Capacity Approximations for Multiuser MIMO-MRC with Antenna Correlation

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Abstract—This paper investigates the capacity of multiuser MIMO-MRC systems in spatially correlated environments. We present new capacity approximations which are shown to be accurate. The approximations are based on new simple expansions which we derive for the maximum eigenvalue of correlated Wishart matrices. We show that for a large number of users there is a capacity offset due to correlation. Through this, we show that correlation is beneficial for capacity. Our results are confirmed through comparison with Monte-Carlo simulations.

Index Terms—MIMO Systems, Diversity Methods, Correlation, Capacity Analysis, Rayleigh Channels, MIMO MRC

I. INTRODUCTION

Wireless communication systems employing multiple-input multiple-output (MIMO) antennas offer significant capacity advantages over single antenna systems. Their use can be extended to multiuser environments where each user is equipped with multiple antennas and are either transmitting to, or receiving from a base station also equipped with multiple antennas. This corresponds to the multiple-access and broadcast channel respectively.

Assuming the channel is known at both the transmitter and receiver, at low signal to noise ratios (SNR) the capacity achieving access scheme corresponds to having only one user active at any one time [1, 2]. Moreover, the low SNR single user waterfilling solution is for the active user to transmit/receive along a single eigenmode, corresponding to the largest singular value of their channel matrix. A practical method which can realize this scheme in a MIMO system is by using maximum ratio combining (MRC) receivers.

MIMO-MRC has been studied in the single user case in uncorrelated [3, 4] and semi-correlated Rayleigh channels [5] (i.e. channels with transmit or receive correlation, but not both). Recently, double-correlated Rayleigh channels have been considered by deriving the eigenvalue statistics of double-correlated Wishart matrices [6].

For the multiuser case, the capacity of MIMO-MRC in an uncorrelated environment was considered in [7], and was shown to scale double logarithmically with the number of users. Systems with transmit correlation and a single receive antenna per user were analyzed in [8], and capacity was shown to scale logarithmically with the correlation coefficient. These results, however, do not extend easily to the case of MIMO-MRC systems with either semi- or double-correlation.

In addition, the capacity expressions given in [7, 8] are loose for a practical number of users (< 50).

In this paper, we consider MIMO-MRC systems with antenna correlation at either or both the transmitter and receiver ends. We present new accurate capacity approximations by first deriving new expansions for the cumulative distribution function (c.d.f.) of the maximum eigenvalue of uncorrelated, semi, and double-correlated Wishart matrices, and then using results from order statistics. Our results apply for both broadcast and multiple-access channels. We also analyze the large user capacity offset between the different correlation scenarios. Our results show that the capacity scales logarithmically with the maximum eigenvalue of the correlation matrix. This generalizes the results of [7, 8] to arbitrary correlation models and multiple antennas. Finally, we confirm our results by Monte-Carlo simulations.

II. SYSTEM MODEL AND CAPACITY

Consider a multiuser MIMO-MRC system with $K$ users each having $N_t$ transmit and $N_r$ receive antennas. The $k$th user has a channel of the following form:

$$ H_k = R_k H_{w,k} S^\dagger \sim CN_{N_t,N_r}(0_{N_t \times N_r}, R \otimes S) $$  \hspace{1cm} (1)

where $R_k \in CN_{r \times N_r}$ and $S \in CN_{r \times N_t}$ are the receive and transmit correlation matrices respectively with unit diagonal entries, and $H_{w,k} \sim CN_{N_r,N_t}(0_{N_r \times N_t}, I_{N_r} \otimes I_{N_t})$. This is the common Kronecker structure used to model correlation (e.g. as used in [6, 9]).

We define $\lambda_{\text{max}}$ to be the maximum eigenvalue of $H_k^\dagger H_k$, and define $\lambda_{\text{max},K} \triangleq \max(\lambda_{\text{max}}^{(1)}, \ldots, \lambda_{\text{max}}^{(K)})$. We also use $k_{\text{max}}$ to denote the user which has the largest overall eigenvalue.

