Non-isolated quasi-degrees

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Received 15 August 2008, revised 1 December 2008, accepted 3 December 2008
Published online 16 November 2009

Key words Computably-enumerable degrees, quasi-reducibility, 2-c.e. degrees, isolated degrees.

MSC (2000) 03D28, 03D30

We show that non-isolated from below 2-c.e. \( Q \)-degrees are dense in the structure of c.e. \( Q \)-degrees. We construct a 2-c.e. \( Q \)-degree, which can’t be isolated from below not only by c.e. \( Q \)-degrees, but by any \( Q \)-degree. We also prove that below any c.e. \( Q \)-degree there is a 2-c.e. \( Q \)-degree, which is non-isolated from below and from above.

1 Introduction

A set \( A \subseteq \omega \) is called 2-computably enumerable (2-c.e.), if there are computably enumerable sets \( A_1 \) and \( A_2 \) such that \( A_1 - A_2 = A \).

A set \( A \subseteq \omega \) is called quasi-reducible to a set \( B \subseteq \omega \) (\( A \leq Q B \)), if there is a computable function \( g \) such that for all \( x \in \omega \) we have \( x \in A \) if and only if \( W_g(x) \subseteq B \). This reducibility was introduced by Tennenbaum (see [1], p.207) as an example of a reducibility which differs from \( T \)-reducibility on the class of computably enumerable sets.

A set \( A \) is quasi-equivalent to a set \( B \) (\( A \equiv Q B \)), if \( A \leq Q B \) and \( B \leq Q A \). It is not hard to see that the relation \( \equiv Q \) is an equivalence relation. The class of all sets, which are quasi-equivalent to a set \( A \), is called the quasi-degree (\( Q \)-degree) of \( A \) and usually is denoted by a small italic letter \( a \).

The \( Q \)-degree of a 2-c.e. set is called a properly 2-c.e. degree, if it doesn’t contain any c.e. sets.

The interest to the study of the algebraic structure of \( Q \)-degrees arose after results of Dobritsa and Belegradek (see [2]). It follows from these results that every obtained property of the structure gives a property of the classes of finitely generated subgroups of algebraically closed groups. The further study of the algebraic structure of \( Q \)-degrees was conducted in [3], [4] and [5].

In this paper we study the isolation property of the algebraic structure of \( Q \)-degrees. The results of the paper supplement the results of [6]. In [6] we showed that isolated from below and from above 2-c.e. \( Q \)-degrees are dense in the structure of c.e. \( Q \)-degrees; in this paper we show that non-isolated from below 2-c.e. \( Q \)-degrees are also dense in the structure of c.e. \( Q \)-degrees and that non-isolated from above 2-c.e. \( Q \)-degrees are downward dense in the structure of c.e. \( Q \)-degrees.

We adopt the standard notational conventions, found, for instance, in [6]. In particular, we write \([s]\) after functionals and formulas to indicate that every functional or parameter therein is evaluated at stage \( s \).

We also use the following definition:

Definition 1.1

(i) A degree \( d \) is called isolated from below, if there is a c.e. degree \( b < Q d \) such that for all c.e. degree \( a \), if \( a \leq Q d \), then \( a \leq Q b \). A degree \( d \) is called non-isolated from below, if there is no such c.e. degree \( b < Q d \).

(ii) A degree \( d \) is called isolated from above, if there is a c.e. degree \( b > Q d \) such that for all c.e. degree \( a \), if \( a \geq Q d \), then \( b \leq Q a \). A degree \( d \) is called non-isolated from above, if there is no such c.e. degree \( b > Q d \).

We also assume in the statements of our theorems that none of the sets below is \( \omega \), since \( \omega \) has a \( Q \)-degree strictly below that of any other set.

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2 Non-isolated from below degrees

The study of isolated from below $T$-degrees was started in 1995 by Cooper and Yi (unpublished). They established the following results:

(i) There exists an isolated from below 2-c.e. $T$-degree

(ii) There exists a non-isolated from below properly 2-c.e. $T$-degree.

Arslanov, Lempp and Shore in [7] showed that non-isolated from below $T$-degrees are dense in the structure of c.e. $T$-degrees. The following theorem shows that this results holds for $Q$-degrees too.

**Theorem 2.1**

Given c.e. degrees $v <_Q u$ there is a properly 2-c.e. degree $d$ such that $v <_Q d <_Q u$, and, for any c.e. degree $w$, if $w <_Q d$, then $\alpha <_Q w$ for some c.e. degree $\alpha <_Q d$.

**Proof.** Let $V \in v$ and $U \in u$ be c.e. sets. We construct c.e. sets $D_1, D_2, D_2 \subseteq D_1$, such that the degree of $(D_1 - D_2) \oplus V$ will be the desired degree.

To ensure that $D_1 - D_2 <_Q U$ we construct a uniformly c.e. sequence of c.e. sets $\{U_x\}_{x \in \omega}$ such that $x \in D_1 - D_2$ if and only if $U_x \subseteq U$. For every $x \in \omega$ we enumerate at any stage $s$, if $x \notin (D_1 - D_2)[s]$, some $y \notin U_s$ in $U_s$. Afterwards we let $x$ to be enumerated in $D_1$, only if $y$ is enumerated in $U$. If we need to enumerate $x$ in $D_2$, we just enumerate some fixed $u \notin U$ in $U_x$. The reducibility $D_1 - D_2 \leq Q U$ is guaranteed by the computable function $f$, defined in the following way: $W_f(x) = U_x$.

To ensure that the degree of $(D_1 - D_2) \oplus V$ is non-isolated from below we satisfy for every $c, i$ the requirement:

$$P_{(c,i)} : W_e \leq Q (D_1 - D_2) \oplus V \text{ via } \Phi_i \Rightarrow (\exists \text{ c.e. } A_{e,i} \leq Q (D_1 - D_2) \oplus V)(\forall j)(A_{e,i} \leq Q W_e \text{ via } \Phi_j).$$

Here $\{W_e, \Phi_i, \Phi_j\}_{e \in \omega}$ is some enumeration of all possible three-tuples of c.e. sets $W_e$ and partially-computable functions $\Phi_i$ and $\Phi_j$.

To ensure that the degree of $(D_1 - D_2) \oplus V$ is a properly 2-c.e. degree we satisfy for all $c, i, j \in \omega$ the requirement:

$$N_{(c,i,j)} : D_1 - D_2 \leq Q W_e \text{ via } \Phi_i \& W_e \leq Q (D_1 - D_2) \oplus V \text{ via } \Phi_j \Rightarrow U \leq Q V.$$

By conventions of the theorem, $U \not\leq Q V$, thus this requirement guarantees that either $D_1 - D_2 \not\leq Q W_e$ via $\Phi_i$, or $W_e \not\leq Q (D_1 - D_2) \oplus V$ via $\Phi_j$.

**The basic module for the requirement $N_{(c,i,j)}$.**

To satisfy the requirement we shall define a partially computable function $s_{c,i,j}$ such that either $D_1 - D_2 \not\leq Q W_e$ via $\Phi_i$, or $W_e \not\leq Q (D_1 - D_2) \oplus V$ via $\Phi_j$, or $s_{c,i,j}$ is computable and $U \leq Q V$ via $s_{c,i,j}$. During the construction we construct a uniformly c.e. sequence of c.e. sets $\{Z_{c,i,j,n}\}_{n \in \omega}$, and for all $n$ define $s_{c,i,j}(n)$ as the index of $Z_{c,i,j,n}$.

For a convenience we first rewrite the requirement in a following way:

$$\exists x \Phi_i(x), \text{ or}$$

$$\exists x (x \notin D_1 - D_2 \& W_{\Phi_i(x)} \subseteq W_e), \text{ or}$$

$$\exists x (x \in D_1 - D_2 \& W_{\Phi_i(x)} \not\subseteq W_e), \text{ or}$$

$$\exists y \Phi_j(y), \text{ or}$$

$$\exists y (y \notin W_e \& W_{\Phi_j(y)} \subseteq (D_1 - D_2) \oplus V), \text{ or}$$

$$\exists y (y \in W_e \& W_{\Phi_j(y)} \not\subseteq (D_1 - D_2) \oplus V), \text{ or}$$

$$\forall k (k \in U \leftrightarrow Z_{c,i,j,k} \subseteq U).$$

We use an $\omega$-sequence of cycles $0, 1, 2, \ldots$, where each cycle acts as follows:

(1) Choose an unused witness $z_k \notin D_1$, and define $U_{z_k} = k$.

