Abstract. We observe that the splitting of a given logic $L$ into sublogics $L_i$ can be seen as a covering of $L$ by the $L_i$ in a category of logics and take the first steps in exploring the possibility of applying the related language of Grothendieck topologies in this setting.

1 Introduction

The goal of combining two logics $L_1$ and $L_2$ has in [1] been described as to obtain “the smallest logic system for the combined language which is a conservative extension of both $L_1$ and $L_2$”. To achieve this goal, D. Gabbay in [2] proposed the mechanism of fibring which in [3] and subsequent articles was recognised as the construction of a colimit in an appropriate category of logics. Following this hint the notions of modulated fibring ([4]) and metafibring ([1]) have been presented as colimit constructions in different categories.

It was also observed that a presentation of a logic as a colimit of other logics can be seen as a splitting of the logic into sublogics, possibly helping to understand the more complex logic. We observe that the situation when a logic $L$ is completely determined by the translation of other logics $L_i$ into $L$ can be seen as a “covering” of $L$ by the $L_i$. To turn this intuition into a mathematical statement it is necessary to choose a category of logics ($\mathcal{L}$) and give a rigorous definition of covering. We discuss the advantages and disadvantages of some possible such choices.

Furthermore, a notion of covering induces a Grothendieck topology on the (essentially) small full subcategory $\mathcal{L}_{fp}\subseteq\mathcal{L}$ of finitely presentable logics. With this notion of sheaf we are able to relate logics (=filtered colimits of finitely presentable logics) and sheaves within the category $\text{Set}^{(\mathcal{L}_{fp})^{op}}$ of presheaves (=colimits of contravariant hom functors of finitely presentable logics).

To realize this description we start with a brief ad hoc approach and then proceed to some adaptations of this approach. A common category $\mathcal{L}$ of logics is given by taking as objects pairs $(\Sigma,\vdash)$ with $\Sigma$ a signature (i.e. a set of connectives with given arity) and $\vdash$ a consequence relation on $F(\Sigma)$ (the formula algebra

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freely generated by $\Sigma$ over a predetermined enumerable set of variables). As morphisms $f : (\Sigma, \vdash) \longrightarrow (\Sigma, \vdash')$ sequences of functions $(f_n : \Sigma_n \longrightarrow \Sigma'_n)_{n \in \omega}$ such that the induced formula algebra mapping $\hat{f} : F(\Sigma) \longrightarrow F(\Sigma')$ is a translation, i.e. $\Gamma \vdash \phi$ implies $\hat{f}(\Gamma) \vdash' \hat{f}(\phi)$. A covering of $(\Sigma, \vdash)$ could be defined as a set of morphisms $f_i : (\Sigma_i, \vdash_i) \longrightarrow (\Sigma, \vdash)$ such that the consequence relation on $L(\Sigma)$ generated by the images $f_i(\vdash_i)$ is stronger or equal than $\vdash$.

Every logic is a filtered colimit of “finite type” logics, so, identifying the category $L_{fp}$ of such logics with the representable functors in $\text{Set}(L_{fp})^{op}$, we obtain an equivalence of $\text{Flat}((L_{fp})^{op}, \text{Set})$ (the filtered colimits of representables) with $\mathcal{L}$ and an embedding of both into $\text{Set}(L_{fp})^{op}$. Also, the above notion of covering gives rise to a Grothendieck topology on $L_{fp}$ and a corresponding category of sheaves which is linked to $\text{Set}(L_{fp})^{op}$ by the usual adjoint pair of the inclusion and sheafification functors. Altogether we get the following picture:

Throughout this paper, we take a look at some variations of this ad hoc example: How well the categories in the above diagram relate to each other depends strongly on the choice of $\mathcal{L}$ and the coverings. For instance, we can choose $\mathcal{L}$ such that it fulfills certain (co)completeness conditions or require a covering to be an epimorphic family (which implies that all representable functors are separated). Other choices are possible without affecting the overall categorial picture; for example, requiring the morphisms in a covering to be monic (which amounts to allowing only coverings by proper linguistic fragments). A good part of these variations depend only on the underlying category of signatures.

We finally mention that the above considerations should apply equally well to different ambient categories for fibring like those of $\pi$-institutions or the metafibring categories from [1], but we shall not pursue that here.

2 On the categories of signatures and logics

In this section we will define a category of signatures $\mathcal{S}$ and a category of logics $\mathcal{L}$ built above it and prove the latter to be locally presentable. Since local presentability leads to a number of desirable categorial properties, this is an interesting (and welcome) result in its own. Moreover, it enables us to embed $\mathcal{L}$ into the category of presheaves over the essentially small category of finitely
presentable logics (to be defined in section 2.2) and thus enables the application of sheaf theoretic notions to logics as done in section 3.

2.1 The category $\mathcal{S}$

The category $\mathcal{S}$ is the category of signatures and morphisms of signatures. Consider $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ an enumerable set (written in a fixed order) [5].

**What is $\mathcal{S}$?** Objects: A signature $\Sigma$ is a sequence of sets $\Sigma = (\Sigma_n)_{n \in \omega}$ such that $\Sigma_i \cap \Sigma_j = \emptyset$ for every $i < j < \omega$. We will write $|\Sigma| = \bigcup_{n \in \omega} \Sigma_n = \bigcup_{n \in \omega} \Sigma_n \times \{n\}$.

We will write $F(\Sigma)$ for the set of all (propositional) formulas built with signature $\Sigma$ over the variables in $X$. The notion of complexity $l(\varphi)$ of the formula $\varphi$ is the usual: (i) $l(\varphi) = 1$ if $\varphi \in X \cup \Sigma_0$; and (ii) $l(\varphi) = 1 + l(\psi_0) + \ldots + l(\psi_{n-1})$ if $\varphi = c(\psi_0, \ldots, \psi_{n-1})$ where $c \in \Sigma_n$ and $n > 0$.

We will call a signature $\Sigma$ a finite type if $|\Sigma|$ is finite.

Morphisms: If $\Sigma, \Sigma'$ are signatures, then a morphism $f : \Sigma \rightarrow \Sigma'$ is a sequence of functions $f = (f_n)_{n \in \omega}$ where $f_n : \Sigma_n \rightarrow \Sigma'_n$.

If $f : \Sigma \rightarrow \Sigma'$ then there is only one function $\hat{f} : F(\Sigma) \rightarrow F(\Sigma')$ (called the extension of $f$) such that:

* $\hat{f}(x) = x$ if $x \in X$;
* $\hat{f}(c) = f_0(c)$ if $c \in \Sigma_0$;
* $\hat{f}(c(\psi_0, \ldots, \psi_{n-1})) = f_n(c)(\hat{f}(\psi_0), \ldots, \hat{f}(\psi_{n-1}))$ if $c \in \Sigma_n, n > 0$.

Composition is componentwise. The extension to the formula algebra of a composition is the extensions’ composition.

Identities are the sequences of identities on each level $n, n \in \omega$. The extension of an identity is the identity function on the formula algebra.

**Some facts about $\mathcal{S}$**

**Remark 1.** About substitution:

(i) For any function $\sigma : X \rightarrow F(\Sigma)$, called substitution, there is only one extension $\hat{\sigma} : F(\Sigma) \rightarrow F(\Sigma)$ such that $\hat{\sigma}$ is a “homomorphism”:\n
$\hat{\sigma}(x) = \sigma(x)$, for every $x \in X$ and $\hat{\sigma}(c_n(\psi_0, \ldots, \psi_{n-1}) = \sigma(c_n)(\hat{\sigma}(\psi_0), \ldots, \hat{\sigma}(\psi_{n-1}))$, for every $c_n \in \Sigma_n, n \in \omega$.

(ii) Let $f : (\Sigma, \vdash) \rightarrow (\Sigma', \vdash')$ a $\mathcal{L}$-morphism. Then for any substitution $\sigma : X \rightarrow F(\Sigma)$, there is another substitution $\sigma' : X \rightarrow F(\Sigma')$ such that $\hat{\sigma'} \circ \hat{f} = \hat{f} \circ \hat{\sigma}$.

**Proof.** Item (i) follows directly by induction on complexity. For (ii) take $\sigma' : X \rightarrow F(\Sigma')$ such that $\sigma' = \hat{f} \circ \sigma$ and the claim follows by induction on complexity. □

**Proposition 1.** $\mathcal{S}$ is a complete and cocomplete category.
Proof. Observe that $\mathcal{S}$ is equivalent to the functor category $\text{Set}^{\mathcal{N}}$ where $\mathcal{N}$ is the discrete category with object class $\mathcal{N}$. Hence, $\mathcal{S}$ has all the small limits and colimits and they are componentwise.