For the broadcast channel, the transmitter sends only to user $k_{\text{max}}$. For the multiple access channel, user $k_{\text{max}}$ is the only one to transmit. In both cases, full channel state information (CSI) is assumed to be known at the receiver, but only the eigenvector corresponding to $\lambda_{\text{max},K}$ is known at the transmitter. Note that only one user is active at any one time.

The received vector is

$$ r = \sqrt{\gamma} H_{k_{\text{max}}} w + n $$  \hspace{1cm} (2)
where $x$ is the transmitted symbol satisfying $E\left[\left|x\right|^2\right] = 1$, $w$ is the $N_t \times 1$ transmit weight vector with $E\left[\left\|w\right\|^2\right] = 1$, $n$ is the additive noise vector $\sim CN_{N_r,1}(0, N_r I)$, and $\tilde{\gamma}$ is the transmit SNR. At the receiver, the signal is multiplied by the vector $w^H H_{\text{max}}^t$, according to the MRC principle, to give
\[ z = w^H H_{\text{max}}^t r = \sqrt{\tilde{\gamma}} w^H H_{\text{max}}^t H_{\text{max}} w x + w^H H_{\text{max}}^t n. \]  
(3)

Thus the SNR at the output of the MRC can be expressed as
\[ \gamma = \frac{\tilde{\gamma}}{w^H H_{\text{max}}^t H_{\text{max}} w}. \]  
(4)

The transmit weight vector which maximizes (4) is denoted $w_{\text{opt}}$, and is given by the eigenvector corresponding to $\lambda_{\text{max},K}$ [3]. This gives
\[ \gamma = \tilde{\gamma} w^H w_{\text{opt}} H_{\text{max}} H_{\text{max}}^t w_{\text{opt}} = \tilde{\gamma} \lambda_{\text{max},K}. \]  
(5)

The resulting capacity for the multiuser MIMO-MRC system is given by
\[ C_{\text{multi}} = E[\log_2 (1 + \tilde{\gamma} \lambda_{\text{max},K})] = \int_0^\infty \log_2 (1 + \tilde{\gamma} x) f_{\lambda_{\text{max},K}}(x) \, dx \]  
(6)

where $f_{\lambda_{\text{max},K}}(\cdot)$ is the probability density function (p.d.f.) of $\lambda_{\text{max},K}$. Using a result from order statistics [10], (6) can be written as
\[ C_{\text{multi}} = K \int_0^\infty \log_2 (1 + \tilde{\gamma} x) f_{\lambda_{\text{max}}}(x) F_{K-1}^{\lambda_{\text{max}}}(x) \, dx \]  
(7)

where $\lambda_{\text{max}}$ is the maximum eigenvalue for an arbitrary user, with p.d.f. $f_{\lambda_{\text{max}}}(\cdot)$ and c.d.f. $F_{\lambda_{\text{max}}}(\cdot)$.

In general, the integral in (7) is hard to solve explicitly, even for the simplest case of a single user system ($K = 1$) with uncorrelated Rayleigh fading [11]. In this paper we approximate (7) by
\[ C_{\text{multi}} \approx \log_2 (1 + \tilde{\gamma} g(K)) \]  
(8)

where $g(K) = F_{\lambda_{\text{max}}}^{-1}\left(\frac{1}{1 + e^{-\frac{1}{K + 1} \frac{\tilde{\gamma}}{\lambda_{\text{max}}}}}ight)$. This expression is obtained by first noting that for a random variable $X$ following a symmetric distribution, with c.d.f. $F_X(\cdot)$ and largest order statistic $X_K$, we have the following inequality [12]
\[ E[X_K] \leq F_X^{-1}\left(\frac{1}{1 + e^{-\frac{1}{K+1} \frac{\tilde{\gamma}}{\lambda_{\text{max}}}}}ight). \]  
(9)

This upper bound has been shown to be tight for a Gaussian distribution [13]. Now defining $m = \max(N_t, N_r)$ and $n = \min(N_t, N_r)$, $C_{\text{multi}} = E[\log_2 (1 + \tilde{\gamma} \lambda_{\text{max}})]$ converges to a Gaussian distribution for fixed $n$ and as $m \to \infty$ [14]. Thus, we substitute $X = \log_2 (1 + \tilde{\gamma} \lambda_{\text{max}})$ into (9), and apply simple algebraic manipulation to obtain (8).

