Spatial Multiplexing with MMSE Receivers in Ad Hoc Networks

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Abstract—The performance of spatial multiplexing systems with linear minimum-mean-squared-error receivers is investigated in ad hoc networks. We present new exact closed-form expressions for the outage probability and transmission capacity. These expressions reveal that from a transmission capacity perspective, single-stream transmission is preferable over multi-stream transmission.

I. INTRODUCTION

Multiple antennas can offer significant performance improvements in wireless communication systems by providing higher data rates and more reliable links. A practical method which can achieve high data rates is to employ spatial multiplexing transmission in conjunction with low complexity linear receivers, such as the minimum-mean-squared-error (MMSE) receiver. The MMSE receiver is particularly important as it uses its receive degrees of freedom (DOF) to optimally trade off strengthening the energy of the desired signal of interest and reducing unwanted interference, such that the signal-to-interference-and-noise ratio (SINR) is maximized. Although MMSE receivers have been well investigated for simple network scenarios [1], a thorough characterization of their performance in more complicated network scenarios, such as ad hoc networks, is currently lacking.

In this paper, we investigate spatial multiplexing systems with MMSE receivers in ad hoc networks. In particular, each transmitting node sends separate data streams to their corresponding receiver. At the receiver, a bank of linear MMSE filters are used to independently decode, without successive interference cancelation (SIC), each transmitted data stream. To facilitate this decoding, the transmitter-receiver channel and the interference covariance matrix are assumed to be known at each receiver. However, we note that no channel information is required at the transmitters.

We consider a scenario where the transmitting nodes are spatially distributed according to a homogeneous Poisson point process (PPP) on an infinite 2-D plane. The transmission capacity will be investigated, which measures the maximum number of successful transmissions per unit area, assuming transmission at a fixed data rate, such that a target outage probability $\epsilon$ is attained for each data stream.

We present two key contributions in this paper. First, we derive exact closed-form expressions for the outage probability and transmission capacity of spatial multiplexing systems with a bank of linear MMSE receivers. Second, using these expressions, we show that from a transmission capacity perspective, single-stream transmission is preferable when the optimal linear MMSE processing strategy is employed.

Prior work on single-stream transmission with multiple receive antennas in ad hoc networks and Poisson distributed transmitting nodes include [2–6], where spectral efficiency and transmission capacity scaling laws were presented for different receiver structures. In [2], receive antennas are used for spatial diversity to increase the desired signal power, while in [3], receive antennas are used to cancel interference from the strongest interferer nodes. In [4–6], for both sub-optimal and MMSE linear receivers, the transmission capacity was shown to scale linearly with the number of receive antennas. In this paper, we extend these prior contributions to derive new outage probability and transmission capacity scaling laws for arbitrary number of data streams using MMSE receivers.

Multi-stream transmission with multiple receive antennas have been considered in [7–10]. In [7], MMSE receivers were used to reduce interference from all interfering nodes. However, analysis was limited to a per-link performance measure and the large antenna regime. In [8, 9], receive antennas were used to cancel interference from the corresponding transmitter, but not the interferers. For these papers, the transmission capacity as $\epsilon \to 0$ was shown to behave as $\frac{1}{\epsilon} \frac{1}{x^2}$, thus indicating that for low outage probability operating values, PZF receivers achieve a higher transmission capacity than the receivers in [8, 9]. This transmission capacity result was used to show that single-stream transmission is preferable over multi-stream transmission. Our work differs from [10] by showing single-stream optimality for the optimal MMSE receiver. We will investigate the performance advantages of the MMSE receiver over the sub-optimal PZF receiver in this paper.

II. SYSTEM MODEL

We consider an ad hoc network comprising of transmitter-receiver pairs, where each transmitter communicates to its...
corresponding receiver in a point-to-point manner, treating all other transmissions as interference. The transmitting nodes are distributed spatially according to a homogeneous PPP of intensity \( \lambda \) (transmitting nodes per unit area) in \( \mathbb{R}^2 \), and each receiving node is randomly placed at a distance \( d_0 \) away from its corresponding transmitter.