(2) Wait for a stage $s_0$ such that $k \in U_{s_0}$ or $\Phi_i(z_k) \downarrow [s_0]$, there exists $y_k \in W_{\Phi_i(z_k)} - W_e[s_0]$ such that $\Phi_j(y_k) \downarrow [s_0]$ and there exists $z_k \in W_{\Phi_j(y_k)} - (D_1 - D_2) \oplus V[s_0].$

(3a) If $k \in U_{s_0}$, then leave $Z_{c,i,j,k} = \emptyset$ and open the cycle $k + 1$. Enumerate in $U_{z_k}$ the fixed element $\bar{u} \notin U$.

(3b) If $z_k = 2n_k + 1$ for some $n_k$, then enumerate $n_k$ in $Z_{c,i,j,k}[s_0]$ and open the cycle $k + 1$. 
(4b) Wait until either $n_k$ is enumerated in $V$, or $k$ is enumerated in $U$. In the former case close all cycles $> k$ and return to the step (2), in the latter case enumerate $x_k$ in $D_1$, close all cycles $> k$, wait until $n_k$ is enumerated in $V$, and if this happens, open the cycle $k + 1$.

(3c) If $z_k = 2n_k$ for some $n_k$, and $n_k \neq x_k$, then enumerate the fixed $\bar{v} \notin V$ in $Z_{c, i, j, k}$, restrain $n_k$ from to be enumerated in $D_1$ and open the cycle $k + 1$.

(4c) If later $k$ is enumerated in $U$, then enumerate $x_k$ in $D_1$ and close all cycles $> k$.

(3d) If $z_k = 2n_k$ for $n_k = x_k$, then enumerate the fixed $\bar{v} \notin V$ in $Z_{c, i, j, k}$ and open the cycle $k + 1$.

(4d) If later $k$ is enumerated in $U$, then enumerate $x_k$ in $D_1$, close all cycles $> k$ and wait for a stage $s_2 > s_1$ such that $y \in W_e[s_2]$.

(5d) If this happens, then enumerate $x_k$ in $D_2$ and enumerate the fixed $\bar{u} \notin U$ in $U_{x_k}$.

The module has the following possible outcomes:

(A) Some (least) cycle $k$ either gets stuck at step (2) or at step (4d), or reaches step (4c), or reaches step (5d). In this case either $\Phi_e(z_k) \uplus$, or $\Phi_f(y) \uplus$, or $x_k \notin D_1 - D_2 & W_{\Phi_e(x_k)} \subseteq W_e$, or $x_k \notin D_1 - D_2 & W_{\Phi_e(x_k)} \subseteq W_e$, or $y \notin W_e \& W_{\Phi_f(y)} \subseteq (D_1 - D_2) \oplus V$, or $y \in W_e \& W_{\Phi_f(y)} \subseteq (D_1 - D_2) \oplus V$. $x_k$ witnesses that $D_1 - D_2 \not\subseteq W_e$ via $\Phi_e$ or witnesses that $W_e \not\subseteq Q (D_1 - D_2) \oplus V$ via $\Phi_f$, the requirement is satisfied.

(B) Some (least) cycle $k$ infinitely many times proceeds from the step (4b) to the step (2). In this case $x_k \notin D_1$ and $W_{\Phi_e(x_k)} \subseteq W_e$, the requirement is satisfied.

(C) Each cycle stops at one of the steps (3a), (3b), (3c), (3d) or (4b), and during the construction we open infinitely many cycles. In this case $U \subseteq Q V \forall s_{e, i, j}$, contrary to conventions of the theorem.

The basic module for the requirement $P_{(e, i)}$.

To ensure that $A_{e, i} \subseteq Q (D_1 - D_2) \oplus V$, we construct a uniformly c.e. sequence of c.e. sets $\{Y_{e, i, x} \mid x \in \omega\}$ such that $x \in A_{e, i}$ if and only if $Y_{e, i, x} \subseteq (D_1 - D_2) \oplus V$. The reducibility $A_{e, i} \subseteq Q (D_1 - D_2) \oplus V$ is guaranteed by the computable function $h_{e, i}$, defined in the following way: $W_{h_{e, i}(x)} = Y_{e, i, x}$.

The requirement $P_{(e, i)}$ is divided up into subrequirements $P_{(e, i, j)}$:

$$P_{(e, i, j)} : W_e \not\subseteq Q (D_1 - D_2) \oplus V \text{ via } \Phi_i \text{ or } A_{e, i} \not\subseteq Q W_e \text{ via } \Phi_j.$$

In satisfying the subrequirement we shall define a partially computable function $g_{e, i, j}$ such that either $W_e \not\subseteq Q (D_1 - D_2) \oplus V \text{ via } \Phi_i$, or $A_{e, i} \not\subseteq W_e \text{ via } \Phi_j$, or $g_{e, i, j}$ is computable and $U \subseteq Q V \forall g_{e, i, j}$. During the construction we construct a uniformly c.e. sequence of c.e. sets $\{X_{e, i, j, n} \mid n \in \omega\}$, and for all $n$ define $g_{e, i, j}(n)$ as the index of $X_{e, i, j, n}$: $W_{g_{e, i, j}(n)} = X_{e, i, j, n}$.

For a convenience we first rewrite the subrequirement $P_{(e, i, j)}$ in a following way:

$$\exists k \left( \Phi_i(k) \uplus k \notin W_e \& W_{\Phi_i(k)} \subseteq (D_1 - D_2) \oplus V \right) \lor$$

$$\exists k \left( \Phi_f(k) \uplus k \notin A_{e, i} \& W_{\Phi_f(k)} \subseteq W_e \right) \lor$$

$$\forall k \left( k \in U \implies X_{e, i, j, k} \subseteq V \right).$$

We use an $\omega$-sequence of cycles $\{0, 1, 2, \ldots\}$, where each cycle $k$ acts as follows:

1. Choose an unused witness $x_k \notin D_1$, $x_k \notin A_{e, i}$ and define $U_{x_k} = \{ k \}, Y_{e, i, x_k} = \{ x_k \}$.

2. Wait for a stage $s_0$ (or $\Phi_f_x(k) \downarrow s_0$), there exists $y_k \notin W_{\Phi_i(x_k)} - W_e[s_0]$ such that $\Phi_i(y_k)$ and there exists $z_k \in W_{\Phi_i(y_k)} - (D_1 - D_2) \oplus V[s_0]$.

3a) If $k \in U_{x_k}$, then leave $X_{e, i, j, k} = \emptyset$ and open the cycle $k + 1$. Enumerate in $U_{x_k}$ the fixed element $\bar{u} \notin U$. (3b) If $z_k = 2n_k + 1$, then enumerate some $n_k$, then enumerate $n_k$ in $X_{e, i, j, k}[s_0]$ and open the cycle $k + 1$.

4b) Wait until $n_k$ is enumerated in $V$ or $k$ is enumerated in $U$. In the former case close all cycles $> k$ and return to the step (2), in the latter case, enumerate $x_k$ in $D_1$, enumerate $x_k$ in $A_{e, i}$, close all cycles $> k$ and wait until $n_k$ is enumerated in $V$. If this happens, open the cycle $k + 1$.

3c) If $z_k = 2n_k$ for some $n_k$ and $n_k \neq x_k$, then enumerate the fixed $\bar{v} \notin V$ in $X_{e, i, j, k}[s_0]$, restrain $n_k$ from to be enumerated in $D_1$ and open the cycle $k + 1$.

4c) If later $k$ is enumerated in $U$, then enumerate $x_k$ in $D_1$, enumerate $x_k$ in $A_{e, i}$ and close all cycles $> k$.

5d) If there is such a stage, then enumerate $x_k$ in $D_2$ and enumerate the fixed $\bar{u} \notin U$ in $U_{x_k}$. 

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The module $\mathcal{P}_{\langle e, i, j \rangle}$ has the following possible outcomes:

(A) Some (least) cycle $k$ either gets stuck at step (2) or at step (4d), or reaches step (4c), or reaches step (5d).

In this case either $\Phi_j(x_k) \uparrow$, or $\Phi_i(y) \uparrow$, or $x_k \not\in A_{e,i}$ & $W_{\Phi_j(x_k)} \subseteq W_e$, or $x_k \in A_{e,i}$ & $W_{\Phi_j(x_k)} \not\subseteq W_e$, or $y \not\in W_e$ & $W_{\Phi_i(y)} \subseteq (D_1 - D_2) \oplus V$, or $y \in W_e$ & $W_{\Phi_i(y)} \not\subseteq (D_1 - D_2) \oplus V$. $x_k$ witnesses that $A_{e,i} \subseteq Q_x W_e$ via $\Phi_i$ or $y$ witnesses that $W_e \not\subseteq (D_1 - D_2) \oplus V$ via $\Phi_i$, the requirement is satisfied. Note, that in case of reaching step (5), we injure the reducibility $A_{e} \subseteq Q_x (D_1 - D_2)$, however we guarantee that $W_e \subseteq (D_1 - D_2) \oplus V$ via $\Phi_i$ and still satisfy the main requirement $\mathcal{P}_{\langle e, i \rangle}$.