To calculate (8), we require the inverse function of $F_{\lambda_{\text{max}}}(\cdot)$. Unfortunately, for uncorrelated, semi- and double-correlated scenarios, $F_{\lambda_{\text{max}}}(\cdot)$ is a complicated determinantal function (see, e.g., [3, 4, 6, 15]), and the inverse calculation is intractable in general. Progress can be made, however, by noting that the argument of the inverse c.d.f. in (8) is close to 1 for typical values of $K$, as shown in Table I. In other words, we are dealing with the tail of the c.d.f. This result motivates us to derive new expansions for $F_{\lambda_{\text{max}}}(\cdot)$ which are accurate in the tail, and admit simple formulation of the required inverse.

We first define the matrices $\Sigma \in C^{n \times mn}$ and $\Omega \in C^{m \times n}$ as follows
\[ \Sigma = \left\{ \begin{array}{ll} R & \text{for } N_r > N_t \\ S & \text{for } N_r \leq N_t \end{array} \right. \quad \Omega = \left\{ \begin{array}{ll} R & \text{for } N_r \leq N_t \\ S & \text{for } N_r > N_t \end{array} \right. \]  
(10)

where $m = \max(N_t, N_r)$ and $n = \min(N_t, N_r)$. With these definitions, note that the maximum channel eigenvalue $\lambda_{\text{max}}$ is statistically equivalent to the maximum eigenvalue of a complex Wishart matrix $X^t X$, where $X$ has the following distribution:

- Uncorrelated fading: $X \sim \mathcal{CN}_{m,n}(0_{m \times n}, I_m \otimes I_n)$
- Semi-correlated fading: $X \sim \mathcal{CN}_{m,n}(0_{m \times n}, I_m \otimes \Omega)$
- Double-correlated fading: $X \sim \mathcal{CN}_{m,n}(0_{m \times n}, \Sigma \otimes \Omega)$

In the following section we will derive new asymptotic expansions for the distribution of $\lambda_{\text{max}}$ for each of these scenarios. We note that although the uncorrelated and semi-correlated scenarios can be seen as special cases of the double-correlated scenario, the results do not decompose easily and will be treated separately.

### III. C.D.F. EXPANSIONS FOR THE MAXIMUM EIGENVALUE OF COMPLEX WISHART MATRICES

The following theorem applies to the uncorrelated fading case.

**Theorem 1**: Let $X \sim \mathcal{CN}_{m,n}(0_{m \times n}, I_m \otimes I_n)$. The c.d.f. of the maximum eigenvalue of the complex Wishart matrix $X^t X$ is given by
\[ F_{\lambda_{\text{max}}}(x) = 1 - \alpha_{\text{id}} e^{-x} \sum_{p=1}^n \sum_{q=1}^n C_{p,q} K_{p,q} \sum_{t=0}^{\tau + p + q - 2} \frac{x^t}{t!} + O(e^{-2x^2 m + 2n - 4}) \]  
(11)

where $\alpha_{\text{id}} = 1/([\prod_{k=1}^m (m-k)![(n-k)!])$, $\tau = m - n$, and $C_{p,q}$ is the $(p,q)$th cofactor of the $n \times n$ matrix $K$ with $(i,j)$th element $K_{i,j} = \Gamma(\tau + i + j - 1)$.

An alternate expansion with slower decaying $O(\cdot)$ terms is
\[ F_{\lambda_{\text{max}}}(x) = 1 - \frac{e^{-x} x^{m+n-2}}{(m-1)!(n-1)!} + O(e^{-x^{m+n-3}}). \]  
(12)
Proof: Omitted due to space limitations.

