An inventory model for deteriorating items with stock-dependent selling rate and partial backlogging under inflation

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This paper presents an inventory model for deteriorating items with stock-dependent selling rate under inflation and time value of money over a finite planning horizon. In the model, shortages are allowed and the unsatisfied demand is partially backlogged at the exponential rate with respect to the waiting time. We establish the theoretical results and provide an efficient solution procedure to find the optimal number of replenishment and the cycle time. Then, the optimal order quantity and the total present value of profits are obtained. A numerical example and sensitivity analysis are presented to illustrate the proposed model and particular cases of the model are also discussed.

Key words: Inventory, stock-dependent demand, deteriorating, partial backlogging, inflation.

INTRODUCTION

It has been observed in certain consumer goods that the demand is usually influenced by the amount of stock displayed on the shelves, that is, the demand rate may go up or down if the on-hand stock level increases or decreases. Such a situation generally arises from a consumer-goods type of inventory. As reported by Levin et al. (1972), a large pile of goods displayed in a supermarket will lead customers to buy more. Silver and Peterson (1985) also noted that sales at the retail level tend to be proportional to the amount of inventory displayed. These observations have attracted many marketing researchers and practitioners to investigate the modeling aspects of this phenomenon. Padmanabhan and Vrat (1995) considered an EOQ model for perishable items with stock-dependent selling rate for sales environment. Under instantaneous replenishment with zero lead time, they present three models: without backlogging, with complete backlogging and partial backlogging for sales environment. However, for the partial backlogging model case, the opportunity cost due to lost sales was not taken into account in Padmanabhan and Vrat (1995). Recently, Hou (2006) extended Padmanabhan and Vrat (1995) by incorporating the effects of deterioration rate, inflation and time value of money to develop an inventory model with stock-dependent consumption rate and with complete backlogging for non-sales environment. However, the uniqueness of the optimal solution for the sales environment and partial backlogging remained further unexplored. In real life, such as fashionable commodities and high-tech products with short product life cycle, the length of the waiting time for the next replenishment is the main factor for deciding whether the backlogging will be accepted or not. Some customers usually are willing to wait until replenishment, especially if the wait will be short, while others are more impatient, and turn to buy from other sellers. To reflect this phenomenon, we allow for shortages, and the rate of backlogged demand decreases exponentially as the waiting time for the next replenishment increases. Dye and Ouyang (2005) developed an inventory model in which the proportion of customers, who would like to accept backlogging, is the reciprocal of a linear function of the waiting time.

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Recently, Abad (1996, 2001) discussed a pricing and lot-sizing problem for a product with a variable rate of deterioration, allowing shortages and partial backlogging. The backlogging rate depends on the time to replenishment, so the more customers must queue up for order, the greater the fraction of sales are lost. However, he does not consider the shortage cost in the objective function. Chern et al. (2008) proposed partial backlogging inventory lot-size models for deteriorating items with fluctuating demand under inflation. Other researchers such as Chang and Dye (1999), Papachristos and Skouri (2003), Wu et al. (2006), Min and Zhou (2009), and Hsieh et al. (2010) have modified inventory policies by considering the time-proportional partial backlogging rate. On the other hand, incorporating the effects of deteriorating items and stock-dependent demand to develop the order-level lot size inventory models were discussed by Datta and Pal (1990), Pal et al. (1993), Padmanabhan and Vrat (1995), Giri et al. (1996), Sarker et al. (1997), Giri and Chaudhuri (1998), Balkhi and Benkherouf (2004), Pal et al. (2006), Hsieh and Dye (2010), and Chang, et al. (2010). Other issues relating to deterioration or batch sizing problem were also addressed by Ghare and Schrader (1963), Shah (1977), Heng et al. (1991), Wee (1995), Goyal and Giri (2001), Teng et al. (2002), and Sana (2010). In addition, the effects of inflation and time value of money on ordering policy over a finite time horizon are also considered in this paper. The pioneer research in this direction was Buzacott (1975), who developed an EOQ model with inflation, subject to different types of pricing policies. Recently, Hou and Lin (2006) incorporated the effects of inflation and time value of money to develop an inventory model for deteriorating items with price and stock-dependent selling rates. Other inventory models with inflation, time value of money and deterioration are presented in Ray and Chaudhuri (1997), Chen (1998), Sarker et al. (2000), Chung and Lin (2001), Wee and Law (2001), Balkhi (2004), Jaggi et al. (2006) and their references.

Based on above situations, the problem of determining the optimal replenishment policy for deteriorating items with stock-dependent selling rate and partial backlogging under inflation and time discounting is considered in this study. Therefore, the proposed model can be seen as an extension of Padmanabhan and Vrat (1995) and Hou (2006).