(B) Some (least) cycle $k$ infinitely many times proceeds from the step (4b) to the step (2). In this case $x_k \not\in A_{e,i}$ and $W_{\Phi_i(x_k)} \subseteq W_e$, the requirement is satisfied.

(C) Each cycle stops at one of the steps (3a), (3b), (3c), (3d) or (4b), and during the construction we open infinitely many cycles. In this case $U \subseteq Q_x V$ via $s_{e,i,j}$, contrary to conventions of the theorem.

**Interactions between the requirements.**

Combine the requirements $\{N_{e,i,j} \}_{e,i,j \in e}$ and $\{P_{e,i,j} \}_{e,i,j \in e}$ in a one $\omega$-list $\{Q_n \}_{n \in \omega}$: $Q_0 = N_0, Q_1 = P_0, Q_n = N_1, Q_1 = P_{1 \ldots}$.

The only possible conflict in activities of strategies of $N_{e,i,j}$ and $N_{\langle e',i',j' \rangle}$, or of strategies of $N_{e,i,j}$ and $P_{\langle e',i',j' \rangle}$, or of strategies of $P_{\langle e',i',j' \rangle}$ and $P_{\langle e'',i'',j'' \rangle}$ for some $e,i,j,e',i',j'$ is the following: one strategy needs to enumerate at one of the steps (4b), (4c) or (4d) of some cycle $k$ an element $x_k$ in $D_1$ which is restrained by the other strategy at the step (3c) of some cycle $k'$ (i.e. $x_k = n_{k'}$).

Let, for instance, $N_{e,i,j}$ needs to enumerate $x_k$ in $D_1$ at one of the steps of a cycle $k$, and $P_{\langle e',i',j' \rangle}$ restrains to do it, since $x_k$ is equal to $n_{k'}$ of some cycle $k'$ (the other three possibility can be handled in the same way).

Let $(e,i,j) < (e',i',j')$. If $N_{e,i,j}$ needs to enumerate $x_k$ at the step (4c) or (4d), then we enumerate $x_k$ in $D_1$ and initialize the requirement $P_{\langle e',i',j' \rangle}$. In this case the requirement $N_{e,i,j}$ will have the (A)-outcome and will never injure requirements of lower priority again. If $N_{e,i,j}$ needs to enumerate $x_k$ at the step (4b), it is possible that $N_{e,i,j}$ has the (B)-outcome because of some cycle $l < k$ and will open infinitely many cycles which need to enumerate various elements in $D_1$ at the step (4b). This conflict is settled by a small modification of the strategy: enumerate $x_k$ in $D_1$, and if later $k'$ will be enumerated in $U$, we wait until $n_k$ is enumerated in $V$ and open the cycle $k + 1$. If it doesn’t happen, then $N_{e,i,j}$ is satisfied once and for all by the witness $x_k$ and the function $g_{e',i',j'}$ of the requirement $P_{\langle e',i',j' \rangle}$ will not reduce $U$ to $V$ only on $k'$, and since there are only finitely many requirements with higher priority, it will not reduce $U$ to $V$ only on a finite set of elements. If $n_k$ is enumerated in $V$, then enumerate $x_k = n_k$ in $D_2$, this will satisfy the requirement $P_{\langle e',i',j' \rangle}$.

If $(e,i,j) > (e',i',j')$, and $P_{\langle e',i',j' \rangle}$ has the (B)-outcome, then it is possible that the strategy will open infinitely many cycles and restrain infinitely many elements. Is conflict is also settled by a small modification of the strategy: enumerate $x_k$ in $D_1$ and if later $k'$ is enumerated in $U$, then enumerate $x_k = n_{k'}$ in $D_2$. In this case the strategy of the requirement $P_{\langle e',i',j' \rangle}$ will stop its activity at the step (4e) of the cycle $k'$, and we will have $x_{k'} \in D_1, x_{k'} \not\in (D_1 - D_2) \oplus V$, and regardless of whether $y_k$ belongs to $W_e$ or not, the requirement $P_{\langle e',i',j' \rangle}$ will have the (A)-outcome, and we just initialize the requirement $N_{e,i,j}$.

**Construction.** Define the set of injuries of the reducibility $U \subseteq Q_x V$ for the requirements $P_{\langle e, i, j \rangle}$:

$$\{ \langle m \notin U \mid \exists (e', i', j') (\langle e', i', j' \rangle < \langle e, i \rangle) \wedge X_{e,i,j,m,s} \not\subseteq V_k \wedge (\langle m = v_{e',i',j',k'} \vee m = v_{e',i',j',k''} \rangle) \},$$

where $v_{e',i',j',k'}$ is a witness of $N_{\langle e',i',j' \rangle}$, and $v_{e',i',j',k''}$ is a witness of $P_{\langle e',i',j' \rangle}$. Define for the requirements $P_{\langle e, i, j \rangle}$ the length function $l(e, i, j, s) = \max \{ l \mid l \leq n \} I \not\subseteq M_{e,i,j,s} \rightarrow (l \not\subseteq M_{e,i,j,s} \rightarrow (l \not\subseteq M_{e,i,j,s} \wedge X_{e,i,j,l} \not\subseteq V[s])) + 1$ and the maximal length function $m(e, i, j, s) = \max l(e, i, j, t)$. By the same way define the set $M_{e,i,j,s}^{l(e, i, j, s)}$ and functions $l'(e, i, j, s)$ and $m'(e, i, j, s)$ for the requirements $N_{e,i,j}$.

**Stage $s = 0$.** For all $e, i, j, n \in \omega$ define $X_{e,i,j,n,0} = Y_{e,i,j,n,0} = Z_{e,i,j,n,0} = U_{e,i,j,n,0} = U_{e,i,j,0} = A_{e,i,j,0} = D_{1,0} = D_{1,0} = 0$.

For every cycle $k$ of every requirement $N_{e,i,j}$ choose a witness $v_{e,i,j,k}^{\langle 0, e, i, j, k \rangle}$ for every cycle $k$ for every requirement $P_{\langle e, i, j \rangle}$ choose a witness $\theta_{e,i,j,k}^{\langle 1, e, i, j, k \rangle} = \langle 1, e, i, j, k \rangle$. Fix elements $\hat{u} \not\in U, \hat{v} \not\in V$ and $d \not\in D_1$.

**Stage $s + 1$.** For all $e, i, j < s$, if $U_{e,i,j,s} = \emptyset$ and $x \not\in D_{1,s}$, then enumerate $\hat{u}$ in $U_{x,s+1}$; if $Y_{e,i,j,s} = \emptyset$ and $x \not\in A_{e,i,j,s}$, then enumerate $d$ in $Y_{e,i,j,s+1}$. For all $(e, i, j) < s$, if $k = l(e, i, j, s) \not\subseteq U_s$ and there is no $P$-associated with $k$ witness, then $P$-associate with $k$ the least $t = t_{e,i,j,k}^{l(e,i,j,s)} \not\subseteq D_{1,s}$, which is greater than all numbers so far mentioned in the construction, and
define $U_{v,s+1} = \{k\}$. For all $(e,i,j) < s$, if $k = l'(e,i,j,s) \not\in U_s$ and there is no $N$-associated with $k$ witness, then $N$-associate with $k$ the least $v = v_{e,i,j,k}^s \not\in D_{1,s}$, which is greater than all numbers so far mentioned in the construction, and define $U_{v,s+1} = \{k\}$.

For all $e, i, j, k < s$, if there is a $P$-associated with $k$ witness $t = t_{e,i,j,k}^s \not\in D_{1,s}$, $k \in U_s$ and $Z_{e,i,j,k} = 0$, then define $U_{t,s+1} = U_{v,s} \cup \{t\}$. For all $e, i, j, k < s$, if there is a $N$-associated with $k$ witness $v = v_{e,i,j,k}^s \not\in D_{1,s}$, $k \in U_s$ and $Z_{e,i,j,k} = 0$, then define $U_{v,s+1} = U_{v,s} \cup \{t\}$.

For all $(e,i,j) < s$, if $k = l(e,i,j,s) \not\in U_s$, $t = t_{e,i,j,k}^s \not\in \Phi_j(t)$, there exists $y \in W_{\Phi_j(t)} - W_v[s]$, such that $\Phi_j(y) \in [s]$, there exists $z = z_{e,i,j,k}^s \in W_{\Phi_j(y)} - (D_1 - D_2) \oplus V[s]$ and

(a) if $z = 2n + 1$ for some $n = n_{e,i,j,k}^s$, then define $X_{e,i,j,k,s+1} = X_{e,i,j,k,s} \cup \{n\}$,
(b) if $z = 2n$ for some $n = n_{e,i,j,k}^s \neq t$, then define $X_{e,i,j,k,s+1} = X_{e,i,j,k,s} \cup \{t\}$,
(c) if $z = 2n$ for $n_{e,i,j,k} = t$, then define $X_{e,i,j,k,s+1} = X_{e,i,j,k,s} \cup \{t\}$.