Note that (12) can also be obtained by integrating the p.d.f. derived in [7].

The following theorem applies to semi-correlated fading.

**Theorem 2:** Let $\mathbf{X} \sim \mathcal{CN}_{m,n}(\mathbf{0}_{m \times n}, \mathbf{I}_m \otimes \Omega)$ where $\Omega \in \mathbb{C}^{n \times n}$ is Hermitian positive-definite with eigenvalues $\omega_1 < \ldots < \omega_n$. Then the c.d.f. of the maximum eigenvalue of the complex Wishart matrix $\mathbf{X}^\dagger \mathbf{X}$ is given by

$$F_{\lambda_{\text{max}}} (x) = 1 - \alpha_{\text{semi}} e^{-\frac{x}{\omega_1}} \sum_{p=1}^{n} D_{p,n} V_{p,n} \sum_{i=0}^{m-p} \left( \frac{x}{\omega_i} \right)^i \frac{1}{i!} + \mathcal{O}(e^{-x/\omega_1}),$$  \hspace{1cm} (13)

where $\alpha_{\text{semi}} = 1/\det(\mathbf{V})$, and $D_{p,n}$ is the $(p, n)^{\text{th}}$ cofactor of the $n \times n$ Vandermonde matrix $\mathbf{V}$ with $(i, j)^{\text{th}}$ element $V_{i,j} = 1/\omega_{i,j}^{-1}$.

An alternate expansion with slower decaying $\mathcal{O}(\cdot)$ terms is

$$F_{\lambda_{\text{max}}} (x) = 1 - \frac{\omega_n^{-m} x^{m-1} e^{-\frac{x}{\omega_1}}}{(m-1)! \prod_{i=1}^{n} (\omega_n - \omega_i)} + \mathcal{O}(e^{-x/\omega_1}).$$  \hspace{1cm} (14)

**Proof:** See Appendix A.

The following theorem applies to double-correlated fading.

**Theorem 3:** Let $\mathbf{X} \sim \mathcal{CN}_{m,n}(\mathbf{0}_{m \times n}, \Sigma \otimes \Omega)$ where $\Sigma \in \mathbb{C}^{n \times n}$ and $\mathbf{X} \sim \mathcal{CN}_{m \times n}$ are Hermitian positive-definite with eigenvalues $\omega_1 < \ldots < \omega_n$ and $\sigma_1 < \ldots < \sigma_m$ respectively. Then the c.d.f. of the maximum eigenvalue of the complex Wishart matrix $\mathbf{X}^\dagger \mathbf{X}$ is given by

$$F_{\lambda_{\text{max}}} (x) = 1 - \frac{e^{-\frac{x}{\varphi}} \omega_n^{m-1} \sigma_{m-1} \prod_{i=1}^{m} (\omega_n - \omega_i)}{\prod_{i=1}^{m} (\sigma_m - \sigma_i)} + \mathcal{O}(e^{-x/\varphi}).$$  \hspace{1cm} (15)

where $\varphi = \omega_n \sigma_{m-1}$ if $\sigma_m < \omega_n$ and $\varphi = \omega_{n-1} \sigma_m$ if $\sigma_m > \omega_n$.

**Proof:** Omitted due to space limitations.

Since we are interested in the tail of the c.d.f., we define $\hat{F}_{\lambda_{\text{max}}} (x)$ to be $F_{\lambda_{\text{max}}} (x)$ given in the theorems above, but without the $\mathcal{O}(\cdot)$ term. Note that in all three cases, $\hat{F}_{\lambda_{\text{max}}} (x)$ is of the form $1 - e^{-ax} J(x)$, where $a$ is a non-negative real number and $J(x)$ is a polynomial.