In this paper, we investigate network-wide performance. To characterize this performance, it is sufficient to focus on a typical transmitter-receiver pair, denoted by index 0, with the typical receiver located at the origin. The transmitting nodes, with the exception of the typical transmitter, constitute a marked PPP, which by Slivnyak’s theorem has the same distribution as the original PPP [11] (i.e., removing the typical transmitter from the transmit process has no effect). This is denoted by \( \Phi = \{ (D_\ell, H_\ell), \ell \in \mathbb{N} \} \), where \( D_\ell \) and \( H_\ell \) model the location and channel matrix respectively of the \( \ell \)th transmitting node with respect to (w.r.t) the typical receiver. The transmitted signals are attenuated by a factor \( 1/r^\alpha \) with distance \( r \), where \( \alpha > 2 \) is the path loss exponent.

We consider a spatial multiplexing system where each transmitting node sends \( N_t \) independent data streams through \( N_t \) different antennas to its corresponding receiver, which is equipped with \( N_r \) antennas. Focusing on the \( k \)th stream, the received \( N_r \times 1 \) signal vector at the typical receiver can be written as

\[
y_{0,k} = \sqrt{\frac{1}{d_0^\alpha}} h_{0,k} x_{0,k} + \sqrt{\frac{1}{d_0^\alpha}} \sum_{q=1}^{N_t} h_{0,q} x_{0,q} + \sum_{D_i \in \Phi} \sqrt{\frac{1}{|D_i|^{\alpha}}} \sum_{q=1}^{N_t} h_{\ell,q} x_{\ell,q} + n_{0,k} \tag{1}
\]

where \( x_{\ell,q} \) is the symbol sent from the \( q \)th transmit antenna of the \( \ell \)th transmitting node satisfying \( \mathbb{E}[|x_{\ell,q}|^2] = P \), \( h_{\ell,q} \) is the \( q \)th column of \( H_\ell \) \( \sim \mathcal{C}\mathcal{N}_{N_r,N_t} \) \( (0_{N_r \times N_t}, I_{N_r}) \) and \( n_{0,k} \) \( \sim \mathcal{C}\mathcal{N}_{N_r,1} \) \( (0_{N_r \times 1}, I_{N_r}) \) is the complex additive white Gaussian noise vector. We see in (1) that the received vector includes: (a) the desired data to be decoded, (b) the self-interference from the typical transmitter and (c) the interference from the other transmitting nodes.

To obtain an estimate for \( x_{0,k} \), we consider the use of MMSE linear receivers. The data estimate is thus given by

\[
\hat{x}_{0,k} = h_{0,k}^H R_{0,k}^{-1} y_{0,k},
\]

where the SINR is written as

\[
\text{SINR}_{0,k} = \frac{\gamma}{d_0^{\alpha}} h_{0,k}^H R_{0,k}^{-1} h_{0,k} \tag{2}
\]

and \( \gamma = \frac{P}{N_0} \) is the transmit SNR. We assume that each receiving node has knowledge of the corresponding transmitter channel \( H_0 \) and the interference (plus noise) covariance matrix \( R_{0,k} \). The practicalities of this assumption are discussed in [5].

### III. Outage Probability

We consider the per-stream outage probability, defined for the \( k \)th stream as the probability that the mutual information for the \( k \)th stream lies below the data rate threshold \( R_k \). At the receiver, the MMSE filter outputs are decoded independently.

We assume the data rate thresholds for all streams are the same and equal to \( R \). The outage probability for each stream can thus be written as

\[
F_Z(z, \lambda) = \Pr(\text{SINR} \leq z) \tag{4}
\]

where \( z = 2^R - 1 \) is the SINR threshold. Note that we have dropped the subscript \( k \) and 0 from the SINR term as the per-stream outage probability is the same for each stream at each receiving node.