So, a finite planning horizon inventory model for deterioration items with stock-dependent selling rates and stockout cost (includes the backorder cost and lost sale cost) is developed where the unsatisfied demand is backlogged and the fraction of shortages backordered increases exponentially as the waiting time for the next replenishment decreases. In further discussion, the model is illustrated with a numerical example adopted from Hou (2006). In addition, the sensitivity analysis of the optimal solution with respect to parameters of the system and special cases are also discussed.

ASSUMPTIONS AND NOTATION

To develop the mathematical model of inventory replenishment schedule, the notation adopted in this paper is listed thus:

1) $H$: planning horizon
2) $T$: replenishment cycle
3) $N$: number of replenishment during the planning horizon (a decision variable); $N = H/T$.
4) $T_j$: the $j$-th replenishment time; $T_j = T_{j-1}+T$ ($j = 1, 2, ..., N$) where $T_N = H$ and $T_0 = 0$.
5) $t_i$: time with positive inventory (a decision variable)
6) $T - t_i$: time when shortage occurs
7) $I(t)$: the level of positive inventory at time $t$, where $0 \leq t \leq t_i$
8) $I(t)$: the level of negative inventory at any time $t$, where $0 \leq t \leq T$
9) $Q$: the 2nd, 3rd, ..., Nth replenishment lot size (units)
10) $l_i$: maximum inventory level per cycle
11) $\alpha$: deterioration rate fraction of the on-hand inventory
12) $r$: discount rate, representing the time value of money
13) $i$: inflation rate
14) $R$: the net discount rate of inflation; $R = r - i$
15) $A$: the replenishment cost per order, $/order$
16) $c$: the purchasing cost per unit, $/unit$
17) $s$: the selling price per unit, where $s > c$, $/unit$
18) $c_1$: the inventory holding cost per unit per unit time, $/unit/unit time$
19) $c_2$: the shortage cost per unit per unit time, $/unit/unit time$
20) $c_3$: the cost of lost sales (i.e., goodwill cost) per unit, $/unit$

In addition, the listed assumptions are made:

1. The selling rate $D(t)$ at time $t$ is assumed to be $\alpha + \beta I(t)$, where $\alpha$ is a positive constant, $\beta$ is the stock-dependent selling rate parameter, $0 \leq \beta \leq 1$, and $I(t)$ is the inventory level at time $t$.
2. A single item is considered with a constant rate of deterioration over a known and finite planning horizon of length $H$.
3. The replenishment rate is infinite and lead time is zero.
4. Shortages are allowed and backlogged partially. We adopt the concept used in Abad (1996), that is, the unsatisfied demand is backlogged, and the fraction of shortages backordered is $e^{-\alpha x}$, where $x$ is the waiting time up to the next replenishment and $\delta$ is a positive constant.
5. Product transactions are followed by instantaneous cash flow.
MATHEMATICAL MODEL

To establish the total present value function of profits over a finite planning time horizon $H$, we suppose that the planning horizon $H$ is divided into $N$ equal parts of length $T = H/N$. Hence, the reorder times over the planning horizon $H$ will be $T_j = jT$ ($j = 0, 1, 2, ..., N$).

The first replenishment lot size of $I_m$ is replenished at $T_0 = 0$. During the time interval $[T_j, t_{j+1}]$ ($j = 0, 1, 2, ..., N-1$), the inventory is depleted due to stock-dependent selling rate and deterioration until it is zero at $t = t_j$ ($j = 1, 2, ..., N$). During the time interval $[t_j, T_j]$ ($j = 1, 2, ..., N$), the inventory level only depends on demand, and unsatisfied demand is backlogged at a rate $e^{-\delta (T-j)}$, where $t \in [T_j, T_j]$ ($j = 1, 2, ..., N$).

A realization of the inventory level in the system is given in Figure 1. Hence, the inventory level during the first replenishment cycle can be represented by the following differential equations:

$$\frac{dI_1(t)}{dt} + \beta I_1(t) = -[\alpha + \beta I_1(t)] \quad 0 \leq t \leq t_1$$  \hspace{1cm} (1)

$$\frac{dI_2(t)}{dt} = -\alpha e^{-\delta (T-t)} \quad t_1 \leq t \leq T$$  \hspace{1cm} (2)

with the boundary conditions $I_1(t_1) = 0$ and $I_2(t_1) = 0$, the solution of (1) and (2) can be represented by:

$$I_1(t) = \frac{\alpha}{\beta + \theta} \left[ e^{(\beta + \theta)(t_1-t)} - 1 \right] \quad 0 \leq t \leq t_1$$  \hspace{1cm} (3)

$$I_2(t) = -\frac{\alpha}{\delta} \left[ e^{-\delta (T-t)} - e^{-\delta (T-t_1)} \right] \quad t_1 \leq t \leq T$$  \hspace{1cm} (4)