Fig. 1 shows $\hat{F}_{\lambda_{\text{max}}} (x)$ for the uncorrelated (11), semi- (13), and double-correlated (15) scenarios. The correlation matrices were constructed using the practical channel model from [16] with its associated correlation parameters $\theta_1$, $\theta_2$, $\sigma_2^2$, $\sigma_t^2$, $d_\theta$ and $d_\tau$. Clearly our new simple expansions are accurate in each case.

**IV. CAPACITY APPROXIMATIONS**

This section presents new approximations to the capacity for the three correlation scenarios we are considering. The approximations are derived by substituting $\hat{F}_{\lambda_{\text{max}}} (x)$, derived from the theorems in Section III, into (8).

We evaluate $\hat{g}(K) = \frac{1}{a} \left( \ln (J(\hat{g}(K))) + \ln(N(K)) \right)$ until convergence where $N(K) = 1 + e^{\sum_{i=1}^{K-1} \frac{x}{\tau}}$, and where for the uncorrelated case

$$a = 1, \quad J(x) = \alpha_{\text{uni}} \sum_{p=1}^{n} \sum_{q=1}^{n} C_{p,q} K_{p,q} \prod_{\ell=0}^{\tau+p+q-2} x^\ell,$$  \hspace{1cm} (17)

for the semi-correlated case

$$a = \frac{1}{\omega_n}, \quad J(x) = \alpha_{\text{semi}} \sum_{p=1}^{n} D_{p,n} V_{p,n} \prod_{i=0}^{m-p-1} \frac{(ax)^i}{i!},$$  \hspace{1cm} (18)

and for the double-correlated case

$$a = \frac{1}{\omega_n \sigma_m}, \quad J(x) = \frac{1}{\prod_{i=1}^{m} (\sigma_m - \sigma_i)} \prod_{i=1}^{m} (\omega_n - \omega_i).$$  \hspace{1cm} (19)

Simulations indicate that (16) has a fast rate of convergence. The exact rate, however, depends on the number of antennas and correlation in the system. For example, using the same semi-correlated scenario and antenna configuration in Fig. 1, running the iteration only 5 times for $\hat{F}_{\lambda_{\text{max}}} (x) = 0.833$ results in an almost negligible error of only 0.06%.

We denote $g_{\text{conv}}(K)$ to be the converged value from (16). Substituting this into (8) in place of $g(K)$ gives an approximation to capacity as follows

$$C_{\text{multi}} \approx \log_2 (1 + g_{\text{conv}}(K)).$$

Fig. 2 shows the capacity for uncorrelated fading, for different $N_t$ and $N_r$. The analytical approximation curves were obtained using (16), (17) and (20). We see that these curves accurately approximate the Monte Carlo simulated curves in all cases. For comparison, we also present the previous capacity approximation from [7], for the case of $2 \times 2$
antennas. We clearly see that our new approximation is much more accurate for all practical $K$ values.

Fig. 2. Capacity of multiuser MIMO-MRC in uncorrelated Rayleigh fading, with SNR $\bar{\gamma} = 0$ dB.

Fig. 3 shows the capacity for semi-correlated fading (receive correlation), for different $N_t$ and $N_r$. The analytical approximation curves were obtained using (16), (18) and (20). Again, we see that the analytical curves accurately approximate the Monte Carlo simulated curves in all cases.

Fig. 4 shows the capacity for double-correlated fading, for different $N_t$ and $N_r$. The analytical approximation curves were obtained using (16), (19) and (20). Again, we see that the analytical curves accurately approximate the Monte Carlo simulated curves in all cases.

related to the level of receive correlation. We see that the capacity increases with decreasing angular spread (i.e. increasing correlation). This agrees with previous observations given for single-user MIMO-MRC systems in [11]. Although not shown, similar results have also been obtained for the semi-correlated case.