Before deriving an expression for the outage probability, we first introduce some notation and concepts from number theory. The integer partitions of a positive integer \( k \) are defined as the different ways of writing \( k \) as a sum of positive integers [12]. For example, the integer partitions of 4 are: i) 4, ii) 3+1, iii) 2+2, iv) 2+1+1 and v) 1+1+1+1. We denote \( h_{i,j,k} \) as the \( i \)th summand of the \( j \)th integer partition of \( k \), \( |h_{i,j,k}| \) as the number of summands in the \( j \)th integer partition of \( k \) and \( |h_{j,k}| \) as the number of integer partitions of \( k \). For example, when \( k = 4 \), we have \( h_{2,3,4} = 2, h_{2,4,4} = 1, |h_{3,4}| = 2 \) and \( |h_4| = 5 \).

It is also convenient to introduce the concept of non-repeatable integer partitions, defined as the integer partitions without any repeated summands. For example, the non-repeatable partitions of 4 are given by i) 4, ii) 3+1, iii) 2, iv) 2+1 and v) 1. We denote \( g_{i,j,k} \) as the number of times the \( i \)th summand of the \( j \)th non-repeatable integer partition of \( k \) is repeated in the \( j \)th integer partition of \( k \) and \( |g_{j,k}| \) as the number of summands in the \( j \)th non-repeatable partition of \( k \). For example, when \( k = 4 \), we have \( g_{1,3,4} = 2, g_{1,5,4} = 4 \) and \( |g_{3,4}| = 1 \). With these definitions, we present the following theorem for the outage probability\(^3\).

**Theorem 1:** The per-stream outage probability of spatial multiplexing systems with MMSE receivers is given by

\[
F_Z(z, \lambda) = 1 - e^{-\frac{\gamma}{d_0^{\alpha}}} \frac{e^{-\Theta N_r \lambda}}{(1 + z)^{N_r - 1}} \sum_{p=0}^{N_r - 1} \left( \frac{\gamma}{(u - 1)!} \sum_{q=0}^{\min(p, N_r - 1)} \left( N_t - 1 \right)^q z^q \right) \prod_{j=1}^{q} \Xi_j, p-q(1 - \Theta N_r \lambda)^{|h_{j,p-q}|} \tag{5}
\]

where

\[
\Xi_{j,p,q} = \frac{\prod_{i=1}^{|h_{i,j}|} \prod_{i=1}^{|h_{i,j}|} (N_t - q + 1)(q - k)}{\prod_{i=1}^{|h_{i,j}|} g_{i,j,p,q}!} \tag{6}
\]

\(^3\)Note that the outage probability for the specific case where \( N_t = 1 \) was recently derived independently in [6].
The transmission capacity is given in the following theorem.

**Theorem 2:** The transmission capacity of spatial multiplexing systems with MMSE receivers as \( \epsilon \to F_{SU}(z) \) is given by

\[
c(\epsilon) = \frac{N_t R}{\Theta_{N_t} \Omega} \left( \epsilon - F_{SU}(z) \right) + o \left( \left( \epsilon - F_{SU}(z) \right) \right)
\]

where \( \ell = \left\lfloor \frac{N_t}{N_t - 1} \right\rfloor \), \( \Psi_{p,\ell} \) is the set of all integer partitions of \( p \) with \( \ell \) summands, \( \left\lfloor \cdot \right\rfloor \) denotes the floor function and

\[
\Omega = \frac{1}{\ell!} \sum_{q=0}^{N_t-1} \left( \frac{N_t-1}{q} \right) \sum_{p=\ell}^{N_t-1-q} \sum_{j \in \Psi_{p,\ell}} \Xi_{j,p}.
\]

**Proof:** The result is proven by taking a first order expansion of the outage probability in (5) around \( \lambda = 0 \), followed by substituting the resultant expression into (8). The full proof is omitted due to space limitations. 

By observing that the exponent of \( \epsilon \) in (10) is a decreasing function of the number of data streams, we see that for low outage probability operating values, the transmission capacity is maximized when only a single data stream is used for transmission. This indicates that to maintain a desired outage for a fixed number of data streams per unit area \( N_t \lambda(\epsilon) \), it is preferable to have a high density of single-stream transmissions rather than a low density of multi-stream transmissions when using the optimal MMSE receiver in ad hoc networks.