For all $(e,i,j) < s$, if $k = l'(e,i,j,s) \not\in U_s$, $t = u_{e,i,j,k}^s \neq v$, then define $X_{e,i,j,k,s+1} = X_{e,i,j,k,s} \cup \{v\}$.

For all $(e,i,j) < s$, if there exists $k < m(e,i,j,s) \in U_s$ such that $Z_{e,i,j,k} \not\in V[s]$, $t = t_{e,i,j,k}^s$, then define $D_{1,s+1} = D_{1,s} \cup \{t\}$, $A_{e,i,j,s+1} = A_{e,i,j,s} \cup \{t\}$ and initialize all requirements $N_{e',i',j'}$ and $P_{e',i',j'}$ such that $(e,i,j) < (e',i',j')$. If this happens, we say that the requirement $R_{e,i,j}$ has received an attention.

For all $(e,i,j) < s$, if there exists $k < m(e,i,j,s) \in U_s$ such that $Z_{e,i,j,k} \not\in V[s]$, $v = v_{e,i,j,k}^s$, then define $D_{1,s+1} = D_{1,s} \cup \{v\}$ and initialize all requirements $N_{e',i',j'}$ and $P_{e',i',j'}$ such that $(e,i,j) < (e',i',j')$. If this happens, we say that the requirement $R_{e,i,j}$ has received an attention.

For all $(e,i,j) < s$, if there exists $x \in (D_1 - D_2)[s]$ such that $\Phi_j(x) \in [s]$ and there exists $y \in W_{\Phi_j(x)} \cap W_v[s]$ such that $\Phi_j(y) \in [s]$ and $2x \in W_{\Phi_j(x)}[s]$, then define $D_{2,s+1} = D_{2,s} \cup \{x\}$ and $U_{x,s+1} = U_{x,s} \cup \{k\}$.

For all $(e',i',j') < s$, if there exists $x \in (D_1 - D_2)[s]$ such that $x = v_{e',i',j',k'}^s$, or $x = t_{e',i',j',k'}^s$ for some $k'$ and for some $e, i, j$ such that $(e,i,j) \leq (e',i',j')$ there exists $k \in U_s$ such that $\Phi_j(t_{e,i,j,k}) \in \{t\}$, there exists $y \in W_{\Phi_j(t_{e,i,j,k})[s]}$ such that $\Phi_j(y) \in \{t\}$ and there exists $z \in W_{\Phi_j(y)}[s]$ such that $2x = z$, then define $D_{2,s+1} = D_{2,s} \cup \{x\}$, $U_{x,s+1} = U_{x,s} \cup \{k\}$ and initialize all requirements $N_{e',i',j',m'}$ and $P_{e',i',j',m'}$ such that $(e,i,j) < (e',i',j',m')$. If this happens, we say that the requirement $R_{e,i,j}$ has received an attention.

For all $(e',i',j') < s$, if there exists $x \in (D_1 - D_2)[s]$ such that $x = v_{e',i',j',k'}^s$, or $x = t_{e',i',j',k'}^s$ for some $k'$ and for some $e, i, j$ such that $(e,i,j) \leq (e',i',j')$, there exists $k \in U_s$ such that $\Phi_j(v_{e,i,j,k}^s) \in \{t\}$, there exists $y \in W_{\Phi_j(v_{e,i,j,k}^s)}[s]$ such that $\Phi_j(y) \in \{t\}$ and there exists $z \in W_{\Phi_j(y)}[s]$ such that $2x = z$, then define $D_{2,s+1} = D_{2,s} \cup \{x\}$, $U_{x,s+1} = U_{x,s} \cup \{k\}$ and initialize all requirements $N_{e',i',j',m'}$ and $P_{e',i',j',m'}$ such that $(e,i,j) < (e',i',j',m')$. If this happens, we say that the requirement $R_{e,i,j}$ has received an attention.

For all $(e,i,j) < s$, if there is $k \in U_s$ such that $x \in (D_1 - D_2)[s]$ for $x = v_{e,i,j,k}^s$ or $x = t_{e,i,j,k}^s$ and for some $e',i',j'$ such that $(e,i,j) \leq (e',i',j')$, there is $k' \in U_s$ such that $x = n_{e',i',j',k'}^s$ and $n_{e',i',j',k'} \in V_s$, then define $D_{2,s+1} = D_{2,s} \cup \{x\}$ and $U_{x,s+1} = U_{x,s} \cup \{k\}$.

**Verification.** Define $D_1 = \bigcup_{e \in \omega} D_{1,s}$, $D_2 = \bigcup_{e \in \omega} D_{2,s}$ and for all $e, i, A_{e,i} = \bigcup_{s \in \omega} A_{e,i,s}$. Let us prove that $\mathcal{Q}$-degree of the set $D = (D_1 - D_2) \oplus V$ satisfies the conditions of the theorem.

**Lemma 2.2** Let $Q_{n}$ be satisfied for all $n < (e,i,j)$, then $R_{e,i,j}$ will be satisfied and only finitely many times will receive an attention.

Let $s$ be the least stage such that the requirements $Q_{n}$, $n < (e,i,j)$, don’t receive any attention after it. Let $D_1 - D_2 \leq Q_{e} \cup W_v \leq_\mathcal{Q} (D_1 - D_2) \oplus V$ via $\Phi_j$. We show that in this case $U \leq \mathcal{Q} V$ via $\Phi_j$ defined as indices of sets $Z_{e,i,j,n} = W_{s_{e,i,j}}[n] = Z_{e,i,j,n}$. Let us suppose that this function doesn’t reduce $U$ to $V$ at some (the least) element $k$. 


Let us suppose that \( k \notin U \). Consider \( v = v_{e,i,j,k}^{n+1} \notin D_{1,e,i,j,k} \): since \( D_1 - D_2 \leq Q W_e \), there exists a stage \( t_0 \) such that \( \Phi_i(v) \upharpoonright [t_0] \), there exists \( y \in W_{\Phi_i(v)} - W_e[t_0] \), \( y \notin W_e \), such that \( \Phi_j(y) \upharpoonright [t_0] \) and there exists \( z \in W_{\Phi_j(y)} - (D_1 - D_2) \oplus V[t_0] \) such that either \( z = 2n + 1 \) for some \( n \) and \( n \notin V \), or \( z = 2n \) and \( n \notin D_1 - D_2[t_0] \). In the former case, by the construction, \( n \in Z_{e,i,j,k,t_0+1} \not\subseteq V \). In the latter case, by the construction, \( n \in Z_{e,i,j,k,t_0+1} \not\subseteq V \).

Let us suppose that \( k \in U \) and there exists \( x \in Z_{e,i,j,k} \). Let \( t_0 > s \) be the stage such that \( x \in \{z \in Z_{e,i,j,k} : t \leq s \} \) and \( x \notin Z_{e,i,j,k} \). Let \( t_0 > s \) be the stage such that \( x \in \{z \in Z_{e,i,j,k} : t \leq s \} \). Then \( t_0 > s \) be the stage such that \( x \in \{z \in Z_{e,i,j,k} : t \leq s \} \). Then \( t_0 > s \) be the stage such that \( x \in \{z \in Z_{e,i,j,k} : t \leq s \} \).

**Lemma 2.3** Let \( Q_n \) be satisfied for all \( n < \langle e, i, j \rangle \), then \( P_{e,i,j} \) will be satisfied and only finitely many times will receive an attention.

Let \( s \) be the least stage such that the requirements \( Q_{n+1} \), \( n < \langle e, i, j \rangle \), don’t receive any attention after it. Let \( W_s \leq Q (D_1 - D_2) \oplus V \) via \( \Phi_i \) and \( A_{e,i} \leq Q W_e \) via \( \Phi_j \). We show that in this case \( U \leq Q V \) via \( A_{e,i} \) defined as indices of sets \( X_{e,i,j,k,n} : W_{A_{e,i,j,k,n}} = X_{e,i,j,k,n} \). Let us suppose that this function doesn’t reduce \( U \) to \( V \) at some (the least) element \( k \).