Hence, the maximum inventory level during the first replenishment cycle is:

$$I_m = I_2(0) = \frac{\alpha}{\beta + \theta} \left[ e^{(\beta + \theta)H/N} - 1 \right]$$  \hspace{1cm} (5)

During the time interval $[T_j, t_{j+1}]$ ($j = 0, 1, 2, ..., N-1$), the replenished inventory is being consumed due to selling and deterioration. During the time interval $[t_j, T_j]$ ($j = 1, 2, ..., N$) the inventory level only depends on demand, the unsatisfied demand during time interval $[t_j, T_j]$ will be partially backordered at the exponential rate with respect to the waiting time. Under instantaneous cash transaction, the present value of sales revenue during the first cycle is:

$$S = \int_0^{T_1} \left[ \alpha e^{\alpha t} \right] dt + \int_0^{T_1} \left[ \alpha e^{\alpha t} \right] dS = \frac{\alpha}{\beta + \theta} \left[ e^{(\beta + \theta)H/N} - e^{(\beta + \theta)0/N} \right] + \frac{\alpha}{\beta + \theta} \int_0^{T_1} \left[ e^{(\beta + \theta)H/N} - 1 \right]$$  \hspace{1cm} (6)

Since replenishment in each cycle is done at the start of each cycle, the present value of ordering cost during the first cycle is:

$$C_r = A$$  \hspace{1cm} (7)

Inventory occurs during time interval $[T_j, t_{j+1}]$ ($j = 0, 1, 2, ..., N-1$), therefore, the present value of holding cost during the first replenishment cycle is:

$$C_h = c_h \int_0^{T_1} I_1(t) e^{-\delta t} dt = \frac{c_h \alpha}{(\beta + \theta)(\beta + \theta + R)} \left[ e^{(\beta + \theta)H/N} - e^{(\beta + \theta)0/N} \right] + \frac{c_h \alpha}{(\beta + \theta)R} \left[ e^{-\delta H/N} - 1 \right]$$  \hspace{1cm} (8)
Note that the unsatisfied demand during time interval \([t_j, T_j]\) will be partially backordered at the exponential rate with respect to the waiting time \(t\) and \(t \in [t_j, T_j]\) \((j = 1, 2, \ldots, N)\). So, the present value of shortage cost during the first cycle is:

\[
C_s = c_1 \int_0^T \frac{c_1 \alpha}{\delta} \left[ e^{-\delta(T-t)} - e^{-\delta(T-T_1)} \right] \cdot e^{-Rt} \, dt
\]

\[
= c_1 \frac{c_1 \alpha}{\delta} \left[ \frac{e^{-RT} - e^{-\delta(T-T_1)-R_1}}{R(\delta-R)} - e^{-Rt} \right]
\]

\[
= c_1 \frac{c_1 \alpha}{R} \left[ e^{-Rt} - e^{-\delta(T-T_1)} \right] - c_1 \frac{c_1 \alpha}{\delta} \left[ e^{-RT} - e^{-\delta(T-T_1)-R_1} \right]
\]

(9)

and the opportunity cost due to lost sales during the first cycle is:

\[
C_o = c_1 \alpha \int_0^T \left[ 1 - e^{-\delta(T-t)} \right] \cdot e^{-Rt} \, dt
\]

\[
= c_1 \alpha \left[ e^{-Rt} - e^{-\delta(T-T_1)} \right] - c_1 \frac{c_1 \alpha}{R} \left[ e^{-RT} - e^{-\delta(T-T_1)-R_1} \right]
\]

(10)

Replenishment is done at \(t = T_j\) \((j = 0, 1, 2, \ldots, N)\), the replenishment items are consumed by selling as well as deterioration during time interval \([t_j, T_j]\) \((j = 1, 2, \ldots, N)\). So, the present value of purchase cost during the first cycle is:

\[
C_p = cL_m + c e^{-RT} \int_{t_1}^T \alpha e^{-\delta(T-t)} \, dt
\]

\[
= c \frac{\alpha}{\beta + \theta} \left( e^{(\beta+\theta)t_1} - 1 \right) + c \frac{\alpha}{\delta} \left[ 1 - e^{-\delta(T-T_1)} \right]
\]

(11)

Consequently, the total present value of profits during the first replenishment cycle can be formulated as:

\[
\pi = S_a - C_s - C_h - C_s - C_o - C_p
\]

(12)

There are \(N\) cycles during the planning horizon. Since inventory is assumed to start and end at zero, an extra replenishment at \(t = H\) is required to meet the unsatisfied demand of the last cycle in the planning horizon. Therefore, there are \(N+1\) replenishments in the entire time horizon \(H\), the first replenishment lot size is \(I_m\), and the 2nd, 3rd, ..., \(N\)th replenishment lot size is:

\[
Q = I_m + \int_{t_1}^T \alpha e^{-\delta(T-t)} \, dt
\]