We will write the constructions but will omit the (standard) verifications:

**Limits** Let $\mathcal{I}$ be a small category and $D : \mathcal{I} \rightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{\delta^h_i} \Sigma^j)_{(i \xrightarrow{h} j) \in \mathcal{I}}$ a diagram. Then $(\Sigma^i, (\pi^i)_{i \in \text{obj}(\mathcal{I})})$ is the limit of this diagram if we take:
- $\Sigma_n = \{c = (c_i)_{i \in \text{obj}(\mathcal{I})} \in \prod_{i \in \text{obj}(\mathcal{I})} \Sigma_i^i :$ for every $\mathcal{I}$-arrow $(i \xrightarrow{h} j), \delta_n^h(c_i) = c_j\}$;
- $\pi^i : \Sigma_n \rightarrow \Sigma_n^i$ such that if $c = (c_i)_{i \in \text{obj}(\mathcal{I})} \in \Sigma_n$ then $\pi^i(c) = c_i$, $n \in \omega$ and $i \in I$. In fact, (a) By construction, $\Sigma \in \text{Obj}(\mathcal{S})$ and $\pi^i \in \mathcal{S}(\Sigma, \Sigma^i)$, $i \in \text{obj}(\mathcal{I})$.
- (b) For every $(i \xrightarrow{h} j) \in \text{Mor}(\mathcal{I})$, $\pi^j = \delta_n^h \circ \pi^i$ since that, for every $n \in \mathcal{N}$ and $c = (c_i)_{i \in \text{obj}(\mathcal{I})} \in \Sigma_n$, we have $\pi^j(c) = \delta_n^h(\pi^i(c))$.
- (c) $(\Sigma, (\pi^i)_{i \in \text{obj}(\mathcal{I})})$ is the universal cone over the diagram $D$; if $(\Sigma', (\alpha^i)_{i \in \text{obj}(\mathcal{I})})$ is a commutative cone over the diagram $D$ then, for every $n \in \omega$, define $\alpha_n : \Sigma'_n \rightarrow \Sigma_n$, if $i \in \Sigma_n$ we have $\pi^i(c) = \delta_n^h(\pi^i(c))$, and it follows that $\alpha_n$ is a well defined function, $\alpha$ is a signature morphism such that $\alpha^i = \pi^i \circ \omega$, $i \in I$ and $\alpha$ is the only signature morphism satisfying that property.

We now describe the most relevant kind of colimit in this work:

**Filtered colimits** Let $\langle I, \leq \rangle$ be a directed ordered set and $D : \langle I, \leq \rangle \rightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{\delta^h_i} \Sigma^j)_{(i \leq j) \in I}$ a diagram. Then $(\Sigma^i, (\gamma^i)_{i \in I})$ is the colimit of this diagram if we take:
- $\Sigma_n = \bigsqcup_{i \leq j} (\Sigma^i)^j_n / \sim_n$ where, if $c_i \in \Sigma_n^i$, $c_j \in \Sigma_n^j$ $(c_i, i) \sim_n (c_j, j)$ iff there is $k \geq i, j$ such that $(f_{ik})^n_i(c_i) = (f_{jk})^n_j(c_j), n \in \omega$; it follows from the directness assumption that $\sim_n$ is an equivalence relation on $\bigsqcup_{i \leq j} (\Sigma^i)^j_n$;
- $\gamma^i_n : \Sigma_n^i \rightarrow \Sigma_n$ such that if $c_i \in \Sigma_n^i$ then $\gamma^i_n(c_i) = [(c_i, i)], n \in \omega$ and $i \in I$. In fact, (a) By construction, $\Sigma \in \text{Obj}(\mathcal{S})$ and $\gamma^i \in \mathcal{S}(\Sigma^i, \Sigma), i \in I$.
- (b) For every $(i \leq j) \in \langle I, \leq \rangle$, $\gamma^j = \gamma^i \circ f^i_j$. Hence, for every $n \in \mathcal{N}$ and $c_i \in \Sigma^i_n$, we have $(c_i, i) \sim_n (f^i_j(c_i), j)$.
- (c) $(\Sigma, (\gamma^i)_{i \in I})$ is the universal cocone over the diagram $D$: if $(\Sigma', (\alpha^i)_{i \in I})$ is a commutative cocone over the diagram $D$ then, for every $n \in \omega$, define $\alpha_n : \Sigma'_n \rightarrow \Sigma_n$, if $c \in \Sigma_n$ choose $i \in I$ such that $c = [(c_i, i)]$ and take $\alpha_n(c) = \alpha^i_n(c_i)$. It follows that $\alpha_n$ is a well defined function, $\alpha$ is a signature morphism such that $\alpha^i = \omega \circ \gamma^i$, $i \in I$ and $\alpha$ is the only signature morphism satisfying that property.

**Colimits** Let $\mathcal{I}$ be a small category and $D : \mathcal{I} \rightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{h} \Sigma^j)_{(i \xrightarrow{h} j) \in \mathcal{I}}$ a diagram. Then $(\Sigma, (\gamma^i)_{i \in \text{obj}(\mathcal{I})})$ is the colimit of this diagram if we take:
- $\Sigma_n = \bigsqcup_{i \xrightarrow{h} j} (\Sigma^i)_n^j / \sim_n$ where $\sim_n$ is the smallest equivalence relation on $\bigsqcup_{i \xrightarrow{h} j} (\Sigma^i)_n^j$ such that for every $\mathcal{I}$-arrow $(i \xrightarrow{h} j)$ if $c_i \in \Sigma^i_n$ then $(c_i, i) \sim_n (f^h_n(c_i), j)$;
- $\gamma^i_n : \Sigma_n^i \rightarrow \Sigma_n$ such that if $c_i \in \Sigma_n^i$ then $\gamma^i_n(c_i) = [(c_i, i)], n \in \omega$ and $i \in \text{obj}(\mathcal{I})$. In fact,
(a) By construction, \( \Sigma \in \text{Obj}(\mathcal{S}) \) and \( \gamma^i \in \mathcal{S}(\Sigma^i, \Sigma) \), \( i \in \text{obj}(\mathcal{I}) \).

(b) For every \( \mathcal{I} \)-arrow \( (i \xrightarrow{h} j) \), \( \gamma^i = \gamma^j \circ f^h \) since that, for every \( n \in \mathbb{N} \) and \( c_i \in \Sigma^i_n \), we have \( (c_i, i) \sim_n (f^h_n(c_i), j) \).

(c) \((\Sigma, (\gamma^i)_{i \in \text{obj}(\mathcal{I})})\) is the universal cocone over the diagram \( D \); if \((\Sigma', (\alpha^i)_{i \in \text{obj}(\mathcal{I})})\) is a commutative cocone over the diagram \( D \) then, for every \( n \in \omega \), define \( \alpha_n : \Sigma_n \rightarrow \Sigma'_n \), if \( c \in \Sigma_n \), choose \( i \in \text{obj}(\mathcal{I}) \) such that \( c = [(c_i, i)] \) and take \( \alpha_n(c) = \alpha^i_n(c_i) \). It follows that \( \alpha_n \) is a well defined function, \( \alpha \) is a signature morphism such that \( \alpha^i = \alpha \circ \gamma^i \), \( i \in \text{obj}(\mathcal{I}) \) and \( \alpha \) is the only signature morphism satisfying that property. \( \square \)

**Remark 2.** About monomorphisms and epimorphisms in \( \mathcal{S} \): Let \( f : \Sigma \rightarrow \Sigma' \) a signature morphism.

(i) \( f \) is a \( \mathcal{S} \)-monomorphism iff for every \( n \in \omega \), \( f_n : \Sigma_n \rightarrow \Sigma'_n \) is injective;

(ii) \( f \) is a \( \mathcal{S} \)-epimorphism iff for every \( n \in \omega \), \( f_n : \Sigma_n \rightarrow \Sigma'_n \) is surjective;

If \( f \) is a \( \mathcal{S} \)-epimorphism then \( \hat{f} : F(\Sigma) \rightarrow F(\Sigma') \) is injective;

If \( f \) is a \( \mathcal{S} \)-epimorphism then \( \hat{f} : F(\Sigma) \rightarrow F(\Sigma') \) is surjective.

\( \mathcal{S} \) is a **locally presentable category** We have just seen that the category \( \mathcal{S} \) is complete and cocomplete. Besides, it has other nice categorial property: it is a *finitely accessible category*. Then \( \mathcal{S} \) is a finitely locally presentable category (a complete and cocomplete finitely accessible category).

**Remark 3.** Additional facts on filtered colimits in \( \mathcal{S} \): Let \( D : (I, \leq) \rightarrow \mathcal{S} \), \((\Sigma^i \xrightarrow{f^i} \Sigma'^i)_{(i \leq j) \in I} \) be a directed diagram and let \((\Sigma', (\alpha^i)_{i \in I})\) be a commutative cocone over the diagram \( D \):

(i) \((\Sigma', (\alpha^i)_{i \in I})\) is the universal colimit cocone of diagram \( D \) iff:

* \( \Sigma'_n = \bigcup_{i \in I} \alpha_n^i[\Sigma_n^i] \), \( n \in \omega \);

* If \( c_i \in \Sigma^i_n, c_j \in \Sigma^j_n \) are such that \( \alpha^i_n(c_i) = \alpha^j_n(c_j) \), then there is \( k \geq i, j \) such that \( f^k_n(c_i) = f^k_n(c_j) \), \( n \in \omega \).

(ii) \((\Sigma', (\alpha^i)_{i \in I})\) is the universal colimit cocone of diagram \( D \), as we see above, for every \( n \in \omega \), \( \Sigma'_n = \bigcup_{i \in I} \alpha^i_n[\Sigma^i_n] \). It follows easily from the directness condition, by induction on complexity, that any formula in the colimit signature can be “obtained at given defined time”, that is, \( F(\Sigma') = \bigcup_{i \in I} F(\Sigma^i) \) and, analogously, any finite set of formulas in the colimit signature can be “obtained at a given defined time”.

(iii) If, for every \((i \leq j) \in I \) and every \( n \in \omega \), \( f^j_n : \Sigma^i_n \rightarrow \Sigma^j_n \) is injective, then if \((\Sigma', (\alpha^i)_{i \in I})\) is the universal colimit cocone of diagram \( D \) so for every \( n \in \omega \), \( \alpha^i_n : \Sigma^i_n \rightarrow \Sigma'_n \) is injective.

