Kinematics, Workspace and Static Analyses of 2-DOF Flexure Parallel Mechanism

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Abstract
In this paper, we propose a 2-DOF parallel mechanism with flexure hinges. The kinematics and static analyses of this mechanism are studied in order to determine the kinematics properties, workspace, holding forces of the actuators and the reaction forces at the flexure hinges. These factors are useful for design process of this mechanism. The pseudo-rigid-body model (PRB model) of this mechanism is formulated. The theoretical results are compared with that obtained from ANSYS.

1 Introduction
The requirement of high accuracy positioning systems is increasing. These systems are used in optical devices, precision machine tools, and semiconductor manufacturing machines. In order to obtain high accuracy, the structure of these positioning systems must have high natural frequency, high rigidity and can produce high-precision motion. Due to the serial arrangement of actuators, serial positioning stages have large accumulated errors, some products of this stacked stage are shown in Figure 1. Parallel mechanisms, on the contrary, possess rigid architectures and can limit the accumulation of errors. Especially, parallel mechanisms that contain flexure hinges have no error due to backlash or friction and there is not need for lubrication [1]. Examples of flexure mechanisms can be seen in the micro-positioner for deep ultraviolet lithography application [2] and the fine-motion stage of the two-stage positioning device [3].

Figure 1: 3-DOF stacked stages (X-Y-θZ)

The flexure hinge is a mechanical member, which substitutes a conventional revolute joint in order to produce a limited angular motion about one axis. A flexure hinge is usually obtained by machining one or two cutouts in a blank material [4]. The merits of using flexure hinges are vacuum compatibility, no backlash, no non-linear friction, simple structure and easy to manufacture. However, insufficient flexibility and stiffness around the non-working axes may cause unintended motions of the output if the flexure hinge bears high load. In addition, the rotation motion of the link may be achieved by deflecting part of the hinge instead of the joint, therefore the analysis and control of such a mechanism with flexure hinges are more complicated [1,5]. Another disadvantage is that the reachable area of a flexure mechanism is limited due to not just by link lengths and geometry, but also material strength of the flexure member [6]. With all above characteristics, the mechanisms with flexure hinges will be only suitable for the high accuracy positioning devices with small workspace and payload.

Figure 2: 2-DOF parallel flexure mechanism

The motivation of our research is to enlarge the working range of the flexure mechanism that currently is limited within 200 μm. For the purpose of getting the larger workspace with high positioning accuracy, we propose a 2-DOF XY parallel mechanism with flexure hinges as a planar positioning system shown in Figure 2. Two linear motors, with the resolution of 0.5 μm and the range of 10 mm, are used to actuate two sliders 1 and 6. The mechanism has high accuracy thanks to the fine resolution of linear actuators and structure without backlash and friction. It also has a workspace larger...
than a 10mmx10mm square. In order to eliminate absolutely the parasitic motion $\theta_Z$, the links 1, 2, 3 and 4 form a parallelogram. The shape of flexure hinges can be right circular, elliptical or corner-filleted. Here we choose the first one because the displacement accuracy of a right circular hinge is the best (the center of flexure is displaced very smaller than another types) [4]. The proposed mechanism can be used in optics as a fine adjusting device.

In order to simplify the analysis and design, the pseudo-rigid-body (PRB) model is proposed for modeling the kinematics input-output behavior of flexure mechanism [7]. Her and Chang [1] give the linear scheme for analyzing the kinematics of a flexure mechanism with a closed loop.

This article presents the workspace determination of mechanisms with flexure hinges based on the PRB model. The pseudo-rigid-body model, or the pseudo-linkage model, or the linear scheme expresses a methodology of treating each flexure hinge as a revolute joint with a torsion spring. This model makes analysis of mechanisms with flexure hinges easier and faster. In the meantime, analysis of the real mechanism needs the finite element method or the elliptic integral solution. This model also makes us easily see the effects of parameters on mechanism design and significantly simplify the design process [6]. When using this model, the stiffness of the torsion spring is calculated from the analytical formulations of circular notched hinges, which were firstly given by Paros and Weisbord [8]. A finite element modeling is established in ANSYS and used for displacement, stress, and reaction analysis and simulation. The mechanism is considered as a monolithic rigid body. The results of displacement and holding payloads from two models are compared with each other. The comparisons show that the PRB model is useful for our mechanism design.