Figs. 2 and 3 plot the transmission capacity vs. outage probability and path loss exponents respectively when thermal noise is neglected, i.e., \( \gamma \to \infty \). We observe in both figures that the transmission capacity is a decreasing function of the number of transmit antennas for all outage probabilities and path loss exponents. In Fig. 2, the ‘Analytical’ curves are plotted using (10), and closely match the ‘Numerical’ curves for outage probabilities as high as \( \epsilon = 0.1 \), which were obtained by numerically taking the inverse of \( F_Z(z, \lambda) \) w.r.t. \( \lambda \), and substituting the resulting expression into (8). Fig.
2 indicates that the optimality of single-stream transmission is not just applicable to small outage probability operating values, but the whole range of outage probabilities considered, i.e. $0.0001 \leq \epsilon \leq 0.8$.

Fig. 3 indicates that with all other parameters kept fixed, the transmission capacity increases with the path loss exponent. This is interesting since whilst the quality of the desired transmit-receive links is reduced by a higher path-loss factor, the effect of interference is also reduced. Our results indicate that this interference reduction has a more significant effect on the overall network performance.

V. COMPARISON WITH THE PZF SCHEME IN [10]

In this section, we investigate key performance differences between the MMSE receiver we consider in this paper, and the (sub-optimal) PZF receiver considered in [10]. In particular, the PZF receiver cancels a fixed and finite subset of “strong” interferers, whereas the MMSE receiver aims to reduce interference from all interferers when maximizing the SINR. It is thus clear that the optimal MMSE receiver will have better performance, regardless of the operating SNR. As there are an infinite number of interferers4, the PZF receiver is incapable of completely canceling the interference from all interferers.

The performance advantages of MMSE over PZF receivers can be seen in Figs. 4 and 5, which respectively plot the outage probability vs. SINR threshold $z$, and the transmission capacity vs. number of antennas $N_r$. For both figures, single-stream transmission is considered, i.e., $N_t = 1$, and thermal noise is neglected5. All curves are generated by Monte-Carlo simulations. We observe that for both figures, the MMSE receiver significantly outperforms the PZF receiver.

VI. CONCLUSION

We derived outage probability and transmission capacity expressions for spatial multiplexing systems with optimal MMSE linear receivers in random ad-hoc networks. These expressions are in closed-form, applying for arbitrary numbers of transmit and receive antennas. We showed that from a transmission capacity perspective, it is preferable to have a high density of single-stream transmissions than a low density of multi-stream transmissions when using the optimal MMSE receiver in ad hoc networks.

APPENDIX

The outage probability, conditioned on $x_i = |D_i|^\alpha < a$, where $x_i$ are independent and identically uniformly distributed

\begin{align*}
&\text{Number of receive antennas, } N_r  \\
&\text{Transmission capacity, } c(\epsilon)  \\
&\text{SINR threshold, } z \text{ dB}  \\
&\text{Outage probability, } F(z, \lambda/N_t)  \\
&\text{Path loss exponent, } \alpha  \\
&\text{Transmission capacity vs. path loss exponent for spatial multiplexing systems with MMSE receivers, and with } N_r = 4, d_0 = 1 \text{ and } \epsilon = 0.001.  \\
&\text{Outage probability vs. SINR threshold, with } N_t = 1, N_r = 4, \alpha = 3 \text{ and } d_0 = 1.  \\
&\text{Transmission capacity vs. number of receive antennas, with } N_t = 1, \epsilon = 0.01, \alpha = 3, z = 1 \text{ dB and } d_0 = 1.
\end{align*}
with \( i = 1, \ldots, L \), is given by [1]

\[
F_{Z|\{x_1, \ldots, x_L\}}(z, \lambda) = 1 - e^{-\frac{\lambda}{\alpha} \sum_{p=0}^{N_1-1} \left( \sum_{u=1}^{N_u-p} \left( \frac{z_d^u}{\gamma} \right)^{v-1} \right)} \times z^p d_0^{p+1} P_p(x_1, \ldots, x_L, \lambda)
\]

(12)

where

\[
P_p(x_1, \ldots, x_L, \lambda) = \frac{C_p(x_1, \ldots, x_L, \lambda)}{(1 + z)^{N_1-1} \prod_{i=1}^{L} (1 + d_0^i x_i^{-1} z)^{N_i}}
\]

(13)

and \( C_p(x_1, \ldots, x_L, \lambda) \) is the coefficient of \( z^p \) in the polynomial expansion of \((1 + d_0^{-\alpha} z)^{N_1-1} \prod_{i=1}^{L} (1 + x_i^{-1} z)^{N_i}\).