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3 A category \( C \) is **finitely accessible** if it has filtered colimits and a set \( A \) of objects that are finitely presentable (\( A \subseteq C_{fp} \)) and every object in the category is a filtered colimit of a diagram of finitely presentable objects in that set \( A \); an object \( c \) is called finitely presentable if the hom functor \( \text{Hom}_C(c, ?) \) preserves all filtered colimits. (See [6], [7], [8])
We have to prove that for every signature morphism $i$, such that for each $n$, $(\Sigma, (\Sigma', \Sigma)_{\Sigma' \in I})$ is surjective: $\Sigma \rightarrow \Sigma_n$ is injective, we have $c = d$.

\[ \forall i \in I \text{ and } n \in \omega. \text{ Let } c_i, d_i \in \Sigma_n \text{ be such that } \alpha_n(c_i) = \alpha_n(d_i). \text{ Then, by item (i), there is } k \geq i \text{ such that } f_n^k(c_i) = f_n^k(d_i) \text{ and, as } f_n^k : \Sigma_n \rightarrow \Sigma_k \text{ is injective, we have } c_i = d_i. \]

**Proposition 2.** Every signature is a directed colimit of finite type signatures.

**Proof.** Consider $I$ the set of every $\Sigma'$ such that for every $n \in \omega$, $\Sigma_n \subseteq \Sigma_n$ and $|\Sigma'| \leq f_{\Sigma_n} |\Sigma|$. Take the pointwise order relation $\Sigma' \leq \Sigma''$ in $I$ iff for every $n \in \omega$, $\Sigma_n \subseteq \Sigma_n$. Then:

- $(I, \leq)$ is a directed ordered set;
- The obvious diagram $D : (I, \leq) \rightarrow S : (\Sigma' \leq \Sigma'') \mapsto (\Sigma' \hookrightarrow \Sigma'')$ is such that $(\Sigma', (\Sigma' \hookrightarrow \Sigma)_{\Sigma' \in I})$ is a $D$-cone commutative;
- By the characterization in Fact 3.(i), $(\Sigma, (\Sigma' \hookrightarrow \Sigma)_{\Sigma' \in I})$ is a colimit cone over $D$, so $\Sigma$ is a directed colimit of finite type (sub)signatures.

\[ \forall i \in I \text{ and a signature morphism } h : \Sigma' \rightarrow \Sigma_i \text{ such that } (\Sigma' \hookrightarrow \Sigma)_{\Sigma' \in I} = (\Sigma' \xrightarrow{h} \Sigma_i \xrightarrow{\gamma_i} \colim_{\Sigma' \in I} \Sigma_i). \]

$k$ is surjective: We have to prove that for every signature morphism $h : \Sigma' \rightarrow \Sigma_i$ such that $(\Sigma' \xrightarrow{h} \colim_{\Sigma' \in I} \Sigma_i) = (\Sigma' \xrightarrow{h} \Sigma_i \xrightarrow{\gamma_i} \colim_{\Sigma' \in I} \Sigma_i)$, then there is only one $i, i_1 \in I$ such that $h_{n_i} \Sigma_n \subseteq \gamma_i \Sigma_n$. There is only one finite set $\{n_0, \ldots, n_{t-1}\} \subseteq \mathbb{N}$ such that for every $r < t$, $\Sigma_{n_r}$ is a directed ordered set, and $\gamma_r$ is a directed ordered set, take an $i_1 \in I$ such that $h_{n_i} [\Sigma_n] \subseteq \gamma_{n_1} [\Sigma_n]$. Then just take, for each $n \in \omega$, $h_n : \Sigma_n \rightarrow \Sigma_i$ such that for each $c'_n \in \Sigma_i$, $h_n(c'_n) \in \Sigma_n$ is such that $h_n(c'_n) = ([h_n(c'_n), i])$ and so $h = \gamma_i \circ h'$.

$k$ is injective: We have to prove that for every signature morphism $h : \Sigma' \rightarrow \Sigma_i$ such that there are $i_0, i_1 \in I$ and signature morphism $h_{n_0} : \Sigma' \rightarrow \Sigma_{n_0}$, $h_{n_1} : \Sigma' \rightarrow \Sigma_{n_1}$ such that $\gamma_{n_0} \circ h_{n_0} = h = \gamma_{n_1} \circ h_{n_1}$, then there is a $j \geq i_0, i_1$ and a $h' : \Sigma' \rightarrow \Sigma_j$ such that $f_{n_0}^j \circ h_{n_0} = h' = f_{n_1}^j \circ h_{n_1}$.

As for each $i \in \{i_0, i_1\}$ and $n \in \omega$, $h_n : \Sigma_n \rightarrow \Sigma_i$ is such that for each $c'_n \in \Sigma_n$, $h_n(c'_n) \in \Sigma_i$ is such that $[(h_n(c'_n), i_0)] = h_n(c'_n) = ([h_n(c'_n), i_1])$, because $h = \gamma_i \circ h'$, then there is $n \geq i_0, i_1$ such that $f_{n_0}^j (h_{n_0}(c'_n)) = f_{n_1}^j (h_{n_1}(c'_n)) \in \Sigma_n^j$. As $\Sigma_n$ is a finite set there is only a finite set $\{n_0, \ldots, n_{t-1}\} \subseteq \mathbb{N}$ such that for every $r < t$, $\Sigma_{n_r} \neq \emptyset$. As $(I, \leq)$ is a directed ordered set, take an $i \geq j_0, \ldots, j_n$. As $D$ is a diagram, for each $n \in \omega$, $f_{n_0}^j \circ h_{n_0} = f_{n_1}^j \circ h_{n_1}$. Take $h' = f_{n_1}^j \circ h'$, $i \in \{i_0, i_1\}$.
Let $\Sigma$ be a finitely presentable logic. Then, by Prop. 2, there is a directed diagram of logics $D : (I, \leq) \rightarrow S$ , $(\Sigma^i \xrightarrow{f_{ij}} \Sigma^j)_{(i \leq j) \in I}$ such that there is an isomorphism $h : I \xrightarrow{\cong} \text{colim}_{i \in I} \Sigma^i$. Then, as the canonical morphism is invertible $k : \text{colim}_{i \in I} S(\Sigma, \Sigma^i) \xrightarrow{\cong} S(\Sigma, \text{colim}_{i \in I} \Sigma^i)$, there is a factorization of $h : (\Sigma \xrightarrow{h} \text{colim}_{i \in I} \Sigma^i) = (\Sigma \xrightarrow{h^i} \Sigma^i \xrightarrow{\gamma^i} \text{colim}_{i \in I} \Sigma^i)$. Then, as $h$ is an isomorphism, $h^i : \Sigma \rightarrow \Sigma^i$ is an $S$-section. Specifically, there is a sequence of injections $(h^i_n : \Sigma_n \hookrightarrow \Sigma_n)_{n \in \omega}$, so, as $|\Sigma^i|$ is finite, then $|\Sigma|$ is finite. \hfill \Box

Theorem 1. The category $S$ is a finitely locally presentable category, that is, $S$ is an accessible category that is cocomplete and complete.

Proof. Direct consequence of Props. 2, 3 and 1. \hfill \Box

Corollary 1. (i) The Yoneda functor $Y : S_{fp} \rightarrow \text{Set}(S_{fp})^{op}$ has an extension to a functor $Y' : S \rightarrow \text{Set}(S_{fp})^{op}$, $\Sigma \mapsto Y'(\Sigma) = \mathcal{L}(\iota(\cdot), \Sigma)$ that is full and faithful.

(ii) Let $\text{Flat}(S_{fp}, \text{Set})$ be the full subcategory of $\text{Set}(S_{fp})^{op}$ with objects being the functors that are filtered colimits of representable functors. Then $\text{Flat}(S_{fp}, \text{Set})$ is the “essential image” of $Y'$ and so its restriction functor $E : S \rightarrow \text{Flat}(S_{fp}, \text{Set})$ is an equivalence of categories.

(iii) $\text{Flat}(S_{fp}, \text{Set})$ coincides with the category of $\text{Set}$-valued functors that preserve finite limits.

(iv) $Y'$ has a left adjoint.

Proof. For (i) and (ii) see [7] The. 5.3.5 (p. 265) or [6] The. 2.26 (p. 83) or [8] Obs. 1.6 (p. 46). For (iii) and (iv) see [6] The. 1.46 (p. 38). \hfill \Box

2.2 The category $\mathcal{L}$

The category $\mathcal{L}$ is the category of propositional logics and translations as morphisms. This is a category “built above” the category $S$, that is, there is a (forgetful) functor $U : \mathcal{L} \rightarrow S$.

What is $\mathcal{L}$? Objects $A$ logic is an ordered pair $l = (\Sigma, \vdash)$ where $\Sigma$ is an object of $S$ and $\vdash$ codifies the “consequence operator” on $F(\Sigma)$: $\vdash$ is a binary relation $\subseteq \text{Parts}(F(\Sigma)) \times F(\Sigma)$ such that $\text{Cons}(\Gamma) = \{ \varphi \in F(\Sigma) : \Gamma \vdash \varphi \}$ for every $\Gamma \subseteq F(\Sigma)$ gives a structural finitary closure operator on $F(\Sigma)$:

- (inflationary) $\Gamma \subseteq \text{Cons}(\Gamma)$;
- (increasing) $I_0 \subseteq I_1 \Rightarrow \text{Cons}(I_0) \subseteq \text{Cons}(I_1)$;
- (idempotent) $\text{Cons}(\text{Cons}(\Gamma)) \subseteq \text{Cons}(\Gamma)$;
- (finitary) $\text{Cons}(\Gamma) = \bigcup \{ \text{Cons}(\Gamma') : \Gamma' \subseteq_{\text{fin}} \Gamma \}$;
- (structural) $\sigma(\text{Cons}(\Gamma)) \subseteq \text{Cons}(\sigma(\Gamma))$; for every substitution $\sigma : X \rightarrow F(\Sigma)$.
In [9] Łoś and Suszko give a characterization of usual provability relation\(^4\) (\(\vdash\)) by the consequence sets with the properties above.