2 Modeling of the Proposed 2-DOF Flexure Parallel Mechanism

2.1 Pseudo-rigid-body model

The pseudo-rigid-body model is proposed as shown in Figure 3. The hinges are replaced by the revolute joints and the torsion springs. Due to the torsion deformation of the spring with stiffness of $K_{ij}$, the links $i, j$ act on each other by the torsion moments $M_{ij}$. These moments are caused by the changing of the relative angles of the links and can be calculated based on the stiffness of the torsion spring $K_{ij}$ and the difference $(\phi_k - \phi_{k0})$ of the current position angles $\phi_k$ and their initial values $\phi_{k0}$ ($k=2,4,5$). These moments can be expressed as follows

\[
\begin{align*}
M_{12} &= M_{21} = K_{12} (\phi_2 - \phi_{20}) \\
M_{23} &= M_{32} = K_{23} (\phi_3 - \phi_{20}) \\
M_{14} &= M_{41} = K_{14} (\phi_4 - \phi_{40}) \\
M_{34} &= M_{43} = K_{34} (\phi_4 - \phi_{40}) \\
M_{35} &= M_{53} = K_{35} (\phi_5 - \phi_{30}) \\
M_{56} &= M_{65} = K_{56} (\phi_5 - \phi_{30})
\end{align*}
\]

(1)

\[\text{Figure 3: Pseudo-rigid-body model}\]

The stiffness is calculated via the formulation given by Paros and Weisbord [8] as follows

\[K = \frac{9\pi t^{1/2}}{2Et^{3/2}}\]

(2)

where $E$ is the Young’s modulus of the material. In the meantime, $t$ (width of the notch), $b$ (thickness), and $r$ (radius of circular notch) are the dimension of the flexure hinge as indicated in Figure 4.

\[\text{Figure 4: Structure of flexure hinge with right circular notched hinge}\]

2.2 Kinematics analysis

Based on Figure 3, two vector-loop equations can be written as:

\[
\begin{align*}
\vec{OB} &= \vec{OA} + \vec{AB} = \vec{OF} + \vec{FE} + \vec{EC} + \vec{CB}
\end{align*}
\]
Expressing the equations in the global fixed coordinate frame Oxy gives:

\[
\begin{align*}
\dot{x} &= x_1 + b \cos \phi_2 = x_6 - d \cos \phi_5 - a \\
\dot{y} &= b \sin \phi_2 = f - d \sin \phi_5 - c
\end{align*}
\]  
(3)

where \(a, b, c, d\) and \(f\) are the dimension of the PRB model \((a=AD=BC, b=AB=CD, c=CE, d=EF, f=FG)\) and \(x_1, x_6\) are active joint variables.

The position of the platform is described by the position of point \(B(x, y)\) and the inverse kinematics analysis gives:

\[
\begin{align*}
x_1 &= x - \sqrt{b^2 - y^2} \\
x_6 &= x + a + \sqrt{d^2 - (f-c-y)^2}
\end{align*}
\]  
(4)

\[
\phi_2 = a \sin \left( \frac{y}{b} \right) \\
\phi_5 = 2\pi + a \sin \left( \frac{f-c-y}{d} \right)
\]

Differentiating equations (3) with respect to time gives

\[
\begin{align*}
\dot{x} &= \dot{x}_1 - b \sin \phi_2 \dot{\phi}_2 = \dot{x}_6 + d \sin \phi_5 \dot{\phi}_5 \\
\dot{y} &= b \cos \phi_2 \dot{\phi}_2 = -d \cos \phi_5 \dot{\phi}_5
\end{align*}
\]  
(5)

From equation (5) we have angular derivatives:

\[
\begin{align*}
\dot{\phi}_2 &= \frac{(\dot{x}_6 - \dot{x}_1) \cos \phi_5}{b \sin(\phi_5 - \phi_2)} \\
\dot{\phi}_5 &= \frac{(\dot{x}_6 - \dot{x}_1) \cos \phi_2}{d \sin(\phi_5 - \phi_2)}
\end{align*}
\]

