A Lattice-Theoretic Analysis of Vector Perturbation for Multi-User MIMO Systems

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Abstract—This paper considers the use of multiple transmit antennas to deliver independent data streams to multiple users. In particular, we examine a multi-user technique known as vector perturbation. We provide a new lattice-theoretic approach to analyze its performance in the presence of Rayleigh fading. Vector perturbation is based on performing a channel inversion, with the additional step of perturbing the data signal prior to linear precoding to significantly reduce the required transmit power. To analyze such systems it is necessary to calculate the resulting average energy of the sphere-encoded signal vector, as this determines the signal-to-noise ratio (SNR) at the output of the demodulator. Previous results presented in the literature were partially analytic, requiring further numerical evaluation. Here, we derive a concise approximation to the output SNR. We also provide tight upper and lower bounds on the bit error rate for the reception of QAM symbols using the required modulo demodulator, as a function of the average energy of the sphere-encoded signal vector.

I. INTRODUCTION

The ever-increasing demand for higher data rates in wireless systems has made it necessary to turn to the spatial dimension to improve the throughput of wireless channels. Towards this end, much research has been directed into the field of multiple-input multiple-output (MIMO) communications. These systems use multiple transmit and receive antennas to exploit the diversity of a rich multipath environment to produce point-to-point communication systems which exhibit a capacity that increases linearly with the minimum number of the transmit and receive antennas. More recent theoretical results (e.g. \cite{1,2}) have shown that such gains are also possible in the multi-user case, \textit{i.e.} when a cellular basestation or wireless LAN access point transmits using multiple antennas to \textit{K} non-colocated users, each with a single receive antenna. A promising practical technique known as vector perturbation has been proposed that achieves near-capacity \cite{3}. Vector perturbation is an attractive design choice compared to other multi-user techniques. It has a lower implementation complexity than dirty-paper coding methods \cite{2}, and has better performance than channel inversion or Tomlinson-Harashima precoding \cite{3}.

Vector perturbation is an extension of the simple channel inversion precoder, which simply premultiplies the data vector by the pseudoinverse of the channel matrix. With vector perturbation, the original data vector is constrained to lie within a \textit{K}-dimensional complex hypercube of side length 1, and is modified by addition of a \textit{perturbation vector} consisting of complex-valued integers. The perturbation vector is chosen so as to minimize the transmit power for each channel use. This minimization problem is an instance of the well-studied NP-hard problem of finding the closest lattice point (for an overview of such algorithms see \cite{4}). A common method to perform the search is the sphere-decoding algorithm. The resulting \textit{sphere-encoded} signal vector is then scaled to satisfy a power constraint. Finally, a modulo decoding process is applied independently at each user’s receiver which completely removes the perturbation.

We consider the case of a long term power constraint \cite{3,5}, which is an upper bound on the average of the transmitted energy over packets, or more specifically channel instances. To analyze and implement these systems we are required to calculate the average energy, \(\mathcal{E}_{se}\), of the sphere-encoded signals, as this determines the SNR at the output of the demodulator. Due to the fact that sphere-encoding is an NP-hard problem, it is unsurprising that calculating \(\mathcal{E}_{se}\), exactly has proved intractable, although some useful partially numerical results have been derived. A lower bound on \(\mathcal{E}_{se}\) for arbitrary \textit{K} and constellation was proposed in \cite{3}. This bound requires numerical simulation of the system to calculate, but does provide insight into the resulting direction of the perturbation vector. A suboptimal transmission method exploiting this property of the perturbation vector was proposed in \cite{6}. More recently, an expression for \(\mathcal{E}_{se}\) in the limit as \(K \rightarrow \infty\), was derived for a general class of constellations using the replica analysis method of statistical physics \cite{5}. However, this result requires numerical evaluation of a pair of coupled fixed-point (integral) equations, and therefore the insight into \(\mathcal{E}_{se}\) is limited.

In this paper we derive an approximation to \(\mathcal{E}_{se}\) for vector perturbation systems operating in a Rayleigh fading environment and analyze the bit error rate as a function of \(\mathcal{E}_{se}\). To make the analysis tractable we first assume that the data is uniformly distributed. We provide numerical simulation results to show that the value of \(\mathcal{E}_{se}\) obtained for uniformly distributed inputs is a good first order approximation to that obtained using standard square QAM constellations. Additionally, we
show that uniformly distributed inputs provide an upper bound on the value of $E_{se}$ which can be achieved by application of a dither signal. We then provide a new lower bound on $E_{se}$ for the case of uniformly distributed inputs over i.i.d. Gaussian fading channels, using a lattice-theoretic approach. We propose that this new result can be used as a good approximation to $E_{se}$ for other constellations. Finally, as an application of this result, we provide new tight upper and lower bounds on the resulting bit error rate for the modulo decoder, given $E_{se}$.