We will say that a logic \(l = (\Sigma, \vdash)\) is of finite type when \(|\Sigma|\) is a finite set and \(\vdash\) “is determined”, in the above sense, by a finite set of axioms and inference rules.

Morphisms: If \(l = (\Sigma, \vdash), l' = (\Sigma', \vdash')\) are logics then a translation morphism 
\[f : l \rightarrow l'\]
is a signature morphism \(f : \Sigma \rightarrow \Sigma'\) that "preserves the consequence relation", that is, for every \(\Gamma \cup \{\psi\} \subseteq F(\Sigma), \Gamma \vdash \psi\) then \(f[\Gamma] \vdash' f(\psi)\).

We will say that a morphism \(f : l \rightarrow l'\) is a conservative translation morphism if for every \(\Gamma \cup \{\psi\} \subseteq F(\Sigma), \Gamma \vdash \psi\) if and only if \(f[\Gamma] \vdash' f(\psi)\).

Composition and Identities: Like in \(S\).

Some facts about \(\mathcal{L}\)

**Definition 1. Direct image and inverse image**

Let \(f : \Sigma \rightarrow \Sigma'\) an \(S\)-morphism:

(i) Inverse image: if \(l' = (\Sigma', \vdash') \in \text{obj}(\mathcal{L})\) then, for every \(\Gamma \cup \{\psi\} \subseteq F(\Sigma), \Gamma \vdash \psi\) if and only if \(f[\Gamma] \vdash' f(\psi)\);

(ii) Direct image: if \(l = (\Sigma, \vdash) \in \text{obj}(\mathcal{L})\) then, for every \(\Gamma' \cup \{\psi'\} \subseteq F(\Sigma'), \Gamma' \vdash \psi'\) if there is a finite sequence of \(\Sigma'\)-formulas \((\phi_0, \ldots, \phi_t)\) such that:

* \(\phi_t = \psi'\);

* for every \(p \leq t:\)
  ** \(\phi_p^t\) is a hypothesis\(^*: \phi_p^t \in \Gamma'\);
  ** \(\phi_p^t\) is an instance of an \(l\)-axiom\(^*: \theta_p \in F(\Sigma)\) such that \(\vdash \theta_p\) and there is a substitution \(\theta' : X \rightarrow F(\Sigma')\) such that \(\tilde{\theta'}(\tilde{f}(\theta_p)) = \phi_p^t\);
  ** \(\phi_p^t\) is a direct consequence of an instance of \(l\)-inference rule applied over previous members in the sequence\(^*: \) there is \(\Delta_p' \cup \{\theta_p\} \subseteq F(\Sigma)\) such that \(\Delta_p' \vdash \theta_p\) and there is a substitution \(\sigma' : X \rightarrow F(\Sigma')\) such that \(\sigma'(\tilde{f}(\theta_p)) = \phi_p^t\) and \(\tilde{\sigma'}(\tilde{f}(\Delta_p)) \subseteq \{\phi_0, \ldots, \phi_{j-1}\}\).

**Definition 2.** The “pointwise definition” of order between consequence relations on each signature \(\Sigma: \vdash_0 \leq \vdash_1\) iff the identity function \((\Sigma, \vdash_0) \rightarrow (\Sigma, \vdash_1)\) is a translation morphism.

**Remark 4.** About direct image and inverse image

Let \(f : \Sigma \rightarrow \Sigma'\) an \(S\)-morphism:

(i) If \(l' = (\Sigma', \vdash') \in \text{obj}(\mathcal{L})\) then \(f^*[l'] = (\Sigma, \vdash, f^*[\vdash]) \in \text{obj}(\mathcal{L})\);

(ii) If \(l = (\Sigma, \vdash) \in \text{obj}(\mathcal{L})\) then \(f^*[l] = (\Sigma', \vdash, f^*[\vdash]) \in \text{obj}(\mathcal{L})\).

(iii) Let \(l = (\Sigma, \vdash)\) and \(l' = (\Sigma', \vdash')\) logics \((l, l' \in \text{obj}(\mathcal{L}))\). The following are equivalent:

\(^4\) A formula is demonstrable from a given set of hypothesis iff there is a finite sequence of formulas such that the last one is the thesis and each formula is an hypothesis or an instance of axiom or is obtained from the previous formulas in the sequence by an instantiation of a (finitary) inference rule.
(ii)* \vdash \leq f^*(l^-);  
(ii)_{tm} f : (\Sigma, \vdash) \rightarrow (\Sigma', \vdash') is a translation morphism;
(ii)* f_*(l) \leq' \vdash'.

Proof. (i) The proof of (i)* is analogous to the proof of item Limits (a) in Prop. 4; (i), is analogous to the item Colimits (a) in Prop. 4. (ii) The equivalence (ii)* \iff (ii)_{tm} follows directly from the definitions; the implication (ii)* \Rightarrow (ii)_{tm} is analogous to item Colimits (b) in Prop. 4; the implication (ii)_{tm} \Rightarrow (ii)* is analogous to item Colimits (c) in Prop. 4. □

Proposition 4. The category \mathcal{L} is complete and cocomplete and the obvious forgetful functor \mathcal{U} : \mathcal{L} \rightarrow \mathcal{S} creates all small limits and colimits.

Proof. We will give here all the constructions, but we will give the proof only in the case of filtered colimits. The full proof is in [10].

Limits\footnote{In [5] there is a similar proof for products, but with another notion of signature morphism.} Let \mathcal{I} be a small category and \mathcal{D} : \mathcal{I} \rightarrow \mathcal{L}, ((\Sigma^i, \vdash_i) \xrightarrow{f^i} (\Sigma'^i, \vdash'_i))_{(i \leq j) \in \mathcal{I}} a diagram, and take \Sigma, (\pi^i)_{i \in \text{obj}(\mathcal{I})} the limit of the underlying diagram (\mathcal{I} \xrightarrow{D} \mathcal{S} \xrightarrow{U} \mathcal{L}). For every \Gamma \cup \{\psi\} \subseteq F(\Sigma), define \Gamma \vdash \psi \iff there is \Gamma^- \subseteq \text{fin} \Gamma such that for every \psi \in \text{obj}(\mathcal{I}) \Gamma^- \vdash \pi^i(\psi) \vdash, then \Gamma = (\Sigma, \vdash) is a logic and (l, (\pi^i)_{i \in \text{obj}(\mathcal{I})}) is the limit of \mathcal{D} in \mathcal{L}. In fact, this follows from (a), (b) and (c) below:
(a) l \in \text{obj}(\mathcal{L});
(b) \pi^i \in \mathcal{L}(l, l') for every \psi \in \text{obj}(\mathcal{I}) ;
(c) If \{l', (\alpha^i)_{i \in \text{obj}(\mathcal{I})}\} is a commutative cocone over the diagram \mathcal{D}, then the unique signature morphism \alpha : \Sigma' \rightarrow \Sigma such that \alpha^i = \pi^i \circ \alpha, \alpha \in \text{obj}(\mathcal{I}) preserves the consequence relation.

Filtered colimits\footnote{This definition works too for the terminal logic \mathcal{L} = (\Sigma, \vdash); where \Sigma is the terminal signature (\text{card}(\Sigma)) = 1, \forall \in \omega), and for every \Gamma \cup \{\psi\} \subseteq F(\Sigma), \Gamma \vdash \psi.} Let (I, \leq) be a directed ordered set and \mathcal{D} : (I, \leq) \rightarrow \mathcal{L}, ((\Sigma^i, \vdash_i) \xrightarrow{f^i} (\Sigma'^i, \vdash'_i))_{(i \leq j) \in I} a diagram, and take \Sigma, (\gamma^i)_{i \in I} the colimit of the underlying diagram (I \xrightarrow{D} \mathcal{S} \xrightarrow{U} \mathcal{L}). For every \Gamma \cup \{\psi\} \subseteq F(\Sigma), define: \Gamma \vdash \psi \iff there is \Gamma^- \subseteq \text{fin} \Gamma and there is \psi \in I such that \Gamma^- \cup \{\psi\} \subseteq \hat{\gamma}^i[I(F(\Sigma^i))] and there is \Gamma^- \cup \{\psi\} \subseteq \text{fin} F(\Sigma^i) such that \hat{\gamma}^i[I](\Gamma^-) = \Gamma^- , \hat{\gamma}^i(\psi) = \psi and \Gamma^- \vdash \psi, then \Gamma = (\Sigma, \vdash) is a logic and (l, (\gamma^i)_{i \in I}) is the colimit of \mathcal{D} in \mathcal{L}. In fact, this follows from (a), (b) and (c) below:
(a) l \in \text{obj}(\mathcal{L});
(b) \gamma^i \in \mathcal{L}(l, l') for every \psi \in \text{obj}(\mathcal{I});
(c) If \{l', (\alpha^i)_{i \in \text{obj}(\mathcal{I})}\} is a commutative cocone over the diagram \mathcal{D} then the unique signature morphism \alpha : l \rightarrow l' such that \alpha^i = \alpha \circ \gamma^i, i \in I preserves the consequence relations:
(a) It follows directly from the definition of ⊢ that it gives a finitary and increasing consequence operator. It is also inflationary because if ψ ∈ Γ, take Γ− = {ψ} and any i ∈ I such that ψ ∈ 0[Σi], take ψi ∈ F(Σi) such that 0[ψi] = ψ and Γ−i = {ψi} then, as ⊢i gives an inflationary operator, Γ−i ⊢i ψi.