(13)

and the last \((N+1)\)th replenishment cost size is:

\[
I_b = \frac{\alpha}{\delta} \left[ 1 - e^{-\delta(T-T_1)} \right]
\]

(14)

So, the total present value of profits over a finite planning time horizon \(H\) is:

\[
P(t_1, N) = \sum_{j=0}^{N-1} e^{-Rt_j} - \alpha e^{-RH}
\]

\[
= \pi \left( 1 - e^{-RRT} \right) - \alpha e^{-RH}
\]

\[
= \left[ \frac{S \alpha}{R} (1-e^{-R_1}) + \frac{S \beta \alpha}{(\beta+\theta)(\beta+R)} (e^{(\beta+\theta)R_1} - e^{-R_1}) + \frac{S \beta \alpha}{(\beta+\theta)R} (e^{-R_1} - 1) \right] e^{-RHT} - AD
\]

\[
+ \left[ \frac{S \alpha}{R} \left( 1 - e^{-R_{11}(N-1)} \right) \right] e^{-RHT}
\]

\[
- \left[ \frac{c_1 \alpha}{R} \left( 1 - e^{-\delta R_{11}(N-1)} \right) \right] e^{-RHT}
\]

\[
= c_1 \alpha \left( e^{-R_1} - 1 \right) + c_1 \frac{c_1 \alpha}{R} \left( 1 - e^{-\delta R_{11}(N-1)} \right)
\]

(15)

where \(\pi\) is derived by substituting (7)-(11) into (12), \(T = H/N\),

\[
D = \frac{e^{RH/N} - e^{-R}}{e^{RH} - 1} \cdot E = 1 - e^{-RH/T} \cdot F = 1 - e^{-RH/N} \cdot F
\]

When the complete backlogging is allowed (that is, \(\delta = 0\)), the total present value of profits in (15) becomes:

\[
P(t_1, N) = \left[ \frac{S \alpha}{R} (1-e^{-R_1}) + \frac{S \beta \alpha}{(\beta+\theta)(\beta+R)} (e^{(\beta+\theta)R_1} - e^{-R_1}) + \frac{S \beta \alpha}{(\beta+\theta)R} (e^{-R_1} - 1) \right] E
\]

\[
+ \left[ \frac{S \alpha}{R} \left( 1 - e^{-R_{11}(N-1)} \right) \right] F - AD - \left[ \frac{c_1 \alpha}{(\beta+\theta)R} \left( e^{(\beta+\theta)R_1} - e^{-R_1} \right) \right] E
\]

\[
+ \left[ \frac{c_1 \alpha}{R} \left( 1 - e^{-\delta R_{11}(N-1)} \right) \right] F - \frac{c_1 \alpha}{\beta+\theta} \left( e^{(\beta+\theta)R_1} - e^{-R_1} \right)
\]

(16)

Then, the (16) is the same as Equation (14) in Hou (2006) with non-sales environment. On the other hand, when the inflation and time value of money are not considered (that is, \(R = 0\)) as well as when complete backlogging is allowed (that is, \(\delta = 0\)) and the number of replenishment per unit times is equal to 1/\(T\) (that is, \(H = 1\)), hence, the total present value of profits in (15) with infinite-time-horizon becomes:
\[
P(T,t) = \frac{1}{T} [(T-t)c_{i} + \alpha \beta + c_{i} \theta] \left[ e^{(\beta + \theta)t} - 1 \right] - A - \frac{\alpha \beta}{2} (T-t)^2 + \alpha (S-c) T + \alpha \theta \left( B - c_{i} \right) \]  
\]  
\[  (17) \]

Note that the (17) is the same as the Equation (19) in Padmanabhan and Vrat (1995) with complete backlogging. Therefore, the model in this paper is an extension of Hou (2006) and Padmanabhan and Vrat (1995).

The objective of this paper is to determine the values of \( t_i \) and \( N \) that maximize \( P(t_i, N) \), where \( t_i \) is a continuous variable and \( N \) is a discrete variable. For a given value of \( N \), the necessary condition for \( P(t_i, N) \) to be maximized is 
\[ dP(t_i, N)/dt_i = 0 \]  
which gives:

\[
dP = \frac{d}{dt}[S\alpha \beta e^{(\beta + \theta)t} - 1 - A - \frac{\alpha \beta}{2} (T-t)^2 + \alpha (S-c) T + \alpha \theta (B - c_i)] = 0 
\]  
\[  (18) \]

Furthermore, proposition 1 in Appendix A shows that the \( P(t_i, N) \) is concave with respect to \( t_i \). So, for a given positive integer \( N \), the optimal value of \( t_i \) can be obtained from (18) by using the Newton-Raphson method. In addition, the following solution procedure is used to derive the optimal \( t_i \) and \( N \) values; this will help managers to determine optimal replenishment policy for the maximum total present value of profits.