II. SYSTEM MODEL

We consider a multi-user, narrowband MIMO system with $N_T$ transmit antennas broadcasting to $K \leq N_T$ spatially dispersed users in the presence of Rayleigh fading. Each user has a single transmit antenna. For each packet, the channel realization $H \in \mathbb{C}^{K \times N_T}$ is chosen independently from other packets, and is assumed to be constant for the duration of a packet. The elements $h_{k,t}$ of $H$ represent the channel between the $k$th user and $t$th transmit antenna and are chosen according to an independent and identically distributed (i.i.d.) zero-mean circularly symmetric complex Gaussian distribution $\mathcal{C}N(0,1)$. We use $(\cdot)'$ to denote matrix transpose, $(\cdot)\dagger$ to denote matrix conjugate transpose, and $(\cdot)^+$ to denote Moore-Penrose pseudoinverse. We also denote the set of Gaussian (complex) integers as $\mathbb{Z}_C$ and use $[\cdot]_C$ to denote the element-wise rounding of a vector to the nearest Gaussian integer vector.

Given the transmitted vector $x = [x_1 \ldots x_{N_T}]' \in \mathbb{C}^{N_T \times 1}$, the received vector $y = [y_1 \ldots y_K]' \in \mathbb{C}^{K \times 1}$ is given by

$$y = Hx + n$$ (1)

where $n = [n_1 \ldots n_K]'$ is the vector of additive noise where each $n_k$ is $\mathcal{C}N(0,N_0)$. The transmitted vector $x$ is a modified “perturbed” and “precoded” form of the data vector $a = [a_1 \ldots a_K]$. A practical choice of constellation is that of $M$-ary square QAM constellations, where the real and imaginary components of each data symbol $a_k$, $k = 1, \ldots, K$ belong to the set

$$\{ \pm \frac{x}{2\mu} : x = 1, 3, \ldots, \mu - 1 \}.$$  

where $\mu \triangleq \sqrt{M}$.

For analytical purposes we will consider the case of uniformly distributed inputs where $a$ is an i.i.d. random variable with probability distribution function $p(a) = \chi_{\mathbb{C}}[a]$ where $\chi(\cdot)$ is the characteristic (indicator) function and $\mathbb{B}^K$ is the $K$-ary Cartesian product of the region

$$\mathbb{B} \triangleq \{ a : |\text{Re}\{a\}| < 0.5, \ |\text{Im}\{a\}| < 0.5 \}.$$ (2)

Clearly, $\text{Vol}\left(\mathbb{B}^K\right) = 1$.

To generate $x$, the data vector $a$ is first perturbed and then precoded to create the sphere-encoded signal vector, $s$, i.e.

$$s = H^+(a + p)$$ (3)

where $p$ is the Gaussian (complex) integer-valued perturbation vector given by

$$p = \arg\min_{q \in \mathbb{Z}_C^K} \|H^+(a + q)\|^2$$ (4)

where $\mathbb{Z}_C$ is the set of Gaussian (complex) integers. The minimization in (4) is the well-studied problem of finding the closest lattice point in an infinite lattice. For an overview of algorithms that solve this problem, we refer the reader to [4]. An optimal approach will have complexity exponential in $K$ e.g. the sphere-decoding algorithm of [7]. As $K$ increases, suboptimal methods of polynomial complexity may be employed, such as the lattice reduction based approach of [8], and the singular value decomposition based approach of [6]. For the purposes of analytical tractability, we assume that the algorithm used to solve (4) gives the optimal solution.

An optimal solution to (4) implies that $s$ will lie within the Voronoi region of the lattice point at the origin. That is, $s \in \mathcal{V}(\Lambda_{H^+},0)$ which denotes the Voronoi region about the origin of the lattice $\Lambda_{H^+}$ with generator matrix $H^+$. We note that without the sphere-encoding operation (i.e. when $p$ is 0), this that corresponds to a simple channel inversion. The value of $E_{se}$ obtained by channel inversion is much larger than for the vector perturbation case, and is actually infinite for the case of $N_T = K$ [9].