Idempotent: Let ψ ∈ F(Σ) such that T ⊢ ψ where T = {θ ∈ F(Σ) : Γ ⊢ θ}, let us prove that T ⊢ ψ.

for each θ ∈ Δ let Γθ ⊢ ψi such that Γθ ⊢ ψi then Γθ = ∪θ∈ΔΓθ is such that Γθ ⊢ ψ and as ⊢ is increasing, for each θ ∈ Δ, Γθ ⊢ ψ. Then, we can choose i ∈ I and Δi ⊈ {ψi} ⊆ fin F(Σi) with 0[Δi] = Δ, 0[ψi] = ψ and Δi ⊢i ψi because: by definition, Δ ⊢ ψ iff there is a subset Δ ⊆ fin Δ and a j ∈ I Δ−j ⊈ {ψi} ⊆ fin F(Σj) with 0[Δ−j] = Δ, 0[ψj] = ψ and Δ−j ⊢j ψj and, as Δ is directed ordered set, there is j ≥ i and Δ ⊆ fin F(Σi) such that 0[Δ−j] ⊆ Δ and 0[Δ−j] = Δ. We have 0[Δ−j] ⊢i 0[ψj] and, as (Σ, (θi)i∈I) is a commutative cocone over the diagram D, taking ψi = 0[ψj] we have, as ⊢i is increasing, Δi ⊈ {ψi} ⊆ fin F(Σi) with 0[Δi] = Δ, 0[ψi] = ψ and Δi ⊢i ψi. Analogously we can choose for each θ ∈ Δ, an iθ ∈ I such that there is Γ−θ ⊈ {θi} ⊆ fin F(Σθ) with 0[Γ−θ] = Γ−θ, 0[θi] = θ and Γ−θ ⊢θ iθ. Then, as (I, ≤) is a directed ordered set, take j ≥ i, iθ, for every θ ∈ Δ. Then, as (Σ, (θi)i∈I) is a commutative cocone over the diagram D : (I, ≤) → L. Hence, with Γ−j = ∪θ∈ΔΓ[θ] ⊆ fin F(Σj) and θi = 0[θi] for every θ ∈ Δ, we have, as ⊢j is increasing, Γ−j ⊢j θj for every θ ∈ Δ, and we can also suppose j such that Δj = {θj : θ ∈ Δ} is such that Δj ⊈ {ψj} ⊆ fin F(Σj) with 0[Δj] = Δj, 0[ψj] = ψ and Δj ⊢j ψj. Finally, as ⊢j is idempotent, we have Γ−j ⊢j ψj and we have 0[Δ[θ]] = ∪θ∈Δ0[θ] ⊆ fin F(Σθ) and 0[Δ[θ]] = ∪θ∈Δ0[θ] ⊆ fin F(Σθ) and 0[ψ] = 0[ψ] = 0[ψ] = 0[ψ].

Structural: Let Γ ⊇ {ψi} ⊆ F(Σ) be such that T ⊢ ψ. We have to prove that for any substitution σ : X → F(Σ) we have 0[σ[Γ]] ⊢ 0[σ[ψ]]. Let Γ− ⊆ fin Γ, i ∈ I be such that Γ− ⊈ {ψ} ⊆ fin F(Σi) and a set Γ−i ⊈ {ψi} ⊆ fin F(Σi) such that 0[Γ−i] = Γ−i, 0[ψi] = ψ and Γ−i ⊢i ψi. Now, as Γ− ⊈ {ψ} is a finite set of Σ-formulas, then also Γ− ⊈ {ψ} ⊆ fin F(Σi) ⊆ fin F(Σ). So there is n ∈ ω such that the X-variables that occur in Γ− ⊈ {ψ} ⊆ fin F(Σi) and i ∈ fin F(Σi) are in the finite set {x0, . . . , xn−1}. Then, as (I, ≤) is a directed ordered set, there is k ≥ i such that σ[[x0, . . . , xn−1]] ∈ fin F(Σ). Now take a substitution σk : X → F(Σk) such that for any m < n, σk(xm) ∈ 0[σk(xm)]. Then, with Γ−k = 0[Γ−i] and ψk = 0[ψi] , we have Γ−k ⊈ {ψk} ⊆ fin F(Σk) such that 0[Γ−k] = Γ−k, 0[ψk] = ψ and Γ−k ⊢k ψk. So, as ⊢k is structural, 0[σk[Γ−k]] ⊢k 0[σk[ψk]]. As 0[σk[Γ−k]] ⊆ fin F(Σk) to prove that 0[σ[Γ]] ⊢ 0[σ[ψ]] it is enough to show that for every Σ-formula θ in the finite set Γ− ⊈ {ψ} ⊆ fin F(Σ) the Σk-formula θk ∈ Γ− ⊈ {ψk} ⊆ fin F(Σk) such that 0[σ[θ]] = θ also satisfies 0[σ[θ]] = 0[σ[θ]]. As we have seen above, θ = θ(x0, . . . , xn−1), then 0[σ[θ]] = θ(σ(x0), . . . , σ(xn−1)) = 0[σ[θ]](σ(x0), . . . , σ(xn−1)) = 0[σ[θ]](σ(x0), . . . , σ(xn−1))...
(b) Let \( j \in I \) and \( \Gamma^j \cup \{ \psi \} \subseteq F(\Sigma') \) be such that \( \Gamma^j \vdash \psi \). As \( \vdash \) is finitary, select \( \Gamma^{-j} \subseteq \text{fin} \Gamma^j \) such that \( \Gamma^{-j} \vdash \psi \). Take \( \Gamma \cup \{ \psi \} \subseteq F(\Sigma) \) such that \( \Gamma = \tilde{\gamma}([\Gamma]) \) and \( \psi = \tilde{\gamma}(\psi') \), then \( \Gamma \vdash \psi \) because: there is a \( \Gamma^{-} \subseteq \text{fin} \Gamma \) (take \( \Gamma^{-} = \tilde{\gamma}([\Gamma^{-j}]) \)) and there is an \( i \in I \) (take \( i = j \)) such that \( \Gamma^{-} \cup \{ \psi \} \subseteq \tilde{\gamma}(F(\Sigma')) \) and there is \( \Gamma^{-1} \cup \{ \psi \} \subseteq \text{fin} F(\Sigma') \) such that \( \tilde{\gamma}([\Gamma^{-1}]) = \tilde{\gamma}([\Gamma^{-}]) \) and \( \tilde{\gamma}(\psi') = \psi \) and \( \Gamma^{-1} \vdash \psi' \).

(c) Let \( \Gamma \cup \{ \psi \} \subseteq F(\Sigma) \) such that \( \Gamma \vdash \psi \), then by definition, there is \( \Gamma^{-} \subseteq \text{fin} \Gamma \) and there is an \( i \in I \) such that \( \Gamma^{-} \cup \{ \psi \} \subseteq \tilde{\gamma}(F(\Sigma')) \) and there is \( \Gamma^{-1} \cup \{ \psi \} \subseteq \text{fin} F(\Sigma') \) such that \( \tilde{\gamma}([\Gamma^{-1}]) = \tilde{\gamma}([\Gamma^{-}]) = \psi \) and \( \Gamma^{-1} \vdash \psi' \).

As \( \alpha^i : l' \longrightarrow l' \) is a translation morphism, \( \hat{\alpha}([\Gamma^{-1}]) \vdash \hat{\alpha}(\psi') \). As \( \alpha^i = \alpha \circ \gamma^i \), \( \hat{\alpha}([\Gamma^{-1}]) \vdash \hat{\alpha}(\gamma^i(\psi')) \). Then \( \hat{\alpha}([\Gamma^{-}]) \vdash \hat{\alpha}(\psi') \), and, as \( \vdash' \) is increasing, \( \hat{\alpha}([\Gamma]) \vdash \hat{\alpha}(\psi') \).