**Step 1:** Start by a discrete variable \( N \), where \( N \) is an integer number equal or greater than 1.

**Step 2:** For different integer \( N \) values, derive \( t_i^* \) from (18). Substitute \((t_i^*, N)\) into (15) to obtain \( P(t_i^*, N) \).

**Step 3:** Repeat step 2 for all possible \( N \) values within the lower and upper bound until the maximum \( P(t_i^*, N) \) is found.

The \((t_i^*, N)\) and \( P(t_i^*, N) \) values constitute the optimal solution and satisfy the following condition:

\[
\Delta P(t_i^*, N^*) < 0 < \Delta P(t_i^*, N^* + 1) 
\]  
\[  (19) \]

where 
\[ \Delta P(t_i^*, N^*) = P(t_i^*, N^* + 1) - P(t_i^*, N^*) \]  

Using the optimal solution procedure described earlier, we can find the maximum inventory level to be:

\[
I_m = \frac{\alpha}{\beta + \theta} \left[ e^{(\beta + \theta)\Delta t} - 1 \right] 
\]  

substitute \((t_i^*, N)\) into (13) to derive the 2nd, 3rd, ..., \( N \)th replenishment lot size, we have:

\[
Q^* = \frac{\alpha}{\beta + \theta} \left[ e^{(\beta + \theta)\Delta t} - 1 \right] + \alpha(H / N - t_i^*) 
\]  

**NUMERICAL EXAMPLE AND SENSITIVITY ANALYSIS**

To illustrate the earlier results, we apply the proposed solution procedure to solve the following numerical example. We use the data in Hou (2006): \( \alpha = 600 \) units/year, \( \beta = 0.05 \), \( \theta = 0.2 \), \( A = $250/\text{order}, c_i = $1.75/\text{unit/year}, c_0 = $3.0/\text{unit/year}, R = 0.2 \), \( H = 10 \) year. Besides, let us set \( s = $18/\text{unit}, c_s = $5.0/\text{unit/year}, and \( \delta = 1 \) adapted to our model. For \( s = 0, c_s = 0, and \( \delta = 0 \) this example yields the one used by Hou (2006). In this example, we find that 

\[
S\beta - c_i - \alpha^2 (\beta + \theta + R) = -3.1 < 0 
\]  

Hence, using the solution procedure described earlier, the results are presented in Table 1.

From this table, we see that the maximum total present value of profits is found when the number of replenishments, \( N \), is 19. The total number of replenishments is therefore \((N + 1)\) or 20. With the nineteenth replenishments, the optimal \( t_i \), \( T-t_i \) and \( T \) values are 0.456, 0.070 and 0.526 years, respectively. The optimal order quantity is 332.045 units, and the optimal total present value of profits is $29569.614.

To study the effects of parameters such as \( \delta, \beta, \theta, \) and \( R \) on the optimal replenishment policy, we consider the different values of all \( \delta, \beta, \theta, \) and \( R \) as 0.05, 0.1, 0.15, and 0.20. Besides we assume \( \delta = (0.0, 0.5, 1.5, 2.0) \). The results are summarized in Tables 2 to 5. Based on Tables 2 to 5, the observations can be made thus:

1. When the backlogging rate \( \delta \) is increasing, the total present value of profits is decreasing. In addition, the optimal order quantity \( Q \) and total present value of profits \( P \) are more sensitive to \( \delta \) when its value is small.
2. When the stock- dependent selling rate \( \beta \) is increasing, both the optimal order quantity \( Q \) and total present value of profits \( P \) will increase. That is, the change in \( \beta \) will lead to the positive changes in \( Q \) and \( P \). Furthermore, \( Q \) and \( P \) are all very sensitive to changes in \( \beta \). As a result, the total present value of profits can be increased through appropriately raising the amount of stock level.
Table 1. Optimal solution with partially backlogging.

<table>
<thead>
<tr>
<th>N</th>
<th>1.</th>
<th>T - 1.</th>
<th>T</th>
<th>Q</th>
<th>P(1,N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>0.482</td>
<td>0.074</td>
<td>0.556</td>
<td>351.442</td>
<td>29564.055</td>
</tr>
<tr>
<td>19*</td>
<td>0.456*</td>
<td>0.070*</td>
<td>0.526*</td>
<td>332.045*</td>
<td>29569.614*</td>
</tr>
<tr>
<td>20</td>
<td>0.434</td>
<td>0.066</td>
<td>0.5</td>
<td>314.672</td>
<td>29563.368</td>
</tr>
</tbody>
</table>

*Optimal solution.