Finally, $s$ is scaled to give:

$$x = \frac{s}{\sqrt{E_{se}}}$$ (5)

where $E_{se} = E_{a,H}||s||^2$ is the average energy of the sphere-encoded vector $s$, where the expectation taken over $a$ and $H$. Similarly, $E_{s|H} \triangleq E[||s||^2 | \mathbf{H}]$ is the average energy of the sphere-encoded vector $s$ given $H$. The scaling ensures the average transmit vector energy $E_{a,H}||s||^2$ is 1. Hence, the resulting signal-to-noise ratio (SNR), $\rho$, at each receive antenna is

$$\rho = \frac{1}{N_0}$$ (6)

and is the same for each packet.

The data is recovered using a modulo demodulator:

$$\hat{a} = \sqrt{E_{se}}y - [\sqrt{E_{se}}y]_C$$ (7)

$$= a + p + \sqrt{E_{se}}n - [a + p + \sqrt{E_{se}}n]_C$$ (8)

$$= a + \sqrt{E_{se}}n - [a + \sqrt{E_{se}}n]_C.$$ (9)

Clearly, it is desirable to investigate $E_{se}$ to determine what effect the choice of transmit and receive antennas has on the noise power at the output of the modulo demodulator. Note that the only channel knowledge required at the user for detection is $E_{se}$, which is independent of the current data and channel state, and can therefore be easily estimated.

III. VECTOR PERTURBATION ANALYSIS

In this section we provide a simple closed-form approximation to the value of $E_{se}$. We first assume the case of uniformly distributed inputs and provide numerical simulation results to demonstrate that this approximates well the
value of $\mathcal{E}_{se}$ for practical QAM constellations. Using lattice-theoretic techniques we then provide an analytic lower bound on $\mathcal{E}_{se|H} \triangleq E_{se}|\|s\|^2 | \mathbf{H}$ under the assumption of uniformly distributed inputs. We then apply a random matrix theory result to derive a lower bound on $\mathcal{E}_{se}$.

### A. Uniformly Distributed Inputs

We now provide numerical simulation results to justify the assumption of uniformly distributed inputs used in our further analysis. This is achieved by showing that $\mathcal{E}_{se}$ does not vary much with the choice of constellation, and is well approximated by uniformly distributed inputs. In addition we will see that $\mathcal{E}_{se}$ does not vary much with $K$, as also observed in [3]. Finally, we show that for any constellation, $\mathcal{E}_{se|H}$, and consequently $\mathcal{E}_{se}$, can always be upper bounded by the value of $\mathcal{E}_{se}$ for uniformly distributed inputs by applying a dither signal.

Figure 1 provides values for $\mathcal{E}_{se}$ obtained by Monte Carlo simulation for systems with $N_T = K$ and $N_T = 2K$ transmit antennas where we consider $K = 2$ to 7 users. We simulate using QPSK, 16-QAM and 64-QAM constellations, and also for uniformly distributed inputs. For each plot over 100000 channel usages are simulated using at least 1000 independently generated channels.

We see that $\mathcal{E}_{se}$ does not vary much with the choice of constellation used, but does approach the value of $\mathcal{E}_{se}$ for the uniform input case as the constellation size increases. This can be explained intuitively by noting that the structure of the constellation is distorted by the precoding operation, so that the elements of $s$ are placed in a distorted fashion over $\mathcal{V}(\Lambda_{H^+}, 0)$, giving a value of $\mathcal{E}_{se|H}$ similar to that obtained by choosing the elements uniformly over $\mathcal{V}(\Lambda_{H^+}, 0)$. We also confirm the observation of [3] that $\mathcal{E}_{se}$ does not vary much with $K$, and converges as $K$ increases.

As an additional result we show how a dither signal can be applied to improve the performance of the system if the chosen constellation results in a value of $\mathcal{E}_{se}$ that is greater than the value obtained using uniformly distributed inputs. The dither signal $\mathbf{d}$ is an i.i.d. uniformly distributed random variable over $\mathbb{B}^K$. This approach is often used in lattice-theoretic quantization and communications problems e.g. [10]. The dither $\mathbf{d}$ is added to $\mathbf{a}$, and then the same precoding operation in (3) is performed,