Let \( k \notin U \). Consider \( t = (t_{e,i,j,k}^{n+1} \in A_{e,i,j,k} \) since \( A_{e,i} \leq Q W_e \), there exists a stage \( s_0 \) such that \( \Phi_j(t) \upharpoonright [s_0] \), there exists \( y \in W_{\Phi_j(t)} - W_e[s_0] \), \( y \notin W_e \), such that \( \Phi_i(y) \upharpoonright [s_0] \) and \( x \in W_{\Phi_j(y)} - (D_1 - D_2) \oplus V[s_0] \) such that either \( z = 2n + 1 \) for some \( n \) and \( n \notin V \), or \( z = 2n \) and \( n \notin D_1 - D_2[s_0] \). In the former case, by the construction, \( n \in Z_{e,i,j,k,s_0+1} \not\subseteq V \). In the latter case, by the construction, \( n \in Z_{e,i,j,k,s_0+1} \not\subseteq V \).

Conversely, let us suppose that \( k \in U \) and there exists \( x \in X_{e,i,j,k} \). Let \( s_0 > s \) be the stage such that \( x \in X_{e,i,j,k,s_0} \) and \( x \notin X_{e,i,j,k,s_0} \) then for \( t = (t_{e,i,j,k}^{n+1} \in A_{e,i,j,k} \) we have \( \Phi_j(t) \upharpoonright [s_0] \), there exists \( y \in W_{\Phi_j(t)} - W_e[s_0] \) such that \( \Phi_i(y) \upharpoonright [s_0] \) and \( x \in W_{\Phi_j(y)} - (D_1 - D_2) \oplus V[s_0] \). Let \( s_1 > s_0 \) be the stage such that \( k \in U_{s_1} - U_{s_1} \). Then, by the construction, \( t = A_{e,i+j,s_1} \).

If \( x = 2n + 1 \) for some \( n \) or \( x = 2n \), then either \( y \notin W_e \) and then \( A_{e,i} \leq Q W_e \) via \( \Phi_j \), or \( y \notin W_e \) and then \( W_e \leq Q (D_1 - D_2) \oplus V \) via \( \Phi_i \), contrary to the hypothesis.

If \( x = 2n \), then either \( y \notin W_e \) and \( v \notin D_2 \) since \( t_1 > s \), and, hence, \( D_1 - D_2 \leq Q W_e \) via \( \Phi_j \), or \( y \notin W_e[t_2] \) for some \( t_2 \) and, by the construction, \( v \in D_{2,t_2+1} \), \( W_e \leq Q (D_1 - D_2) \oplus V \) via \( \Phi_j \), contrary to the hypothesis.

**Corollary 2.4** Non-isolated from below 2-c.e. Q-degrees are dense in the structure of c.e. Q-degrees.

**Proof.** Let \( v \leq Q u \) be two c.e. degrees. By the theorem, there is a properly 2-c.e. degree \( d \) such that \( v \leq Q d \leq Q u \) and, for any c.e. degree \( w \), if \( w \leq Q d \), then \( a \leq Q w \) for some c.e. degree \( a \leq Q d \). By the Definition 1.1 \( d \) is non-isolated from below.

The following theorem shows that there exists a 2-c.e. Q-degree, which can’t be isolated from below by any Q-degree (not only by any c.e. Q-degree).

**Theorem 2.5** There exists a properly 2-c.e. degree \( d \) such that for all \( b \leq Q d \) there exists a c.e. degree \( a \leq Q d \) such that \( a \leq Q b \).

**Proof.** We construct c.e. sets \( D_1, D_2, D_2 \subseteq D_1 \) such that the degree of \( D = (D_1 - D_2) \) will be the desired degree.

Every set \( B \leq Q D_1 - D_2 \) can be presented as \( \{x \in W_{\Phi(x)} \subseteq D_1 - D_2 \} \), where \( \Phi \) is the computable function which reduce \( B \) to \( D_1 - D_2 \). Let \( \Phi \subseteq \{x \in \omega \} \) be the enumeration of all partially-computable functions. For every \( e \) define \( B_e = \{x \Phi(e,x) \upharpoonright \} \) and \( W_{\Phi(x)} \subseteq D_1 - D_2 \), thus the sequence \( \{B_e \} \) will include all sets reducible to \( D_1 - D_2 \). Define the approximation of the set \( B_e \) at the stage \( s \): \( B_{e,s} = \{x \Phi(e,x) \upharpoonright \} \) and \( W_{\Phi(x),s} \subseteq D_1 - D_2 \) for every \( e, s \in \omega \).
For every $c$ we construct a c.e. set $A_c \leq_Q D$ in order to satisfy for every $i$ the following requirement:

$$\mathcal{P}_{(e,i)} : (\exists x)(\Phi_e(x) \not\in W_e) \lor (A_c \leq_Q B_e \text{ via } \Phi_i \Rightarrow D_1 - D_2 \leq_Q B_e).$$

To ensure that $A_c \leq_Q D_1 - D_2$, we construct a uniformly c.e. sequence of c.e. sets $\{U_{e,x}\}_{x \in e}$ such that $x \in A_c$ if and only if $U_{e,x} \subseteq D_1 - D_2$. The reducibility $A_c \leq Q D_1 - D_2$ is guaranteed by the computable function $h_e$, defined in the following way: $W_{h_e(x)} = U_{e,x}$. 

To ensure that the degree of $D_1 - D_2$ is a properly 2-c.e. degree we satisfy for every $e, i, j \in \omega$ the following requirements:

$$\mathcal{N}_{(e,i,j)} : D_1 - D_2 \not\leq_Q W_e \text{ via } \Phi_i \lor W_e \not\leq_Q D_1 - D_2 \text{ via } \Phi_j.$$

The basic module for the requirement $\mathcal{N}_{(e,i,j)}$.

For a convenience we first rewrite the requirement:

$$\exists x \Phi_i(x) \not\in W_e \lor \exists x(x \not\in D_1 - D_2 \land W_{\Phi_i(x)} \subseteq W_e),$$

$$\exists x(x \in D_1 - D_2 \land W_{\Phi_i(x)} \not\subseteq W_e),$$

$$\exists y \Phi_j(y) \not\in W_e \lor \exists y(y \not\in D_1 - D_2),$$

$$\exists y(y \in D_1 - D_2 \land W_{\Phi_j(y)} \not\subseteq W_e).$$

(1) Choose an unused witness $x_k \not\in D_1$.

(2) Wait for a stage $s_0$ such that $\Phi_i(x_k) \not\in [s_0]$ and there exists $y \in W_{\Phi_i(x_k)} - W_e[s_0]$.

(3) Wait for a stage $s_1 > s_0$ such that $\Phi_j(y) \not\in [s_1]$ and there exists $z \in W_{\Phi_j(y)} - (D_1 - D_2)[s_1]$.

(4a) If $z \not= x_k$, then enumerate $x_k$ in $D_1$ and restrain $z$ from to be enumerated in $D_1$.

(4b) If $z = x_k$, then enumerate $x_k$ in $D_1$ and wait for a stage $s_2 > s_1$ such that $y \in W_e[s_2]$.

(5) If this happens, then enumerate $x_k$ in $D_2$.

The basic module for the requirement $\mathcal{P}_{(e,i)}$.

In satisfying the requirement we shall define a partially computable function $g_{e,i}$ such that either $\Phi_e \not\in W_e$ or $A_e \not\leq_Q B_e$ via $\Phi_i$ or $g_{e,i}$ is computable and $D_1 - D_2 \leq_Q B_e$ via $g_{e,i}$.

During the construction we construct a uniformly c.e. sequence of c.e. sets $\{X_{e,i,n}\}_{n \in \omega}$, and for all $n$ define $g_{e,i}(n)$ as the index of $X_{e,i,n}$: $W_{g_{e,i}(n)} = X_{e,i,n}$.

For a convenience we first rewrite the requirement:

$$\exists x(\Phi_e(x) \not\in W_e \lor \Phi_i(x) \not\in W_e \land W_{\Phi_i(x)} \subseteq B_e \lor k \in A_e \land W_{\Phi_i(x)} \not\subseteq B_e),$$

$$\exists k(k \in D_1 - D_2 \land X_{e,i,k} \subseteq B_e).$$

We use an $\omega$-sequence of cycles $\{0, 1, 2, \ldots\}$, where each cycle $k$ acts as follows:

(1) Choose an unused witness $x_k \not\in A_e$ and define $U_{e,x_k} = \{k\}$.

(2) Wait for a stage $s_0$ such that $\Phi_i(x_k) \not\in [s_0]$, there exists $y_k \in W_{\Phi_i(x_k)}$ such that $\Phi_e(y_k) \not\in [s_0]$, and there exists $z_k \in W_{\Phi_e(y_k)} - (D_1 - D_2)[s_0]$ (i.e., $y_k \not\in B_{e,s_0}$).

(3) Define $X_{e,i,k,s_0} = \{y_k\}$, and open the cycle $k + 1$.

(4) If at the stage $s_1 > s_0$, $z_k$ is enumerated in $D_1$, then close all cycles $> k$ and return to the stage (2).