V. C APACITY OFFSET

In order to gain further insights into the effect of correlation on the capacity of multiuser MIMO-MRC systems, it is instructive to examine the capacity behavior in the large-$K$ regime. Note that the asymptotic behavior for the uncorrelated case has been considered previously in [7]; however this result will be included here for completeness.
Proposition 1: For sufficiently large $K$, the capacity of a multiuser MIMO-MRC system behaves as

$$C_{\text{multi}} = \log_2 \ln K + \log_2 \tilde{\gamma} + \text{CO}_\infty$$

$$+ \log_2 \left( 1 + \mathcal{O} \left( \frac{\ln \ln K}{\ln K} \right) \right)$$ (21)

where CO$_\infty$ is the capacity offset, given for the uncorrelated, semi and double-correlated scenarios as follows

$$\text{CO}_\infty = \begin{cases} 0 & \text{uncorrelated} \\ \log_2(\omega_n) & \text{semi-correlated} \\ \log_2(\omega_n \sigma_m) & \text{double-correlated} \end{cases}$$ (22)

Proof: See Appendix B.

As shown in (22), CO$_\infty$ can be interpreted as the offset in capacity taken with respect to the reference uncorrelated channel. Since, for $\Omega \neq I_n$ and $\Sigma \neq I_m$, we have $\omega_n > 1$ and $\sigma_m > 1$ respectively, (22) indicates that the capacity increases with correlation. This shows that correlation at the transmitter and/or receiver is beneficial for capacity.

Fig. 6 shows capacity curves for the uncorrelated and double-correlated case obtained by using the analytical approximations given in (17), (19) and (20), and compares them with the large user capacity expression obtained in (21). The capacity offset is clearly observed.

Another way of interpreting CO$_\infty$ is by viewing the capacity offset as either the reduction in users required to achieve a certain capacity for a given $\tilde{\gamma}$, or the reduction in $\tilde{\gamma}$ needed to achieve a certain capacity for a large number of users. Note that this is a similar concept to the high-SNR power offset considered in [17].

![Capacity vs log2 ln K](image)

Fig. 6. Capacity vs $\log_2 \ln K$ in uncorrelated and double-correlated scenario, with SNR $\tilde{\gamma} = 0$ dB and correlation parameters: $\sigma^2 = \sigma^2_t = \pi/30$, $\theta_t = \theta_r = \pi/2$, $d_t = d_r = 1/2$.

VI. CONCLUSION

We have derived accurate approximations for the capacity of multiuser MIMO-MRC systems with correlation. Our results indicate that correlation increases the capacity, compared with uncorrelated scenarios. We demonstrated that the presence of correlation results in a capacity offset, which we calculate analytically for large numbers of users.

APPENDIX

A. Proof of Theorem 2

We start with the exact maximum eigenvalue c.d.f. expression for semi-correlated Wishart matrices given in [5] which, after some simplification, can be written as

$$F_{\lambda_{\text{max}}}(x) = \alpha_{\text{semi}} \det(\mathbf{L})$$ (23)

where $\mathbf{L}$ is an $n \times n$ matrix with $(i,j)^{\text{th}}$ element

$$L_{i,j} = \frac{1}{\omega_j^{\ell-1}} \left( 1 - e^{-\frac{x}{\omega_j}} \sum_{k=0}^{m-i} \left( \frac{x}{\omega_j} \right)^k k! \right).$$ (24)

Now by repeated application of the multi-linear property of the determinant, we can show that

$$\det(\mathbf{L}) = \det(\mathbf{V}) + \sum_{\ell=1}^{n} (-1)^{\ell} \sum_{\{\beta\}} \det(\mathbf{V}_{\beta})$$ (25)

where $\{\beta\}$ is the set of all subsets

$$\beta_{\ell} = \{\beta_1 < \ldots < \beta_{\ell}\} \subseteq \{1, \ldots, n\},$$ (26)

and $\mathbf{V}_{\beta}$ is an $n \times n$ matrix with $(i,j)^{\text{th}}$ element

$$\mathbf{V}_{\beta_{i,j}} = \begin{cases} V_{i,j} & i \notin \beta_{\ell}, j = 1, \ldots, n \\ V_{i,j} e^{-x} \frac{x}{\omega_{i,j}} \sum_{k=0}^{m-i} \left( \frac{x}{\omega_{i,j}} \right)^k k! & i \in \beta_{\ell}, j = 1, \ldots, n \end{cases}$$ (27)