$$s = H^+ (\mathbf{a} + \mathbf{d} + \mathbf{p})$$

where we recall that $\mathbf{p} \in \mathbb{Z}_C^K$ is the perturbation vector chosen to minimize $\|s\|^2$. It follows that $s$ is uniformly distributed over $\mathcal{V}(\Lambda_{H^+}, 0)$, as is the case for uniformly distributed inputs, and hence $\mathcal{E}_{se|H}$ is achievable. For detection at the receiver, the dither vector can be generated by a pseudo-random noise sequence generator synchronized with one at the transmitter. The scaled dither vector $\sqrt{\mathcal{E}_{se|H}} \mathbf{d}$ is subtracted prior to applying the modulo demodulator.
of dimension $\dim(R)$ and volume $V(R)$. This is the same principal used to derive Zador’s quantization lower bound \cite{11}. Now, the volume and second moment of a region $R$ are related as follows:

$$G(R) \triangleq \frac{\sigma^2(R)}{\dim(R)V(R)^{2/\dim(R)}}$$

where $G(R)$ is the dimensionless second moment \cite[Ch. 21]{12} and $\dim(R)$ is the dimension of $R$. Now, denote $S_L$ as the $L$-dimensional sphere in $\mathbb{R}^L$ of unit volume. The normalized second moment is given by \cite[Ch. 21]{12, 21}

$$G(S_L) = \frac{\Gamma(L/2 + 1)^{2/L}}{(L + 2)\pi}. \quad (13)$$

Hence,

$$E[\|s\|^2]_{\mathbb{H}} \geq 2KG(S_{2K})\det(W)^{-1/K} \quad (14)$$

$$= \frac{KT(G(K + 1)^{1/K}(K + 1)^{1/K})\det(W)^{-1/K}}{(K + 1)^{1/K}}.$$

In Figure 2 we plot the ratio $\mathcal{E}_{se}|_{\mathbb{H}}/\mathcal{E}_{se}|_{\mathbb{H},LB}$ for uniformly distributed inputs, obtained via numerical simulation. We consider the following scenarios: $2 \times 2$, $4 \times 2$, and $4 \times 4$ systems. For each configuration of users and transmit antennas, we simulate 20000 channels, where for each channel instance a packet consisting of 100 i.i.d. symbols is transmitted to each user. We see that for the scenarios considered $\mathcal{E}_{se}|_{\mathbb{H}}/\mathcal{E}_{se}|_{\mathbb{H},LB} = 1.5$dB all the curves are below 10%, implying that our proposed lower bound $\mathcal{E}_{se}|_{\mathbb{H},LB}$ is within 1.5dB of the actual value $\mathcal{E}_{se}|_{\mathbb{H}}$ for at least 90% of the channel instances.

To derive a lower bound on $\mathcal{E}_{se}$ we note that the $h$th moment of a complex Wishart distributed matrix $W \sim \mathcal{W}_K(N_T, I)$ is given by \cite[Thm 2.11]{13}

$$E[\det(W)^h] = \prod_{\ell=0}^{K-1} \frac{\Gamma(N_T + h - \ell)}{\Gamma(N_T - \ell)}. \quad (15)$$

We set $h = -1/K$ and take the expectation of the lower bound in (11) to obtain the following corollary.

**Corollary 1:** The average unnormalized transmit energy $\mathcal{E}_{se}$ of a wireless MIMO vector perturbation system with $N_T$ transmit antennas and $K$ independent users with uniformly distributed inputs satisfies

$$\mathcal{E}_{se} \geq \mathcal{E}_{se,LB} \triangleq \frac{KTK}{(K + 1)^{1/K}} \prod_{\ell=0}^{K-1} \frac{\Gamma(N_T - \frac{1}{K} - \ell)}{\Gamma(N_T - \ell)}. \quad (16)$$

Corollary 1 provides a closed-form lower bound for $\mathcal{E}_{se}$ for arbitrary $K$ for uniformly distributed inputs. We note that this is in contrast to the lower bound for arbitrary $K$ presented in [3] and the asymptotic result in [5], both of which require further numerical evaluation. Importantly, returning to Figure 1, we see that the lower bound is a good approximation for the actual value of $\mathcal{E}_{se}$, and hence using this lower bound to approximate $\mathcal{E}_{se}$ is valid.

**IV. BIT ERROR RATES**

In this section we derive the bit error rate for the multi-user vector perturbation system using Gray-coded square-QAM constellations using the modulo demodulator in (9).