Colimits Let \( \mathcal{I} \) a small category and \( D : \mathcal{I} \longrightarrow \mathcal{L} \), \((\Sigma^i, \vdash_i) \xrightarrow{f^i} (\Sigma^j, \vdash_j)\) \((h,j) \in \mathcal{I} \) a diagram, and take \((\Sigma, (\gamma^i)_{i \in \text{obj}(\mathcal{I})})\) the colimit of the underlying diagram \((\mathcal{I} \xrightarrow{D} \mathcal{L} \xrightarrow{U} \mathcal{S})\); now, for every \( \Gamma \cup \{ \psi \} \subseteq F(\Sigma) \), define: \( \Gamma \vdash \psi \iff \) there is a finite sequence of \( \Sigma \)-formulas \( \phi_0, \ldots, \phi_t \) where \( \phi_t = \psi \) and for every \( p \leq t \) some alternative occurs:

* “\( \phi_p \) is an hypothesis”: \( \phi_p \in \Gamma \);
* “\( \phi_p \) is an axiom”: there are \( i \in \text{obj}(\mathcal{I}) \), \( \theta^i \in F(\Sigma^i), \sigma : X \longrightarrow F(\Sigma) \) such that \( \vdash_i \theta^i \) and \( \phi_p = \hat{\sigma}(\gamma^i(\theta^i)) \); 
* “\( \phi_p \) is a consequence of a inference rule”: there are \( i \in \text{obj}(\mathcal{I}), \Delta^i \cup \{ \theta^i \} \subseteq \text{fin} F(\Sigma^i), \sigma : X \longrightarrow F(\Sigma) \) such that \( \vdash_i \theta^i \) and \( \hat{\sigma}(\gamma^i(\Delta^i)) \subseteq \{ \phi_0, \ldots, \phi_{p-1} \}, \phi_p = \hat{\sigma}(\gamma^i(\theta^i)) \); then \( l = (\Sigma, \vdash) \) is a logic and \((l, (\gamma^i)_{i \in \text{obj}(\mathcal{I})})\) is the colimit of \( D \) in \( \mathcal{L} \).

In fact, this follows from (a), (b) and (c) below:

(a) \( l \in \text{obj}(\mathcal{L}) \); 
(b) \( \gamma^i \in \mathcal{L}(b, l) \), for every \( j \in \text{obj}(\mathcal{I}) \); 
(c) If \( (l', (\alpha^i)_{i \in I}) \) is a commutative cocone over the diagram \( D \) then the unique signature morphism \( \alpha : l \longrightarrow l' \) such that \( \alpha^i = \alpha \circ \gamma^i, i \in \text{obj}(\mathcal{I}) \) preserve the consequence relation:

\[ \square \]

**Proposition 5.** Monomorphisms and epimorphisms in \( \mathcal{L} \):

Let \( U : \mathcal{L} \longrightarrow \mathcal{S} \) be the forgetful functor. Then a morphism \( f \) in \( \mathcal{L} \) is monic (epic) iff \( U(f) \) is monic (epic) in \( \mathcal{S} \).

**Proof.** The right to left implications are clear. Let \( f : l \longrightarrow l' \) be a morphism in \( \mathcal{L} \) such that \( U(f) : U(l) \longrightarrow U(l') \) is not \( \mathcal{S} \)-monic. Then there exist \( \Sigma'' \subseteq \text{obj}(\mathcal{S}) \) and \( g, h : \Sigma'' \longrightarrow U(l) \) such that \( g \neq h \) and \( U(f) \circ g = U(f) \circ h \). These \( g, h \) are \( \mathcal{L} \)-morphisms from the minimal logic over \( \Sigma \) (i.e. the logic whose closure operator is the identity) to \( l \) which satisfy \( g \neq h \) and \( f \circ g = f \circ h \), showing that \( f \) is not \( \mathcal{L} \)-monic. For the “epic” part proceed similarly, taking two “counterexample” arrows in \( \mathcal{S}, g, h : U(l) \longrightarrow U'(l') \) such that \( g \neq h \) and \( g \circ f = h \circ f \). These
\(g, h\) became \(\mathcal{L}\)-morphisms equipping their codomain with the “logic generated” \(^7\) by the direct image logic of these \(\mathcal{S}\)-morphisms. \(\square\)

**Remark 5.** Additional facts on filtered colimits in \(\mathcal{L}\):

Let \(D : (I, \leq) \rightarrow \mathcal{L}\), \((l^i \xrightarrow{f^i} l^j)_{i \leq j} \in I\) a directed diagram and let \((l', (\alpha^i)_{i \in I})\) be a commutative cocone over the diagram \(D\):

1. \(\Sigma^\alpha_n = \bigcup_{i \in I} \alpha^i_n[\Sigma^\alpha_n] \in \omega\);
2. If \(c_i \in \Sigma^\alpha_n\), \(c_j \in \Sigma^\alpha_n\) are such that \(\alpha^i_n(c_i) = \alpha^j_n(c_j)\) then there is a \(k \geq i, j\) such that \((f^k)^n(c_i) = (f^k)^n(c_j)\), \(n \in \omega\).

* For every \(I'' \cup \{\psi'\} \subseteq F(\Sigma^\alpha)\): \(I'' \vdash \psi\:\Leftrightarrow\:\exists\:\Gamma'' \subseteq \text{fin} \Gamma'\) and there is an \(\alpha \in I\) such that \(\Gamma'' \cup \{\psi'\} \subseteq \alpha\[F(\Sigma^\alpha)\]\) and there is \(\Gamma'' \cup \{\psi'\} \subseteq \text{fin} \Gamma'\) such that \(\alpha\[\Gamma''\] = \Gamma''\), \(\alpha\[\psi'\] = \psi'\) and \(\Gamma'' \vdash \psi'\).

* For every \(i \leq j\) \(\in I\) and every \(n \in \omega\), \(f^j_n : l^j_n \rightarrow l^j_n\) is injective and conservative then if \((l', (\alpha^i)_{i \in I})\) is “the” universal colimit cocone of diagram \(D\) so for every \(n \in \omega\), \(\alpha^i_n : l^i_n \rightarrow l^i_n\) is injective and conservative.

**Proof.** We will prove only the part of item (ii) concerning conservativeness: Let \(j \in I\) and \(\Gamma' \cup \{\psi'\} \subseteq F(\Sigma^\alpha)\) such that \(\alpha\[\Gamma'\] \vdash \psi'\). Now take \(\Gamma'' \cup \{\psi'\} \subseteq F(\Sigma^\alpha)\) such that \(\Gamma'' = \alpha\[\Gamma'\], \psi' = \alpha\[\psi'\].\) As \(\Gamma'' \vdash \psi'\) then, by item (i) above, there is \(\Gamma'' \subseteq \text{fin} \Gamma''\) and there is an \(\alpha \in I\) such that \(\Gamma'' \cup \{\psi'\} \subseteq \alpha\[F(\Sigma^\alpha)\]\) and there is \(\Gamma'' \cup \{\psi'\} \subseteq \text{fin} \Gamma'\) such that \(\alpha\[\Gamma''\] = \Gamma''\), \(\alpha\[\psi'\] = \psi'\) and \(\Gamma'' \vdash \psi'\). As \((I, \leq)\) is a directed ordered set, there is a \(k \geq i, j\), then as \(\alpha^i = \alpha^k \circ f^k\), \(\Gamma'' \cup \{\psi'\} \subseteq \alpha^k\[F(\Sigma^\alpha)\]\) and there is \(\Gamma'' \cup \{\psi'\} \subseteq \text{fin} F(\Sigma^\alpha)\) (take \(\Gamma'' = \alpha^k\[\Gamma''\] and \(\psi' = \alpha^k\[\psi'\]\))) such that \(\alpha^k\[\Gamma''\] = \Gamma''\), \(\alpha^k\[\psi'\] = \psi'\) and \(\Gamma'' \vdash \psi'\). \(\square\)

**Proposition 6.** Every logic is a directed colimit of finite type logics.

**Proof.** Let \(I = (\Sigma, \tau) \in \text{obj}\(\mathcal{L}\)\) and take the set \(I\) of every \(l' = (\Sigma', \tau') \in \text{obj}\(\mathcal{L}\)\) such that: \(\Sigma' \subseteq \text{fin} \Sigma\); \(\tau'\) is given by a finite set of axioms and a finite set of inference rules; the signature morphism of inclusion \(\Sigma' \hookrightarrow \Sigma\) is also a translation morphism \(l' \rightarrow l\). then:

(a) This diagram is “directed by inclusions” (clear);

(See item Colimits in Proposition 4.)
(b) \((l, (l') \hookrightarrow l'_{v \in I})\) is the colimit of this diagram.

This follows from the characterization of \(\vdash\) in filtered colimits. It is clear that the first two conditions in Rem. 5.(i) are satisfied, then consider \(\Gamma \cup \{\psi\} \subseteq F(\Sigma)\) such that \(\Gamma \vdash \psi\): take \(\Gamma^{-} \subseteq_{fn} \Gamma\) such that \(\Gamma^{-} \vdash \psi\). Take all symbols in \(\Sigma\) that occurs in formulas in \(\Gamma^{-} \cup \{\psi\}\); this is a finite set \(S \subseteq_{fn} |\Sigma|\) and take the unique subsignature \(\Sigma' \hookrightarrow \Sigma\) such that \(|\Sigma'| = S\). Take \(l' = (\Sigma', \vdash')\) the unique logic that is generated (by substitutions) by the unique basic axiom (if \(\Gamma^{-} = \emptyset\)) or inference rule (if \(\Gamma^{-} \neq \emptyset\) “from hypothesis \(\Gamma^{-}\) conclude \(\psi\)”. then the inclusion \(l' \hookrightarrow l\) is in fact a translation morphism and: \(\Gamma \vdash \psi\) iff there is \(\Gamma^{-} \subseteq_{fn} \Gamma\) such that if \(l' : l \hookrightarrow l\) then \(\Gamma^{-} \cup \{\psi\} \subseteq F(\Sigma')\) and \(\Gamma^{-} \vdash' \psi\) (in \(l'\)). \(\Box\)

Remark 6. Through an analogous argument we can prove that every logic is a filtered colimit of “conservative sublogics” with finite type underlying signature. But the next proposition say that we choose the “correct definition” of finite type logic.