Note that the reason for performing the expansion in (25) is that it allows us to more easily identify dominant terms as $x$ grows large. In particular, if we expand the determinants on the right-hand side of (25) using the definition of the determinant, we see that terms corresponding to the lowest values of $\ell$ will dominate as $x$ increases, since these have the slowest exponential decay. Thus we only consider the case $\ell = 1$ which, when substituted into (23), gives

$$F_{\lambda_{\text{max}}}(x) = \alpha_{\text{semi}} \det(\mathbf{V}) - \alpha_{\text{semi}} \sum_{p=1}^{n} \sum_{q=1}^{n} e^{-\frac{x}{\omega_{pq}}}$$

$$\times D_{p,q} V_{p,q} \sum_{t=0}^{m-p} \left( \frac{x}{\omega_{pq}} \right)^t t! + \mathcal{O}(e^{-\frac{x}{\omega_{n-1}} x^{m-2}}).$$ (28)

Clearly $\alpha_{\text{semi}} \det(\mathbf{V}) = 1$. The desired result (13) now follows by only including $j = n$ in (28), which corresponds to taking only the terms corresponding to the largest eigenvalue $\omega_n$ (i.e. those with the slowest exponential decay).

B. Proof of Proposition 1

We first give the following theorem:

Theorem 4:

$$\lim_{K \to \infty} C_{\text{multi}} = \lim_{K \to \infty} \log_2 \left[ 1 + \tilde{\gamma} F^{-1}_{\lambda_{\text{max}}} \left( \frac{K}{K+1} \right) \right].$$ (29)

Proof: Sketch of Proof: From [12] we get $E(X_r) = F_X^{-1} \left( \frac{r}{K+1} \right)$ which holds for large $K$ where $X$ is a random variable, with
rth order statistic \(X_r\), and c.d.f \(F_X(\cdot)\). Letting \(X = \log_2(1 + \gamma_\lambda_{\text{max}})\), \(r = K\) and performing algebraic manipulation gives (29).

For large \(K\), we need only consider the c.d.f. expansions given in (12), (14) and (15), which are of the form \(F(x) = 1 - \alpha e^{-ax^2}\). In addition, these c.d.f. expansions approach the actual c.d.f. for large \(x\). Now the \(F_{\lambda_{\text{max}}}^{-1}\left(\frac{K}{K+1}\right)\) term in (29) can be solved by substituting \(N(K) = \frac{K}{K+1}\) and \(J(x) = x^t\) into the iteration (16) given in Section IV. As \(K \to \infty\), only one iteration is needed, since more iterations would produce nested \(\ln\) terms which have negligible effect. This gives for large \(K\)

\[
C_{\text{multi}} = \log_2 \left(1 + \frac{\gamma}{a} \left[\ln K + \ln \alpha + \mathcal{O}(\ln \ln K)\right]\right) \tag{30}
\]

\[
= \log_2 \ln K + \log_2 \left(\frac{\gamma}{a}\right) + \log_2 \left(\frac{a}{\gamma \ln K} + 1 + \frac{\ln \alpha}{\ln K} + \mathcal{O}\left(\frac{\ln \ln K}{\ln K}\right)\right)
\]

\[
= \log_2 \ln K + \log_2 \left(\frac{\gamma}{a}\right) + \log_2 \left(1 + \mathcal{O}\left(\frac{\ln \ln K}{\ln K}\right)\right)
\].

Note that the form used in (30) is similar to that given in [7]. The capacity offset is given as \(CO_\infty = \frac{1}{a}\) and the values of \(a\) for the uncorrelated, semi- and double-correlated scenarios can be found in (17), (18) and (19) respectively.

**REFERENCES**