**Lemma 2:** The probability of bit error, $P_b$, for a vector perturbation system with $N_T$ transmit antennas and $K$ users using an $M$-ary square QAM constellation satisfies

1. $P_b \geq P_{b,LB} \triangleq \frac{4}{\log_2 M} \left[ Q\left(\sqrt{2}\varphi\right) - Q\left(\sqrt{2}(2\mu - 1)\varphi\right)\right],$
2. $P_b \leq P_{b,UB} \triangleq \frac{2}{\log_2 M} \left[ Q\left(\sqrt{2}\varphi\right) + (\mu - 2)Q\left(3\sqrt{2}\varphi\right)\right],$
3. $\lim_{\rho \to \infty} P_b = \frac{4}{\log_2 M} Q\left(3\varphi\right).$

where

$$\varphi \triangleq \frac{1}{2\mu} \sqrt{\frac{\rho}{\mathcal{E}_{se}}}. \quad (17)$$

**Proof:** Without loss of generality, we consider just the real component of the received signal at a single user. The resulting probability of bit error is obtained by summing the contribution of the probability of a bit error event over the nearest neighbor region of each point in the repeated PAM constellation of infinite extent. We obtain

$$P_b = \sum_{q=-\infty}^{\infty} \sum_{\ell=1}^{\mu} \frac{\varphi}{\log_2 M} d(q\mu + \ell) G_{q,\ell} \quad (17)$$

where $d(m)$ is the Hamming distance between the $n$th and $(n + m)$th symbol in the Gray-coded repeated $\mu$-ary PAM constellation, and

$$G_{q,\ell} \triangleq \frac{1}{\log_2 M} \left[ F\left(q + 2\ell + 1\right) - F\left(q + 2\ell - 1\right)\right]. \quad (18)$$

where

$$F(x) = 1 - Q(x) = \frac{1}{2} + \frac{1}{2} \text{erf}\left(\frac{x}{\sqrt{2}\sigma^2}\right).$$
is the cumulative distribution function (c.d.f.) of a zero-mean Gaussian random variable with variance \(\sigma^2 = \mathcal{E}_{sc}N_0/2\), and \(Q(x)\) is the \(Q\)-function. We now consider the lower bound of the first part of the lemma,

\[
P_b = \sum_{\ell=1}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{\ell=1}^{\ell} d(qu + \ell) \times (G_{q,\ell} - G_{q,\ell-1})
\]

\[
\geq 2 \sum_{\ell=1}^{\infty} [\text{erf}((2\ell + 1)\varphi) - \text{erf}((2\ell - 1)\varphi)]
\]

\[
= 2[\text{erf}((2\mu - 1)\varphi) - \text{erf}(\varphi)]
\]

where the first inequality follows from the fact that \(d(\ell) \geq 1\) for all the error events, and the second inequality is obtained by considering only \(q = \pm 1\) in the summation. We now consider the upper bound. Expanding (17) gives

\[
P_b = G_{0,1} + \sum_{\ell=2}^{\infty} d(\ell)G_{0,\ell} + \sum_{q=-\infty}^{\infty} \sum_{\ell=1}^{\ell} d(qu + \ell)G_{q,\ell}
\]

\[
\leq 2G_{0,1} + 2(\mu - 1) \sum_{\ell=2}^{\infty} G_{0,\ell} + 2(\mu - 1) \sum_{q=-\infty}^{\infty} \sum_{\ell=0}^{\ell} d(qu + \ell)G_{q,\ell}
\]

\[
= \frac{2}{\log_2 \mu} \left[\text{erf}(3\varphi) - \text{erf}(\varphi)\right] + \frac{2(\mu - 1)}{\log_2 \mu} \text{erfc}(3\varphi)
\]

\[
\leq \frac{2}{\log_2 \mu} \left[\text{erfc}(\varphi) + (\mu - 2) \text{erfc}(3\varphi)\right]
\]

which gives the upper bound. The third part of the proof follows directly by taking the limits of the upper and lower bounds.

In Fig. 3 we plot the probability of bit error as a function of SNR for various vector perturbation systems. We consider the case of \(K = 2\) users using either \(N_T = 2\) or 4 transmit antennas. The values of \(\mathcal{E}_{sc}\) in the bounds are obtained using the Monte Carlo simulation method of Fig. 3. We see that the upper and lower bounds of Lemma 2 are tight, and the limit is reached. It is clear from Lemma 2 and the figure that the effect of changing \(\mathcal{E}_{sc}\) is simply to shift the error rate curve and not change the slope. We see that for this scenario, doubling the number of transmit antennas improves the performance by a factor of approximately 7dB. Comparing this to the effect of increasing the constellation size, we see that we can increase the data rate by 2 bits per user per channel usage, and maintain the same BER by doubling the number of transmit antennas.

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