Table 2. Effects of $\delta$ on the optimal replenishment policy.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>N*</th>
<th>1*</th>
<th>T*</th>
<th>Q*</th>
<th>P*</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>16</td>
<td>0.386</td>
<td>0.625</td>
<td>386.544</td>
<td>30188.301</td>
</tr>
<tr>
<td>0.5</td>
<td>18</td>
<td>0.446</td>
<td>0.556</td>
<td>348.834</td>
<td>29717.127</td>
</tr>
<tr>
<td>1</td>
<td>19</td>
<td>0.456</td>
<td>0.526</td>
<td>332.045</td>
<td>29569.614</td>
</tr>
<tr>
<td>1.5</td>
<td>19</td>
<td>0.456</td>
<td>0.526</td>
<td>333.304</td>
<td>29496.621</td>
</tr>
<tr>
<td>2.0</td>
<td>20</td>
<td>0.460</td>
<td>0.5</td>
<td>316.506</td>
<td>29453.012</td>
</tr>
</tbody>
</table>

Table 3. Effects of $\beta$ on the optimal replenishment policy.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>N*</th>
<th>1*</th>
<th>T*</th>
<th>Q*</th>
<th>P*</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>19</td>
<td>0.456</td>
<td>0.526</td>
<td>332.045</td>
<td>29569.614</td>
</tr>
<tr>
<td>0.10</td>
<td>17</td>
<td>0.523</td>
<td>0.588</td>
<td>378.949</td>
<td>29939.681</td>
</tr>
<tr>
<td>0.15</td>
<td>15</td>
<td>0.609</td>
<td>0.667</td>
<td>441.919</td>
<td>30382.389</td>
</tr>
<tr>
<td>0.20</td>
<td>13</td>
<td>0.724</td>
<td>0.769</td>
<td>530.93</td>
<td>30942.307</td>
</tr>
</tbody>
</table>

Table 4. Effects of $\theta$ on the optimal replenishment policy.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>N*</th>
<th>1*</th>
<th>T*</th>
<th>Q*</th>
<th>P*</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>17</td>
<td>0.529</td>
<td>0.588</td>
<td>361.485</td>
<td>30060.461</td>
</tr>
<tr>
<td>0.10</td>
<td>17</td>
<td>0.522</td>
<td>0.588</td>
<td>365.548</td>
<td>29886.92</td>
</tr>
<tr>
<td>0.15</td>
<td>18</td>
<td>0.488</td>
<td>0.556</td>
<td>348.077</td>
<td>29723.929</td>
</tr>
<tr>
<td>0.20</td>
<td>19</td>
<td>0.456</td>
<td>0.526</td>
<td>332.045</td>
<td>29569.614</td>
</tr>
</tbody>
</table>

Table 5. Effects of $R$ on the optimal replenishment policy.

<table>
<thead>
<tr>
<th>R</th>
<th>N*</th>
<th>1*</th>
<th>T*</th>
<th>Q*</th>
<th>P*</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>17</td>
<td>0.524</td>
<td>0.588</td>
<td>374.486</td>
<td>54767.439</td>
</tr>
<tr>
<td>0.10</td>
<td>17</td>
<td>0.519</td>
<td>0.588</td>
<td>374.05</td>
<td>43732.539</td>
</tr>
<tr>
<td>0.15</td>
<td>18</td>
<td>0.486</td>
<td>0.556</td>
<td>351.788</td>
<td>35625.207</td>
</tr>
<tr>
<td>0.20</td>
<td>19</td>
<td>0.456</td>
<td>0.526</td>
<td>332.045</td>
<td>29569.614</td>
</tr>
</tbody>
</table>

3. When the deterioration rate $\theta$ is increasing, total present value of profits ($P$) is decreasing. In addition, the total present value of profits is more sensitive to $\theta$ when its value is small.

4. When the discount rate of inflation $R$ is increasing, both the optimal order quantity ($Q$) and total present value of profits ($P$) will decrease. In addition, the total present value of profits is more sensitive to $R$ when its value is small.

**SPECIAL CASES**

The important special cases that influence the optimal total present value of profits are described thus:

1. The inflation and time value of money are not taken into account, that is, $R = 0$.
2. The deterioration of inventory items is not considered, that is, $\theta = 0$. 
the total present value of profits is $34417.872 higher than the case the inflation and time value of money are not considered.

2. When the deterioration of inventory items is not considered (that is, θ = 0), the total present value of profits is $677.452 higher than the case deterioration of inventory items is considered.

3. When the stock-dependent selling rate is neglected (that is, β = 0), the total present value of profits is $320.609 lower than the case stock-dependent selling rate is considered.

4. When the complete backlogging is allowed, (that is, δ = 0), the total present value of profits is $618.687 higher than the case backlogging rate is considered.