**Proposition 7.** A logic is finitely presentable if and only if it is of finite type.

**Proof.** (\(\Leftarrow\)) Let \(l' = (\Sigma', \vdash')\) be a logic of finite type, that is \(|\Sigma'|\) is finite and consider \(D : (I, \leq) \rightarrow L\), \((l, \frac{f_{j}}{l_{j}})_{(i \leq j) \in I}\) a directed diagram of logos, then the canonical arrow \(k : \text{colim}_{i \in I} L(l', l_{i}) \rightarrow L(l', \text{colim} \in I l_{i})\) is an isomorphism.

\(\Rightarrow\) Let \(l' = (\Sigma', \vdash')\) be a logic of finite type, that is \(|\Sigma'|\) is finite and \(\Gamma\) be a directed ordered set there is \(a \in \Gamma\) such that, for each \(n \in \omega\) \(h_{n}[\Sigma'_{n} \subseteq \Sigma_{n}]\). As \(\vdash\) is given by a finite set of axioms and a finite set of (finitary) inference rules and \((I, \leq)\) is a directed ordered set there is a \(j \in I\) such that \(j \geq i\) and the finite image set of formulas in this chosen axioms and rules by \(\hat{h} : F(\Sigma') \rightarrow F(\text{colim}_{i \in I} \Sigma')\) are contained in the set \(\hat{\gamma}[F(\Sigma')]\); because \(h\) is a translation morphism with codomain a filtered colimit then we can assume also, by the definition of \(\vdash\) in filtered colimits, that \(j\) is such the images of this axioms and inference rules under \(h\) are in fact \(\vdash\)-derivable in \(l'\). So if we take for each \(n \in \omega\) each \(c_{n} \in \Sigma_{n}\), \(h_{n} : \Sigma_{n} \rightarrow \Sigma'_{n}\) such that \(h_{n}(c_{n}) \in \Sigma'_{n}\) with \(h_{n}(c_{n}) = [(h_{n}(c_{n})), j]\) then \(h : l' \rightarrow l\) is a translation morphism and \(h = \gamma h\); \(k\) is injective; This is analogously to the correspondent part of Proposition 3: in fact, here we need only the information that \(|\Sigma'|\) is a finite set.

(\(\Rightarrow\)) Let \(l = (\Sigma, \vdash)\) be a finitely presentable logic, then, by the proof of the Proposition 6 just above, the logic \(l\) is the colimit of the directed diagram of its finite type sublogics. Then as \(l\) is a finitely presentable logic there is \(l'\), a finite type sublogic of \(l\), such that the identity translation morphism \(\text{id}_{l} : l \rightarrow l\) must factors through the (colimit) canonical inclusion \(l' \hookrightarrow l\), because the canonical morphism \(k : \text{colim}_{v \in I} L(l, l') \rightarrow L(l, \text{colim}_{v \in I} l')\) is surjective: that is there is translation morphism \(h' : l' \rightarrow l'\) such that \((l \xrightarrow{i_{d}} l) = (l \xrightarrow{h'} l' \hookrightarrow l)\), then the inclusion \(l' \hookrightarrow l\) must be an \(S\)-isomorphism. Then \(l' \hookrightarrow l\) and \(h' : l \rightarrow l'\) have as subject the identity \(S\)-morphism. So, because \(l' \hookrightarrow l\) is a translation morphism \(l' \leq l\) and as \(h : l' \rightarrow l\) is a translation morphism \(\vdash \leq l'\). Then we have \(l' = l'\) and \(l\) is a finite type logic. \(\Box\)
**Theorem 2.** The category \( \mathcal{L} \) is a finitely locally presentable category. That is \( \mathcal{L} \) is an accessible category that is cocomplete and complete.

*Proof.* Direct consequence of the Propositions 6, 7 and 4. \( \square \)

**Corollary 2.** (i) The Yoneda functor \( Y : \mathcal{L}_{fp} \longrightarrow \text{Set}(\mathcal{L}_{fp}^{op}) \) has an extension to a functor \( Y' : \mathcal{L} \longrightarrow \text{Set}(\mathcal{L}_{fp}^{op}) \), \( l \mapsto Y'(l) = \mathcal{L}(\iota(l), l) \) that is full and faithful.

(ii) Let \( \text{Flat}(\mathcal{L}_{fp}, \text{Set}) \) the full subcategory of \( \text{Set}(\mathcal{L}_{fp}^{op}) \) with objects the functors that are filtered colimits of representable functors. Then \( \text{Flat}(\mathcal{L}_{fp}, \text{Set}) \) is the “essential image” of \( Y' \), so its restriction functor \( E : \mathcal{L} \longrightarrow \text{Flat}(\mathcal{L}_{fp}, \text{Set}) \) is an equivalence of categories.

(iii) \( \text{Flat}(\mathcal{L}_{fp}, \text{Set}) \) coincides with the category of \( \text{Set} \)-valued functors that preserves finite limits.

(iv) \( Y' \) has a left adjoint.

*Proof.* For (i) and (ii) see [7] The. 5.3.5 (p. 265) or [6] The. 2.26 (p. 83) or [8] Obs. 1.6 (p. 46). For (iii) and (iv) see [6] The. 1.46 (p. 38). \( \square \)

**Remark 7.** \( \mathcal{L} \) is an “algebraic category” and is not a “topological category”:
\( \mathcal{L} \) is an “algebraic category” : all the usual categories in algebra are accessible; \( \mathcal{L} \) has a “topological appeal”: because its objects are some with a particular kind of closure operator and its morphisms are some of the continuous functions relative to the closure operators; \( \mathcal{L} \) is not a “topological category”: the category of commutative \( C^* \)-algebras is \( \omega_1 \)-locally presentable, has as dual the category of compact Hausdorff topological spaces . And there is a general non-duality principle for categories locally presentables: If a category and its opposite are both locally presentables then they are equivalent to a complete lattice \( \mathcal{X} \) : see The. 1.64 in [6] (p. 51).

**Remark 8.** In [10] we verify that categories of algebraizable logics with morphisms the translation morphisms that also preserve the defining equations and the equivalence formulas (see [11] for the definitions) is, too, an accessible category.

**Remark 9.** Our category \( \mathcal{S} \) of signatures is, for its simplicity and good categorial properties, often chosen as a test bed for categorical constructions like fibring of logics, as is the case here. In practice it leads, however, to a too restricted notion of morphism between logics: Two presentations of classical propositional logic taking as primitive connectives (i.e. signatures) \( \{\neg, \wedge\}, \{\neg, \vee\} \) respectively do for example not admit any morphism between each other (since it would have to take \( \wedge \) to \( \vee \)) while they could intuitively be expected to be isomorphic.

To remedy this defect, [5] and others have taken as signature morphisms the substitutions from section 2.1.2 thus allowing to take \( \wedge \) to the derived connective \( \neg(\neg \vee \neg) \). The resulting category \( \mathcal{S} \) of signatures has, however, bad categorial

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8 The three fundamental species of structures of Bourbaki...
properties, for instance does not have all pullbacks nor all colimits, which implies that a category of logics built above it (and thus coming with a limit creating forgetful functor to \( S \)) can not be accessible. The reason for these categorial insufficiencies lies in the fact that \( F(\Sigma) \) (the language freely generated by \( \Sigma \)) is the absolutely free algebra over its signature or equivalently the free such operad\(^9\), so \( S \) is the category of free operads and colimits of free structures are hardly free again. To escape from this dilemma one could take as a category of signatures the category of all operads which is known to have good categorial properties (see e.g. [12]) and thus possibly allows to reproduce the results of this article. This would lead to a category of logics over languages with possibly interdefinable connectives and would thus include a common practice in logic into the formal treatment.

3 Fibrings, coverings and sheaves

3.1 Fibrings and coverings

Now that we have described the objects of \( L \) as colimits of the essentially small category \( L_{fp} \) and thus gained an embedding of \( L_{fp} \) into \( \mathcal{S}et(\Lambda_{fp})^{op} \) (which is the cocompletion of \( L_{fp} \)) we are in the position to introduce Grothendieck topologies on \( L_{fp} \) and apply the related sheaf theoretic notions to the logics in \( L \).

First we will take a look at some possible notions of covering:

**Definition 3.** Let \( l \) a finitely presentable logic and \( H \) a set of translation morphisms with codomain \( l \) and with a domain finitely presentable logic:

(i) \( H \) is a covering(i) of \( l \) iff for every \( \Gamma \cup \{ \psi \} \subseteq F(\Sigma) \) \( \Gamma \vdash \psi \iff \) there is \( \Gamma^{-} \subseteq \text{fin} \, \Gamma \) and there is an \( h \in H \) such that \( \Gamma^{-} \cup \{ \psi \} \subseteq F(\Sigma_{\text{cod}}(h)) \) and there is \( \Gamma^{-} \cup \{ \psi_{h} \} \subseteq \text{fin} \, F(\Sigma_{\text{cod}}(h)) \) such that \( \hat{\gamma}_{h}(\Gamma^{-}) = \Gamma^{-} , \hat{\gamma}_{h}(\psi_{h}) = \psi \) and \( \Gamma^{-} \vdash \text{cod}(h) \psi_{h} \)

Since we have coproducts in \( L \) we can also express properties of families of morphisms with common codomain by means of the induced arrow from the coproduct of the occurring domains:

(ii) \( H \) is a covering(ii) of \( l \) iff the canonical translation morphism \( c_{H} : \coprod_{h \in H} \text{dom}(h) \rightarrow l \), the unique arrow such that for every \( h \in H \), \( \text{dom}(h) \xrightarrow{h} l \) = \( \text{dom}(h) \xrightarrow{h} \coprod_{h \in H} \text{dom}(h) \xrightarrow{c_{H}} l \), is such that \( \vdash = c_{H}(\vdash_{H}) \).