Eliminating angular derivatives in equation (5), we obtain the relation between velocity of the platform and the actuators:

\[
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J \begin{pmatrix} \dot{x}_1 \\ \dot{x}_6 \end{pmatrix}
\]

where \(J\) is the Jacobian matrix:

\[
J = \begin{pmatrix} 
\frac{\cos \phi_2 \sin \phi_5}{\sin(\phi_2 - \phi_5)} & \frac{\sin \phi_2 \cos \phi_5}{\sin(\phi_2 - \phi_5)} \\
\frac{\cos \phi_2 \cos \phi_5}{\sin(\phi_2 - \phi_5)} & \frac{\sin \phi_2 \cos \phi_5}{\sin(\phi_2 - \phi_5)} 
\end{pmatrix}
\]  
(6)

3 Workspace Analysis

The workspace can be determined by implementing the following steps:
- Separate the constraints into small groups: constraints due to architecture, constraints due to range of the actuators and constraints due to the limit of rotation angle of the flexure hinges.
- Determine the reachable area resulting from the groups of constraints.
- Determine the intersection of above areas and thus obtain the workspace of the mechanism.

3.1 Architectural reachability

The reachable area due to mechanism architecture can be determined by the existence of solution to the inverse kinematics problem. From (4) this area is expressed analytically by

\[
\begin{align*}
(x, y) \in R \\
(b &\geq y \geq -b) \\
d + f - c &\geq y \geq f - c - d
\end{align*}
\]  
(7)

3.2 Actuator range

The reachable area due to the range of the actuators can be determined by limiting the range of active variables \(x_1\) and \(x_6\). Without loss of generality, we consider that the nearest position of the actuated link 1 coincides with the origin \(O\). Considering \(S\) as the range of the actuators and \(x_6^{\text{min}}\) as the nearest position of the actuated link 6, we have

\[
0 \leq x_1 \leq S \\
x_6^{\text{min}} \leq x_6 \leq x_6^{\text{min}} + S
\]

Solving these inequalities and taking care of the small motion of mechanisms with flexure hinges give us the analytical expression of the area in (8):

\[
\begin{align*}
x, y &\in R \\
x^2 + y^2 &\geq b^2 \\
x - (x_6^{\text{min}} + S - a) &\geq |y - (f-c)|^2 \geq d^2 \\
x - (x_6^{\text{min}} - a) &\geq |y - (f-c)|^2 \leq d^2
\end{align*}
\]  
(8)

3.3 Limited rotation angle

We define \(\delta_2\) and \(\delta_5\) as the maximum rotation angles of links 2 and 5 relative to the initial position angles \(\phi_20\) and \(\phi_50\). These maximum angles are the deflection angles when the maximum stress reaches to the yield stress \(S_y\).

The maximum stress \(\sigma_{\text{max}}\) occurs at the outermost point of hinge section with minimum thickness \(t_i\)

\[
\sigma_{\text{max}} = \frac{M_i \frac{t_i^3}{12} \delta_i}{K_i t_i^4} = \frac{6K_i \delta_i}{b_1 t_1^4}
\]

where \(I_i = \frac{b_1 t_1^3}{12}\) is the inertial property of the smallest section; \(M_i = K_i \delta_i\) is calculated via the deflection angle \(\delta_i\) with the stiffness \(K_i\) formulated in (2) \(i=2\) or \(5\).
Because of $\sigma_{max} = S_y$, we have $\delta_i = \frac{b_i l_i^2}{6K_i} S_y$. The limits of deflection angles are expressed as

$$\begin{align*}
\varphi_{20} - \delta_2 &\leq \varphi_2 \leq \varphi_{20} + \delta_2 \\
\varphi_{50} - \delta_5 &\leq \varphi_5 \leq \varphi_{50} + \delta_5 
\end{align*}$$

where $\varphi_{20}$ and $\varphi_{50}$ are the initial position angles of links 2 and 5. Hence, the area is expressed by:

$$\{(x, y) \in R \mid y_{\text{min}} \leq y \leq y_{\text{max}}\}$$

$$y_{\text{min}} = \max\{b \sin(\varphi_{20} - \delta_2), f - c - d \sin(\varphi_{50} + \delta_5)\}$$

$$y_{\text{max}} = \min\{b \sin(\varphi_{20} + \delta_2), f - c - d \sin(\varphi_{50} - \delta_5)\}$$