Table 6 shows the comparative results for the optimal solution from Table 1 with the special cases above. Based on the comparative results, the following conclusions can be made:

1. When the inflation and time value of money are not considered (that is, R = 0), the total present value of profits is $34417.872 higher than the case the inflation and time value of money are considered.

2. When the deterioration of inventory items is not considered (that is, θ = 0), the total present value of profits is $677.452 higher than the case deterioration of inventory items is considered.

3. When the stock-dependent selling rate is neglected (that is, β = 0), the total present value of profits is $320.609 lower than the case stock-dependent selling rate is considered.

4. When the complete backlogging is allowed, (that is, δ = 0), the total present value of profits is $618.687 higher than the case backlogging rate is considered.

### Conclusion

In this paper, an inventory model is developed for deteriorating items with stock-dependent selling rate, and partial backlogging under inflation and time discounting. In the model, shortages are allowed and the rate of backlogged demand decreases exponentially as the waiting time for the next replenishment increases. In addition, since the inventory systems need to invest large capital to purchase inventories, which is highly correlated to the return of investment, it is important to consider the effects of inflation and time value of money in inventory decision-making. The proposal model can be seen as an extension of Hou (2006) by considering exponential partial backlogging, and then we incorporate the effects of inflation and time value of money for a finite planning horizon inventory model with deteriorating items. We have given an analytic formulation of the problem on the framework described above and have presented an optimal solution procedure to find optimal replenishment policy. From our research results, we have verified that the effects of stock-dependent selling rate under the conditions of inflation and time value of money result in higher discounted total profit than a policy which does ignore the effects of stock-dependent selling rate. Finally, the sensitivity of the solution to changes in the values of different parameters has been discussed. The results indicate that the effects of inflation and time value of money, stock-dependent selling rate, deterioration of items, and backlogging rate on optimal replenishment policy are significant. Hence, the situations presented should not be ignored in developing the inventory model. Moreover, we know that controlling market demand through the manipulation of selling price is an important strategy for increasing profit. Hence, this model shall be extending to the case of the stock and price-dependent selling rate for further research.

### Acknowledgements

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### References


Appendix

Proof of Proposition

Proposition: \( P(t_1, N) \) is convex with respect to \( t_1 \), provided that \( S\beta - c_1 - c(\beta + \theta + R) < 0 \).

As showed in Equation (15), the total present value of profits for the planning time horizon \( H \) is given by:

\[
P(t_1, N) = \pi \left( \frac{1 - e^{-RT}}{1 - e^{-R}} \right) - Ae^{-RH}
\]

On simplification and summation, we get:

\[
P(t_1, N) = \left( \frac{S\alpha}{R} (1 - e^{-R}) + \frac{S\beta\alpha}{(\beta + \theta)(\beta + \theta + R)} (e^{(\beta + \theta)H} - e^{-R}) + \frac{S\beta\alpha}{(\beta + \theta)R} (e^{-R} - 1) \right)
\]

\[
+ \frac{S\alpha}{\delta} \left[ 1 - e^{-\delta(T - t_1)} \right] e^{-RT} - A - \frac{c_\alpha}{(\beta + \theta)(\beta + \theta + R)} (e^{(\beta + \theta)H} - e^{-R})
\]

\[
- \frac{c_\alpha}{(\beta + \theta)R} (e^{-R} - 1) \left[ 1 - e^{-\delta(T - t_1)} \right] e^{-RT} - \frac{c_\alpha}{R(\delta - R)} \left[ 1 - e^{-(\delta - R)(T - t_1)} \right] e^{-RT}
\]

\[
- \frac{c_\alpha}{\delta} \left[ 1 - e^{-\delta(T - t_1)} \right] e^{-RT} \frac{1 - e^{-RH}}{1 - e^{-R}} - Ae^{-RH}
\]

\[
= \left( \frac{S\alpha}{R} (1 - e^{-R}) + \frac{S\beta\alpha}{(\beta + \theta)(\beta + \theta + R)} (e^{(\beta + \theta)H} - e^{-R}) + \frac{S\beta\alpha}{(\beta + \theta)R} (e^{-R} - 1) \right) E
\]

\[
\quad + \left( \frac{S\alpha}{\delta} - \frac{c\alpha}{\delta} \right) \left[ 1 - e^{-\delta(H/N - t_1)} \right] F - AD
\]

\[
- \frac{c\alpha}{(\beta + \theta)(\beta + \theta + R)} (e^{(\beta + \theta)H} - e^{-R}) + \frac{c\alpha}{(\beta + \theta)R} (e^{-R} - 1) \right) E
\]

\[
- \frac{c\alpha}{\delta} \left[ 1 - e^{-\delta(H/N - t_1)} \right] e^{-RT} + \frac{c\alpha}{R(\delta - R)} \left[ 1 - e^{-(\delta - R)(H/N - t_1)} \right] e^{-RT}
\]