Interactions between the requirements.

The requirements $\mathcal{N}_{(e,i,j)}$ and $\mathcal{N}_{(e',i',j')}$ for some $e, i, j, e', i', j'$ can conflict with each other, if the strategy of $\mathcal{N}_{(e,i,j)}$ needs to enumerate some $x$ in $D_1$, but this $x$ is restrained by $\mathcal{N}_{(e',i',j')}$. Since such requirements acts only finitely many times, this conflict is settled by the priority ordering of the requirements and by using the standard finite priority method.

The requirements $\mathcal{P}_{(e,i)}$ and $\mathcal{N}_{(e',i',j')}$ for some $e, i, e', i', j'$ can conflict with each other, if $\mathcal{P}_{(e,i)}$ at some stage $s_0$ define $X_{e,i,k,s_0} = \{y_k\}$ for some $k$, and at a later stage $s_2 > s_0$ the strategy of $\mathcal{N}_{(e',i',j')}$ needs to enumerate $k$ in $D_1$. To settle this conflict we combine the requirements $\{\mathcal{N}_{(e,i,j)}\}_{e,i,j \in \omega}$ and $\{\mathcal{P}_{(e,i)}\}_{e,i \in \omega}$ in a one $\omega$-list $\{Q_{e,i}\}_{e,i \in \omega}$ and make the following modification of the strategies of the requirements $\{\mathcal{P}_{(e,i)}\}_{e,i \in \omega}$. There are three possibilities, in each of them we are trying to keep the condition $A_e \leq_Q D_1 - D_2$. 

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(A) \((e', i', j') < (e, i)\). In this case enumerate \(k\) in \(D_1\), enumerate some fixed \(d \notin D_1\) in \(E_{e,i,k}\), close the cycle \(k\) and open the cycle \(k + 1\). It injures the reducibility of \(D_1 - D_2\) to \(B_2\) at the element \(k\), however there are only finitely many such injuries, since there are only finitely many \((e', i', j')\) such that \((e', i', j') < (e, i)\), and, hence, we still have \(D_1 - D_2 \leq Q B_2\).

(B) \((e', i', j') > (e, i)\) and \(z_k\) is the witness of \(N_{e', i', j'}\) for some \((e', i', j')\) such that \((e', i', j') < (e, i)\). In this case we enumerate \(k\) in \(D_1\), restrain \(k\) from to be enumerated in \(D_2\), initialize the requirement \(N_{e', i', j'}\), enumerate \(x_k\) in \(A_e\) and wait until \(z_k\) is enumerated in \(D_1\). If this never happens, then the requirement \(P_{(e,i)}\) is satisfied once and for all. If \(z_k\) is enumerated in \(D_1\), open the cycle \(k\) + 1 for the requirement \(P_{(e,i)}\). Since there are only finitely many \(e, i, e', i', j', j''\) such that \((e', i', j') < (e, i)\) and \((e', i', j') < (e, i', j')\), the requirement \(N_{e', i', j'}\) will be injured only finitely many times.

(C) \((e', i', j') > (e, i)\) and \(z_k\) is the witness of \(N_{e', i', j'}\) for some \((e', i', j')\) such that \((e', i', j') > (e, i)\) or \(z_k\) is not a witness for any requirement. In this case enumerate \(k\) in \(D_1\), restrain \(k\) from to be enumerated in \(D_2\), enumerate \(x_k\) in \(A_e\), restrain \(z_k\) from to be enumerated in \(D_1\) and initialize the requirements \(N_{e', i', j'}\) and \(N_{e', i', j''}\). The requirement \(P_{(e,i)}\) is satisfied once and for all, hence, \(N_{e', i', j'}\) and \(N_{e', i', j''}\) will be injured only finitely many times.

**Construction.** Define for every requirement \(P_{(e,i)}\) the set of injuries of the reducibility \(D_1 - D_2 \leq Q B_2\): \(M_{e,i,s} = \{m \in D_1 - D_2 | \exists (e', i', j')(e', i', j') < (e, i)\) & \(X_{e,i,m} \subseteq \exists B_e[s] = v_{e', i', j'}\), where \(v_{e', i', j'}\) is a witness of \(N_{e', i', j'}\). Define the function for \(P_{(e,i)}:\)

\[
l(e, i, s) = \max_{m \leq s} (\forall l \notin M_{e,i,t} \Rightarrow (l \in D_1 - D_2 \Leftrightarrow X_{e,i,l} \subseteq B_e[s])) + 1
\]

and the maximal length function \(m(e, i, s) = \max_{l \leq s} l(e, i, l)\).

**Stage s = 0.** Define \(X_{e,i,0} = U_{e,i,0} = \emptyset\), \(D_0 = D_1 = \emptyset\) and define \(r(e, i, n, 0) = 0, p(e, i, 0) = 0\) for all \(e, i, n \in \omega\). For every requirement \(N_{e,i,j} \) choose a witness \(v_{e,i,j}^0\), \(P(e, i, s) = \max_{(e', i', j')} p(e', i', j')\).

**Stage s + 1.** Define \(R(e, i, n, s) = \max_{r(e', i', n', s)} r(e', i', n', s)\), \(P(e, i, s) = \max_{(e', i') < (e, i)} p(e', i', s)\).

For all \(e, i, s < l, U_{e,i,s} = \emptyset\) and \(x \notin A_{e,s}\), enumerate \(d\) in \(U_{e,s+1}\).

For all \((e, i) < l, s \notin (D - D_2)[s]\) and there is no associated with \(k\) witness, then **associate** with \(k\) the least \(x = x_{e,i,k} \notin A_{e,s}\), which is greater than all numbers so far mentioned in the construction, and define \(U_{e,s+1} = \{k\}\).

For all \((e, i) < l, s \notin (D - D_2)[s], \Phi_{e,i}(x_{e,i,k}) \downarrow\), there exists \(y_{e,i,k} \in W_{\Phi_{e,i}(x_{e,i,k})}\) such that \(\Phi_{e,i}(y) = (D - D_2)[s]\), then define \(X_{e,i,k,s+1} = X_{e,i,k,s} \cup \{y\}\).

We say that the requirement \(N_{e,i,j}\) needs an attention, if \(v > v_{e,i,j} < (D - D_2)[s]\), \(P(e', i', s)\), \(v > R(e, i, j, s)\), \(\Phi_{e,i}(v), \downarrow\), there exists \(y_{e,i,j} \in W_{\Phi_{e,i}(v)}\) such that \(\Phi_{e,i}(y) = (D - D_2)[s]\).

Choose the least \((e_0, i_0, j_0)\) such that \(N_{e_0,i_0,j_0}\) needs an attention, enumerate \(v_{e_0,i_0,j_0}^0\) in \(D_1\) and define \(r(e_0, i_0, j_0, s + 1) = z\). We say that the requirement \(N_{e_0,i_0,j_0}\) has received an attention. For all \(e, i, k < l\) such that \((e_0, i_0, j_0) < (e, i)\), \(x = x_{e,i,k} \notin A_{e,s}\) and \(U_{e,x,s} = v_{e_0,i_0,j_0}^0\), define \(U_{e,s+1} = U_{e,s} \cup \{d\}\).

For all \(e, i\) such that \((e, i) < (e_0, i_0, j_0)\), we say that the requirement \(P_{(e,i)}\) needs an attention, if there exists \(k \leq m(e, i, s)\) such that \(v = v_{e,i,k}^0\), \(k \in D_1 - D_2, x_{e,i,k} \notin A_{e,s}, \Phi_{e,i}(x_{e,i,k}) \downarrow\), there exists \(y = y_{e,i,k} \in W_{\Phi_{(x_{e,i,k})}}\) such that \(\Phi_{e,i}(y) = (D - D_2)[s]\). For all such \(e, i\) enumerate \(x_{e,i,k} \in A_{e,s+1}\) and define \(p(e, i, s + 1) = \max\{p(e, i, s), z_{e,i,k}\}\). We say that the requirement \(P_{(e,i)}\) has received an attention.

For all \((e, i) < s, P_{(e,i)}\) hasn’t received an attention, then define \(p(e, i, s + 1) = p(e, i, s)\).

**Initialize** all requirements \(N_{e', i', j'}\) for all such \((e', i', j')\) that \((e', i', j') > (e_0, i_0, j_0)\) or \((e', i', j') > (e, i)\) for some \(e, i\) such that \(P_{(e,i)}\) has received an attention, by defining \(v_{e', i', j'}^{s+1}\) equal to the least \(v\), which is greater than all numbers so far mentioned in the construction and define \((e', i', j', s + 1) = 0\). For all \((e', i', j')\) such that \(N_{e', i', j'}\) haven’t been initialized, define \(v_{e', i', j'}^{s+1} = v_{e', i', j'}^{s+1}\) and \(r(e', i', j', s + 1) = r(e', i', j', s)\).

