) \lambda_i^{t-1}, & i = 1, \ldots, N_{\text{min}}, \ j = 1 \\ \lambda_i^{N_{\text{min}}-j} & i = 1, \ldots, N_{\text{min}}, \ j = 2, \ldots, N_{\text{min}} \end{cases} \]
where \( h_t(\cdot, \cdot) \) is defined in (18). Now substituting (37) into (31), the m.g.f. becomes
\[ M_\gamma(s) = (-1)^{N_{\text{min}}} (N_0 / E_i)^T \times \mathbb{E}_{\mathbf{A}} \left[ \prod_{i=1}^{N_{\text{min}}} \left( N_0 / E_i + \lambda_i \right) \right] \frac{\det(\mathbf{J})}{\det \left( \lambda_i^{N_{\text{min}}-j} \right)_{i,j=1}^{N_{\text{min}}}}. \] (38)

We see that the remaining expectation is over the \( N_{\text{min}} \) non-zero ordered eigenvalues of \( \mathbf{A} \). These eigenvalues have the same distribution as the ordered eigenvalues of a complex Wishart matrix of dimension \( N_{\text{min}} \times N_{\text{min}} \) and with \( N_{\text{max}} \) degrees of freedom, and thus have the following p.d.f.:
\[ f_{\text{ord}}(\lambda_1, \ldots, \lambda_{N_{\text{max}}}) = \prod_{i=1}^{N_{\text{min}}} \left( e^{-\lambda_i} \lambda_i^{N_{\text{min}}-N_{\text{max}}} \right) \frac{\det \left( \lambda_i^{N_{\text{min}}-j} \right)_{i,j=1}^{N_{\text{max}}} \lambda_i^{N_{\text{min}}}}{\Gamma(N_{\text{min}}) \Gamma(N_{\text{max}})} \] (39)
Using (39) in (38), we obtain
\[ M_\gamma(s) = K_1 \int \cdots \int_{\mathbf{D}} \det \left( \lambda_i^{N_{\text{min}}-j} \right)_{i,j=1}^{N_{\text{min}}} \times \det(\mathbf{J}) \prod_{i=1}^{N_{\text{min}}} g(\lambda_i) \ d\lambda \] (40)
where \( d\lambda = d\lambda_1 d\lambda_2 \cdots d\lambda_{N_{\text{max}}} \), the multiple integral is over the domain \( \mathbf{D} = \{ \infty \geq \lambda_1 \geq \cdots \geq \lambda_{N_{\text{max}}} \geq 0 \} \), \( K_1 \) is defined in (14), and \( g(\lambda_i) = (N_0 / E_i + \lambda_i) \lambda_i^{N_{\text{min}}-N_{\text{max}}} e^{-\lambda_i} \).

The theorem now follows by applying the general integration identity [21, Corr. 2] to (40), performing some simplifications to the resulting determinant using [22, eq. 3.381.4], and then applying Laplace’s expansion along the first column.

References