\[
- \frac{c\alpha}{R} \left[ e^{-R(t_1 - H/N)} - 1 \right] F + \frac{c\alpha}{\delta - R} \left[ 1 - e^{-(\delta - R)(H/N - t_1)} \right] F - \frac{c\alpha E}{\beta + \theta} (e^{(\beta + \theta)H} - e^{-R})
\]

therefore, we have:

\[
\frac{dP(t_1, N)}{dt_1} = \left( S\alpha e^{-R} + \frac{S\beta\alpha}{(\beta + \theta)(\beta + \theta + R)} [(\beta + \theta)e^{(\beta + \theta)H} + Re^{-R}] + \frac{S\beta\alpha}{(\beta + \theta)R} e^{-R} \right) E
\]

\[
- \left( S\alpha - c\alpha e^{-\delta(H/N - t_1)} \right) F - \frac{c\alpha}{(\beta + \theta)(\beta + \theta + R)} [(\beta + \theta)e^{(\beta + \theta)H} + Re^{-R}]
\]

\[
- \frac{c\alpha}{(\beta + \theta)} e^{-R} \right) E - \frac{c\alpha}{R} \left[ e^{-\delta(H/N - t_1)} - e^{-(\delta - R)(H/N - t_1)} \right] F
\]

\[
+ \frac{c\alpha}{e^{-R(t_1 - H/N)}} - e^{-(\delta - R)(H/N - t_1)} \right] F - \frac{c\alpha E}{(\beta + \theta)H} \]
where

\[ D = \frac{e^{-RH_{1N}} - e^{-RH}}{e^{-RH_{1N}} - 1}, \quad E = \frac{1 - e^{-RH}}{1 - e^{-RH_{1N}}}, \quad \text{and} \quad F = \frac{1 - e^{-RH}}{e^{-RH_{1N}} - 1}. \]

and

\[
\frac{d^2 P(t_1, N)}{dt_1^2} = \left\{ \frac{\alpha(S\beta - c_i)}{(\beta + \theta)(\beta + \theta + R)} \left[ (\beta + \theta)^2 e^{(\beta + \theta)t_1} - R^2 e^{-Rt_1} \right] - S\alpha Re^{-Rt_1} + \frac{R\alpha(S\beta - c_i)}{\beta + \theta} e^{-Rt_1} \right\} E
\]

\[- (S\alpha - c\alpha)e^{-\delta(H/N-t_1)} F - \frac{c\alpha}{R} \left[ e^{-\delta(H/N-t_1)} - (\delta - R)e^{-(\delta-R)(H/N-t_1)} \right] F
\]

\[- c\alpha \left[ Re^{R(H/N-t_1)} + (\delta - R)e^{-(\delta-R)(H/N-t_1)} \right] F - c\alpha e(\beta + \theta)e^{(\beta+\theta)t_1}
\]

\[
= \left[ \frac{\alpha(S\beta - c_i)(\beta + \theta)^2}{(\beta + \theta)(\beta + \theta + R)} - c\alpha(\beta + \theta) \right] e^{(\beta+\theta)t_1} E
\]

\[- \frac{c\alpha}{R} \left[ e^{-\delta(H/N-t_1)} - (\delta - R)e^{-(\delta-R)(H/N-t_1)} \right] F
\]

\[- c\alpha \left[ Re^{R(H/N-t_1)} + (\delta - R)e^{-(\delta-R)(H/N-t_1)} \right] F
\]

\[
= \left[ \alpha(\beta + \theta)^2 \left( S\beta - c_i - c(\beta + \theta + R) \right) \right] \left( \frac{1}{(\beta + \theta)(\beta + \theta + R)} \right) e^{(\beta+\theta)t_1} E
\]

\[- \frac{Rc}{(\beta + \theta)(\beta + \theta + R)} e^{-Rt_1} E - (S\alpha - c\alpha)e^{-\delta(H/N-t_1)} F
\]

\[- \frac{c\alpha}{R} \left[ e^{-\delta(H/N-t_1)} - (\delta - R)e^{-(\delta-R)(H/N-t_1)} \right] F
\]

\[- c\alpha \left[ Re^{R(H/N-t_1)} + (\delta - R)e^{-(\delta-R)(H/N-t_1)} \right] F \quad (A3)
\]

If \( S\beta - c_i - c(\beta + \theta + R) < 0 \), the (A3) yields \( \frac{d^2 P(t_1, N)}{dt_1^2} < 0 \). Consequently, \( P(t_1, N) \) is concave with respect to \( t_1 \) provided that \( S\beta - c_i - c(\beta + \theta + R) < 0 \). This completes the proof.