If for some \(e, i, j\) for the requirement \(N_{e,i,j}\) we have \(v_{e,i,j}^0 \in D_1, v_{e,i,j}^{s+1} = z_{e,i,j}\) and \(y_{e,i,j} \in W_{e,s}\), then enumerate \(v_{e,i,j}^{s+1}\) in \(D_2, s + 1\).
**Verification.** Define $D_1 = \bigcup_{e \in \omega} D_{1, e}$, $D_2 = \bigcup_{e \in \omega} D_{2, e}$ and for all $e$ define $A_e = \bigcup_{e \in \omega} A_{s, e}$. Let us prove that the set $D = D_1 - D_2$ satisfies the conditions of the theorem. Order all requirements $\{N_{e,i,j}\}_{e,i,j \in \omega}$ and $\{P_{e,i}\}_{e,i \in \omega}$ in a $\omega$-list $(Q_n)_{n \in \omega}$: $Q_0 = \emptyset$, $Q_1 = \emptyset$, $Q_2 = \emptyset$, $Q_3 = \emptyset$, …

**Lemma 2.6** Let $Q_n$ be satisfied for all $n < \langle e, i, j \rangle$, then $N_{e,i,j}$ will be satisfied and only finitely many times will need an attention.

Let $s$ be the least stage such that the requirements $Q_n$, $n < \langle e, i, j \rangle$, don’t receive any attention after it. Let $D_1 - D_2 \leq W_r$ via $\Phi_4$ and $W_r \leq Q ((D_1 - D_2) \leq W_r)$ via $\Phi_4$. Then for $v = v_{e,i,j}^+$ there exists a stage $t > s$ such that $\Phi_i(v)(t) \downarrow$ and there exists $y \in W_{\Phi_i(v)} \leq ((D_1 - D_2)(t))$. Let $t_0$ be the least of these stages, then $N_{e,i,j}$ receives an attention at the stage $t_0$ and $v$ is enumerated in $D_{1, t_0}$. Since $D_1 - D_2 \leq W_r$ via $\Phi_4$, there exists a stage $t_1 > t_0$ such that $y \in W_r$. If $v \neq z$, then, by the construction, $z \notin D_1$, contrary to the condition $W_r \leq Q (D_1 - D_2)$. If $v = z$, then $z$ will be enumerated in $D_2$ at the stage $t_1$, contrary to the condition $W_r \leq Q (D_1 - D_2)$. The requirement $N_{e,i,j}$ will not need an attention after the stage $t_1$.

**Lemma 2.7** Let $Q_n$ be satisfied for all $n < \langle e, i, j \rangle$, then $P_{e,i,j}$ will be satisfied and only finitely many times will need an attention.

Let $s$ be the least stage such that the requirements $Q_n$, $n < \langle e, i, j \rangle$, don’t receive any attention after it and let $m$ be the greatest natural number used as a witness by the requirements $N_{e,i,j}$, $\langle e', i', j' \rangle < \langle e, i, j \rangle$. Let $\Phi_4$ be computable function and $A_e \leq Q B_e$ via $\Phi_4$. Let us show that in this case $(\forall n > m)(n \in D_1 - D_2 \leftrightarrow X_{e,i,n} \leq B_e)$ and, hence, $D_1 - D_2 \leq Q B_e$ via $g_{e,i}$. It would mean that $D_1 - D_2 \leq Q B_e$ via $g_{e,i}$ which is defined using $g_{e,i}$ in an obvious way.

Let us suppose that there exists $n > m$ such that the condition $(n \in D_1 - D_2 \leftrightarrow X_{e,i,n} \leq B_e)$ doesn’t hold. If $n \notin D_1$, then since $\Phi_4$ is computable and $A_e \leq Q B_e$, there exist $y, z$ and $s$ such that $\Phi_i(n) \downarrow \langle s, y \in W_{\Phi_i(n)} \geq B_e \rangle$. By the construction, $y$ is enumerated in $X_{e,i,n,s+1}$, thus $X_{e,i,n} \nsqsubseteq B_e$. If $n \in D_1$, then there exists $y \in X_{e,i,n} \neg \nsqsubseteq B_e$. By the construction, $y \in X_{e,i,n}$ only if for $x = x_{e,i,n}$ we have: $\Phi_i(n) \downarrow \langle \langle e' \rangle, i', j' \rangle \downarrow \frac{1}{2} \left( X_{e',i',j'} \leq B_e \right)$, and there exists $x_{e,i,n} \in W_{\Phi_i(n)} - (D_1 - D_2)$. Let $t$ be the stage such that $n \in D_{1,t} - D_{1,t-1}$; then since $n > m$, there exists such $i_0, j_0, j_0$ that $(e, i) \not< (e_0, i_0, j_0)$ and $n = v_i^{e_0, i_0, j_0}$. By the construction, the requirement $P_{e,i,j}$ receives an attention at the stage $t$, and $x = x_{e,i,n}$ will be enumerated in $A_e$. Since, by the hypothesis, $A_e \leq Q B_e$, there exists a stage $t_1 > t$ such that $x_{e,i,n} \in D_{1,t_1}$ and $z \notin D_2$, thus $y \in B_e$, contrary to the hypothesis.

**Lemma 2.8** For all $e$, $A_e \leq Q D_1 - D_2$ via a computable function $h_e$ such that $W_{h_e(x)} = U_{e,x}$.

Let $x \notin A_e$ and is not associated with any $k$, then $U_{e,x} = U_{e,x,s} = \{d\} \nsqsubseteq D_1 - D_2$, where $s = x + 1$. Let $x \notin A_e$ and is associated with some $k$, then $k \in U_{e,x}$. If $k \notin D_1 - D_2$, then $U_{e,x} = U_{e,x,s} = \{k\} \nsqsubseteq D_1 - D_2$ for the stage $s$ such that $k \in D_{1,s} - D_{1,s-1}$. Let $x \in A_e$, then it is associated with some number $k \in U_{e,x}$. Let $s$ be the stage such that $x \in A_{e,s} \neg \nsqsubseteq A_{e,s-1}$. By the construction, it could happen only if $k \in D_{1,s}$, $k \notin D_2$ and $k = \nu_{e,i,j}^{e',i',j'}$ for some $e', i', j'$. In this case $e_{e',i',j'}^{e'+1} \neq k$ and $k$ is no longer used in the construction after the stage $s + 1$, it means that $k \notin D_2$, and hence, $k \in D_1 - D_2$ and $U_{e,x} \nsqsubseteq D_1 - D_2$.

### 3 Non-isolated from above degrees

In 1984 Hay and Shore (unpublished) constructed a 2-c.e. $T$-degree $d$ such that there is no any c.e. $T$-degree between $d$ and $0'$, thus, in fact, they proved that there exists an isolated from above $T$-degree.

The systematical study of isolated from above degrees was started by Efremov in [9] and [10]. He showed that for every incomplete c.e. degree $a$ there exists an isolated from above $T$-degree $d$, such that $a <_T d$. He also constructed a noncomputable c.e. degree $b$ such that all 2-c.e. degrees below it are non-isolated from above. Thus, he showed that isolated from above 2-c.e. degrees are not dense in the structure of c.e. $T$-degrees.

In [6] we showed that for $Q$-degrees the converse result holds, namely, 2-c.e. isolated from above $Q$-degrees are dense in the structure of c.e. $Q$-degrees. The following theorem shows that non-isolated from above degrees are downward dense in the structure of c.e. $Q$-degrees.

**Theorem 3.1** Given a c.e. degree $v > Q 0$ there is a properly 2-c.e. degree $d$ such that $d <_Q v$ and for any c.e. degree $w$ there is a c.e. degree $a$ such that $d <_Q a$ and if $d <_Q w$, then $w <_Q a$. 

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Proof. Let \( V \in \mathcal{C} \) be a c.e. set. We construct c.e. sets \( D_1, D_2, D_2 \subseteq D_1 \) such that the set \( D = (D_1 - D_2) \) will have the desired degree.

To ensure that \( D_1 - D_2 \leq Q \) \( V \) we construct a uniformly c.e. sequence of c.e. sets \( \{V_x\}_{x \in \omega} \) such that \( x \in D_1 - D_2 \) if and only if \( V_x \subseteq V \). For every \( x \in \omega \) we enumerate at any stage \( s \), if \( x \notin (D_1 - D_2)[s] \), some element \( y \notin V_x \) in \( V_x \). Later we enumerate \( x \) in \( D_1 \) only if \( y \) is enumerated in \( V \). If we need to enumerate \( x \) in \( D_2 \) we just enumerate some fixed \( \bar{v} \notin V \) in \( V_x \). The reducibility \( D_1 - D_2 \leq Q \) \( V \) is guaranteed by the computable function \( f \) defined in the following way: \( W_{f(x)} = V_x \).