By noting that the number of nodes follows a Poisson distribution with intensity \( \lambda \), the expected value of \( P_p(x_1, \ldots, x_L, \lambda) \) w.r.t. \( x_1, \ldots, x_L \) is given by

\[
E[P_p(x_1, \ldots, x_L, \lambda)] = \frac{e^{-\lambda \sum_{q=0}^{L} \left( N_q - 1 \right)}}{(1 + z)^{N_1-1} \prod_{i=1}^{L} (1 + d_0^i x_i^{-1} z)^{N_i}} \sum_{L=0}^{\infty} \frac{\left( \frac{2 \lambda \gamma}{\alpha} \right)^L}{L!}
\]

 integrals of

\[
\int_0^\infty \cdots \int_0^\infty C_p(x_1, \ldots, x_L, \lambda) \prod_{i=1}^{L} \left( \frac{x_i + d_0^i z}{x_i} \right)^{N_i} dx_1 \cdots dx_L
\]

(14)

To solve the integral in (14), it is convenient to define the following function:

\[
\mathcal{J}_c := \int_0^x \frac{x^{N_1+\frac{2}{\alpha} - 1}}{(x + d_0^i z)^{N_1}} dx
\]

(15)

subject to \( \xi < N_1 + \frac{2}{\alpha} \) where \( \mathcal{F}_1(\cdot; \cdot; \cdot) \) is the regularized generalized Gauss hypergeometric function [13]. Now substituting (15) into (14), we obtain

\[
E[P_p(x_1, \ldots, x_L, \lambda)] = \frac{e^{-\lambda \sum_{q=0}^{L} \left( N_q - 1 \right)}}{(1 + z)^{N_1-1} \prod_{i=1}^{L} (1 + d_0^i x_i^{-1} z)^{N_i}} \sum_{L=0}^{\infty} \frac{\left( \frac{2 \lambda \gamma}{\alpha} \right)^L}{L!} \prod_{i=1}^{L} \left( \frac{x_i + d_0^i z}{x_i} \right)^{N_i} \Delta_L
\]

(16)

where

\[
\Delta_L = \sum_{L=0}^{\infty} \frac{\left( \frac{2 \lambda \gamma}{\alpha} \right)^L}{(L - [h_{j,p}])!} \left( \frac{2 \lambda \gamma}{\alpha} \right)^{[h_{j,p}]} \frac{1}{e^{2 \lambda \gamma \theta_0}}
\]

(17)

To proceed, we take the limit as \( \alpha \to \infty \) in (16), since we are considering an infinite plane. It is thus convenient to note the following two limit functions,

\[
\lim_{a \to \infty} \left( \frac{2 \lambda \gamma}{\alpha} \right)^{[h_{j,p}]} e^{-2 \lambda \gamma \theta_0} = e^{\left( -\frac{2 \lambda \gamma}{\alpha} \right)^{[h_{j,p}]}}
\]

\[
\lim_{a \to \infty} \left( \frac{2 \lambda \gamma}{\alpha} \right)^{[h_{j,p}]} \frac{1}{\Gamma(N_1)} = \exp \left( -\frac{2 \lambda \gamma}{\alpha} \frac{1}{\Gamma(N_1)} \right)
\]

(18)

and

\[
\lim_{a \to \infty} 2 \lambda \gamma \theta_0 \frac{1}{\alpha} = -\left( \frac{d_0^i z}{\alpha} \right)^{[h_{j,p}]} \theta(N_1) \prod_{k=1}^{c} \frac{k - \frac{2}{\alpha}}{N_1 + \frac{2}{\alpha} - k}.
\]

(19)

Substituting (18) and (19) into (16), and substituting the resultant expression into (12), we obtain the desired result.

REFERENCES