### 3.4 Workspace determination

The intersection of the areas, analytically expressed in (7), (8) and (9) gives us the workspace that is indicated by the shaded area in Figure 5.

The angles $\varphi_i$ (i = 2, 4, and 5) are calculated by the kinematics analysis and the moments $M_{ij}$ are calculated via (1).

## 5 Finite Element Analysis

### 5.1 FEM model

A finite element model of the proposed mechanism is built in ANSYS (Figure 6) and its important dimensions are shown in Figure 7. Because the mechanism has only planar motion, we use a 2D model with quadratic quadrilateral element PLANE 82 (eight nodes per element) and the option of plane stress with thickness. This element is a higher order version of the two-dimensional four-node quadrilateral element and is better suited for modeling problems with curved boundary. The shape of the model is smoothened in order to limit the stress concentration. The nodes are non-equally spaced on the model and concentrated near the hinges. The meshing process is divided into two steps: coarse step for the large body and fine step for the area of flexure hinges. This process reduces the number of nodes and the time of solution.

![Figure 6: Finite element model](image)

Figure 6: Finite element model

![Figure 7: Model with dimension](image)

Figure 7: Model with dimension

For simulation, the material was assumed to be an aluminum-alloy-like linearly elastic material with...
Young’s modulus of $E = 71\text{GPa}$ and Poisson’s ratio of $\mu = 0.34$.

### 5.2 Results and discussion

The platform is hardly deformed. Therefore, we can study the displacement of only one node on the platform. Figure 8 shows the plot of the vertical displacement of the platform with respect to the displacements $x_1$ and $x_6$ of the two input actuators in the ranges of 10mm. This figure is built from the data of FEM model analysis. The vertical displacement is commensurate with the difference between $x_1$ and $x_6$. The corresponding Figure 9 is built based on the kinematics analysis of the PRB model. The difference between the two models is illustrated in Figure 10.

Via the reaction report of the finite element model, we calculate the holding payloads $F_1$ and $F_6$ by taking the x-axis projection of the reactions at the actuated links 1 and 6. The result is then compared with the holding payloads computed by equations (10) and (11) of the PRB model.

Figures 11 and 12 show the holding forces computed from FEM model and PRB model. The difference between the FEM model and the PRB model is depicted in Figure 13. The difference is very small and under 5%. From the figures of the displacement and the holding force, we find that the differences between the two models are very small, concretely under 0.1 mm (1%) for the displacement and under 3% for the holding force. The differences increase following the strain of the flexure hinges. In other words, the farther the mechanism moves from the initial position, the larger differences are.

While analyzing the FEM model needs extensive computational consumes, the above comparisons show that the PRB model is useful for analysis and design of the mechanisms with flexure hinges, especially for mechanisms with fine motion. In addition, via the PRB model, we can express the results in analytical forms and can see the effects of parameters such as link lengths, initial position, etc. on the motion of the mechanism.

### 6 Conclusion

The kinematics and static analyses of a 2-DOF parallel mechanism with flexure hinges are investigated. The proposed method uses the pseudo-rigid-body model that simplifies the analysis. The method is verified by using the finite element model, which is considered as the most realistic model of the proposed mechanism. The result of PRB model is precise enough to be used in the actual design. The PRB model is proposed for determining the workspace of a mechanism with flexure hinges. Analytical expression of the result from the PRB
model gives the capability of programming the analysis and design. This method can be used to analyze the dynamics of similar PFM as well.

Figure 11: Holding force $F_1$ via FEM Model

Figure 12: Holding force $F_1$ via PRB model

Figure 13: Difference of holding force in two models

Reference