For every \( c \) we construct a c.e. set \( A_c \geq Q \) \( D \) in order to satisfy for every \( i, j \) the requirement:

\[
\mathcal{R}_{(i,j)}: D_1 - D_2 \leq Q \ W_c \ via \ \Phi_i \ or \ W_c \leq Q \ A_c \ via \ \Phi_j.
\]

To ensure that \( D \leq Q \ A_c \) we construct a uniformly c.e. sequence of c.e. sets \( \{U_{e,k}\}_{k \in \omega} \) such that \( k \in D \) if and only if \( U_{e,k} \subseteq A_c \). For this purpose we fix some infinite co-finite set of integers \( \{a_k\}_{k \in \omega} \), fix some \( a_{-1} \notin A_c \) and for every \( k \) enumerate \( a_k \) in \( U_{e,k} \). If later \( k \) is enumerated in \( D_1 \), we enumerate \( a_k \) in \( A_c \). If later \( k \) is enumerated in \( D_2 \), we enumerate \( a_{-1} \) in \( U_{e,k} \). The reducibility \( D \leq Q \ A_c \) is guaranteed by the computable function \( h_k \) defined in the following way: \( W_{h_k} = U_{e,k} \).

To ensure that the set \( D_1 - D_2 \) has a properly 2-c.e. degree, we satisfy for every \( e, i, j \in \omega \) the following requirement:

\[
\mathcal{N}_{(i,j)}: D_1 - D_2 \leq Q \ W_e via \ \Phi_i \ or \ W_e \leq Q \ D_1 - D_2 via \ \Phi_j.
\]

The basic module for the requirement \( \mathcal{N}_{(i,j)} \).

In satisfying the requirement we shall construct a c.e. set \( U \) such that either \( D_1 - D_2 \leq Q \ W_e \ via \ \Phi_i \), or \( W_e \leq Q (D_1 - D_2) \ via \ \Phi_j \), or \( \omega - V = U \).

For a convenience we first rewrite the requirement:

\( \exists k (\Phi_i(k) \uparrow \land k \notin D_1 - D_2 \land W_{\Phi_i(k)} \subseteq W_e \land k \notin D_1 - D_2 \land \Phi_j(k) \subseteq D_1 - D_2 \lor k \in W_e \land \Phi_j(k) \subseteq D_1 - D_2) \),

\( \forall k (k \notin V \iff k \in U) \).

We use an \( \omega \)-sequence of cycles \( \{0, 1, 2, \ldots\} \), where each cycle acts as follows:

1. Choose an unused witness \( x_k \notin D_1 \) and define \( V_{x_k} = \{k\} \).
2. Wait for a stage \( s_0 \) such that \( k \in V_{x_k} \) or \( \Phi_i(x_k) \downarrow [s_0] \), there exists \( y_k \in W_{\Phi_i(x_k)} - W_e[s_0] \) such that \( \Phi_j(y_k) \downarrow [s_0] \) and there exists \( z_k \in W_{\Phi_j(y_k)} - (D_1 - D_2)[s_0] \).
3. If \( k \in V_{x_k} \), then enumerate in \( V_{x_k} \) the fixed element \( \bar{v} \notin V \) and open the cycle \( k + 1 \).
4. If \( z_k \neq x_k \), then restrain \( z_k \) from to be enumerated in \( D_1 \), enumerate \( k \) in \( U \) and open the cycle \( k + 1 \).
5. If later \( k \) is enumerated in \( V \), then enumerate \( x_k \) in \( D_1 \) and close all cycles \( > k \).
6. If later \( k \) is enumerated in \( V \), then enumerate \( x_k \) in \( D_1 \), close all cycles \( > k \) and wait until \( y_k \) is enumerated in \( W_e \).
7. If this happens, then enumerate \( x_k \) in \( D_2 \) and enumerate some fixed \( \bar{v} \notin V \) in \( V_{x_k} \).

The basic module for the requirement \( \mathcal{R}_{(i,j)} \).

In satisfying the requirement we shall construct a c.e. set \( U' \) such that either \( D_1 - D_2 \leq Q \ W_e \ via \ \Phi_i \), or \( W_e \leq Q A_c \ via \ \Phi_j \), or \( \omega - V = U' \).

We use an \( \omega \)-sequence of cycles \( \{0, 1, 2, \ldots\} \), where each cycle \( k \) acts as follows:

1. Choose an unused witness \( x_k \notin D_1 \) and define \( V_{x_k} = \{k\} \).
2. Wait for a stage \( s_0 \) such that \( k \in V_{x_k} \) or \( \Phi_i(x_k) \downarrow [s_0] \), there exists \( y_k \in W_{\Phi_i(x_k)} - W_e[s_0] \) such that \( \Phi_j(y_k) \downarrow [s_0] \) and there exists \( z_k \in W_{\Phi_j(y_k)} - A_c[s_0] \).
3. If \( k \in V_{x_k} \), then enumerate in \( V_{x_k} \) the fixed element \( \bar{v} \notin V \) and open the cycle \( k + 1 \).
4. If \( z_k \neq x_k \), then enumerate \( k \) in \( U' \), restrain \( z_k \) from to be enumerated in \( A_c \) and open the cycle \( k + 1 \).
5. If later \( k \) is enumerated in \( V \), then enumerate \( x_k \) in \( D_1 \) and close all cycles \( > k \).
(4c) If later $k$ is enumerated in $V$, then enumerate $x_k$ in $D_1$, $a_{x_k}$ in $U_e$, close all cycles $> k$ and wait until $y_k$ is enumerated in $W_e$.

(5c) If this happens, then enumerate $x_k$ in $D_2$, enumerate $a_{-1}$ in $U_e$ and enumerate the fixed $v$ in $V$ in $W_e$.

Interactions between the requirements.

There are only two possible conflicts between $\mathcal{R}_{(e,i,j)}$ and $\mathcal{N}_{(e',i',j')}$ for some $e, i, j, e', i', j'$:

1) If the strategy of $\mathcal{R}_{(e,i,j)}$ needs to enumerate some $x$ in $D_1$, and $x$ is restrained by the strategy of $\mathcal{N}_{(e',i',j')}$;

2) if the strategy $\mathcal{N}_{(e',i',j')}$ needs to enumerate some $x$ in $D_1$, thereby to enumerate $a_x$ in $A_e$, and $a_x$ is restrained by the strategy of $\mathcal{R}_{(e,i,j)}$.

Since each requirement acts only finitely many times, these conflicts are settled by the standard finite injury priority method. We combine the requirements $\{\mathcal{N}_{e,i,j}\}_{e,i,j} \in \omega$ and $\{\mathcal{P}_{e,i,j}\}_{e,i,j} \in \omega$ in a one $\omega$-list $\{Q_n\}_{n \in \omega}$: $Q_0 = N_0, Q_1 = P_0, Q_2 = N_1, Q_3 = P_1, \ldots$. We enumerate $x$ in $D_1$, if it is restrained by a requirement with a lower priority, and initialize the requirement if $x$ is restrained by a requirement with a higher priority.

\[ \square \]

**Corollary 3.2** Non-isolated from above 2-c.e. Q-degrees are downward dense in the structure of c.e. Q-degrees.

**Proof.** Let $v > 0$ be a c.e. degree. By the theorem, there is a properly 2-c.e. degree $d$ such that $d <_Q v$ and for any c.e. degree $w$ there is a c.e. degree $a$ such that $d <_Q a$ and if $d \leq_Q w$, then $w \not<_Q a$. By the Definition 1.1 $d$ is non-isolated from above.

It is not hard to see, that the requirements $\mathcal{N}_{(e',i',j')}$ and $\mathcal{R}_{(e,i,j)}$ from the previous theorem and the weakened requirements $\mathcal{P}_{(e,i,j)}$ from the Theorem 2.1 (where $(D_1 - D_2) \oplus V$ is replaced by $D_1 - D_2$) can be satisfied together at the same time, and, hence, we have the following theorem.

**Theorem 3.3** Given a c.e. degree $u >_Q 0$ there is a properly 2-c.e. degree $d <_Q u$ such that for all c.e. $w$, if $w <_Q d$, then there is a c.e. degree $a <_Q d$ such that $a \not<_Q w$, and for any c.e. degree $w$ there is a c.e. degree $c$ such that $d <_Q c$ and if $d \leq_Q w$, then $w \not<_Q c$.

**Corollary 3.4** Given a c.e. degree $u > 0$ there is a properly 2-c.e. degree $d <_Q u$, which is non-isolated from above and from below.

References