(iii) \( H \) is a covering(iii) of \( l \) iff the canonical translation morphism \( c_{H} : \coprod_{h \in H} \text{dom}(h) \rightarrow l \), the unique arrow such that for every \( h \in H \), \( \text{dom}(h) \xrightarrow{h} l \) = \( \text{dom}(h) \xrightarrow{h} \coprod_{h \in H} \text{dom}(h) \xrightarrow{c_{H}} l \), is such that \( c_{H} \) is a conservative translation morphism that is also an \( L \)-epimorphism.

(iv) \( H \) is a covering(iv) of \( l \) iff the morphism \( c_{H} \) of (ii) is an isomorphism.

\(^9\) An operad is a multicategory with only one object and can be seen as an axiomatization of the behavior of a collection of finitary operations on a set, closed under the formation of derived operations.
The latter is equivalent to saying that $H$ covers $l$ iff $l$ is a fibring in the sense of [3] of the domains of the morphisms in $H$ and thus suggests the more general definition

(i) Given any notion of fibring, define $H$ to be a covering of $l$ iff $l$ is the result of fibring (in the given sense) the domains of the arrows in $H$.

Remark 10. Adopting either of above covering notions (or any other notion which gives conditions only involving the consequence relations) we have some space for “fine tuning” according to the intended meaning of “covering”:

(i) We can require covering families to be epimorphic families (or, equivalently, to be $e_H$ in (ii) to be epic). On the logical side, by Proposition 5, this means that every generating connective of the underlying language of the covered logic will occur in the image of some covering morphism and so rules out coverings by proper linguistic fragments as given by the one-element covering family ($\{\to\}, \text{modus ponens}$) $\rightarrow$ ($\{\neg, \to\}, \text{modus ponens}$). On the sheaf-theoretical side it implies that every representable functor will be a separated presheaf (see next subsection).

(ii) We can require the members of a covering family to be monos, thus (again by Proposition 5) allowing only coverings by proper linguistic fragments but not, for example, ($\{\land_1, \land_2\}, < A \land_1 B \vdash A, A \land_2 B \vdash B >$) $\rightarrow$ ($\{\land\}, < A \land B \vdash A, A \land B \vdash B >$) (where the expressions $< \ldots >$ denote the consequence relations generated by the given rules). This condition amounts to allowing only proper decompositions of logics as coverings and seems appropriate to treat the splitting of logics.

3.2 Sheaves

As mentioned in the introduction, a notion of covering (i.e. a mapping $Cov$ associating to each object $X$ a collection $Cov(X)$ of families of morphisms to $X$) gives rise to a unique Grothendieck topology on $L_{fp}$ \(^\text{11}\). The covering sieves of an object $X$ in this Grothendieck topology are the compositional right ideals of pullbacks to $X$ of covering families in $L_{fp}$ and will also be denoted by $Cov(X)$. The purpose of this section is to investigate what it means for a logic to be a separated presheaf or a sheaf.

We first recall the relevant definitions:

**Definition 4.** Given a presheaf $F \in |\text{Set}(L_{fp})^{op}|$, an object $X \in |L_{fp}|$ and a covering sieve $S = \{f_i : X_i \rightarrow X | i \in I\} \in Cov(X)$, a compatible family is a family $\{s_i \in F(X_i) | i \in I\}$ such that $F(f : X_i \rightarrow X_j)(s_i) = s_j$ for every $i, j \in I$.

A presheaf $F$ is called a separated presheaf (a sheaf, respectively) if for each such $X \in |L_{fp}|$, $S \in Cov(X)$ and compatible family $\{s_i \in F(X_i) | i \in I\}$ there is at most one (exactly one, respectively) $s \in F(X)$ such that $s_i = F(f_i : X_i \rightarrow X)(s)$ for every $i \in I$.

\(^\text{10}\) Some of these covering notions are obviously related: Covering(iv) is the strongest notion between (i), (ii), (iii) and (iv); Covering(iii) and Covering(ii) are stronger than Covering(ii).

\(^\text{11}\) Notation for that kind of site: $(L_{fp}, \text{Fib})$. 

Remark 11. As we see in the diagram in Introduction there is a pair of functors \( \mathcal{L} \rightleftharpoons \text{Sh}(\mathcal{L}_{fp}, \text{Fib}) \) such that \( S' : \mathcal{L} \rightarrow \text{Sh}(\mathcal{L}_{fp}, \text{Fib}) \) preserves finite limits and filtered colimits \(^{12}\). As we know that every logic is a filtered colimit of finitely presentable logic then we get a “good codification” of \( \mathcal{L} \) in \( \text{Sh}(\mathcal{L}_{fp}, \text{Fib}) \), in the sense that codification preserves the glue (the filtered colimits) between fundamental bricks (the FP-logics). Also, we know that the category \( \mathcal{L} \) of all logics has nice categorial properties, but the category \( \mathcal{L} \) itself has not nice logical properties! This “defect” of \( \mathcal{L} \) is fixed by the codification into the category of sheaves, a complete, cocomplete, locally finitely presentable category that has itself nice logical properties; as a elementary topos it has exponential objects and a classifying object.

Now we will take a closer look to the notions of sheaf and separated sheaf and what they says about the behaviour of a logic, we rephrase these definitions into statements about \( F \) in the category \( \text{Set}((\mathcal{L}_{fp})^{op}) \).

Under the Yoneda lemma isomorphism, elements \( s_i \in F(X_i), s \in F(X) \) correspond to unique natural transformations \( \tau_i : \text{Hom}(-, X_i) \rightarrow F, \tau : \text{Hom}(-, X) \rightarrow F \) respectively and \( F(f : X_i \rightarrow X_j)(s_j) = s_i, F(f : X_i \rightarrow X)(s) = s \) translate to \( \tau_i = \tau_j \circ \text{Hom}(-, f), \tau_i = \tau \circ \text{Hom}(-, f_i) \) respectively so that we get the following characterization:

\( F \) is a sheaf iff for each \( X \in |\mathcal{L}_{fp}|, s \in \text{Cov}(X) \) the Yoneda embedded cone \( y(S) \) is universal for \( P \) in the sense that for every family \( \{\tau_i\} \) of morphisms from the domains of \( y(S) \) there exists a unique arrow as in the following diagram:

\[
\begin{array}{ccc}
\text{Hom}(-, X_i) & \xrightarrow{g_i} & \text{Hom}(-, X) \\
\tau_i & \circ & \exists! \\
\text{Hom}(-, X_j) & \xleftarrow{g_j} & \tau_j \\
\end{array}
\]

So that \( F \) be a sheaf roughly says that “for \( F \) all (Yoneda embedded) covering sieves are colimit cones”. From this formulation it is immediate to see the following equivalences which we note in passing:

Remark 12. (i) The representable functors are separated presheaves iff all covering families are epimorphic.

(ii) The representable functors are sheaves iff all covering sieves are colimit cones.

\(^{12}\) Because \( Y' = j \circ E : \mathcal{L} \rightarrow \text{Set}((\mathcal{L}_{fp})^{op}) \) preserves limits and filtered colimits and the “associated sheaf functor” \( a \circ s : \text{Set}((\mathcal{L}_{fp})^{op}) \rightarrow \text{Sh}(\mathcal{L}_{fp}, \text{Fib}) \) preserves finite limits and colimits.
So separateness for a logic \( l \) means that a translation \( h \) from a finitary logic \( l' \) into \( l \) is completely determined by the translations obtained by composing with the morphisms of a covering family of \( l' \) and the sheaf property says that, given translations from the domains of a covering family of \( l' \), such a translation \( l' \to l \) in fact always exists.

**Example 1.** A logic which is not a separated presheaf:
Take as (generating) coverings all families \( \{ f_i : L_i \to L \} \) such that \( \vdash_L \leq \sup\{ f_i^*(\vdash_{L_i}) \} \). Let \( L_1 \) be the logic with signature given by one connective \( c \) and the minimal consequence relation and \( L_2 \) the logic with the signature consisting of two connectives \( c_1, c_2 \) with the same arity as \( c \) and the minimal consequence relation. Then the signature morphism mapping \( c \) to \( c_1 \) is clearly a morphism of logics and moreover gives a (one-element) covering family of \( L \) but we have two different endomorphisms of \( L \) making the diagram below commute, namely the identity and the morphism that maps both \( c_1 \) and \( c_2 \) to \( c_1 \):

\[
\begin{array}{ccc}
\{c\}, \vdash_{\text{min}} & \xrightarrow{c \mapsto c_1} & \{c_1, c_2\}, \vdash_{\text{min}} \\
\downarrow & & \downarrow \text{id} \\
\{c_1, c_2\}, \vdash_{\text{min}} & \xrightarrow{c \mapsto c_1} & \{c_1, c_2\}, \vdash_{\text{min}}
\end{array}
\]

This example (and lots of similar ones) suggests that separateness is a kind of “Occam’s Razor property” in the sense that a separated logic has no redundant (e.g. doubly occurring or interchangeable) connectives. Thus the separation functor \( s \) (the left adjoint to the inclusion of separated presheaves into presheaves) would give a procedure to cut down redundancies in the presentation of a logic – and in fact \( s \) is defined by a quotient construction. Be careful to note that we do not know a priori whether application of \( s \) actually yields a presheaf corresponding to a logic; this has to be investigated separately after fixing a specific notion of covering. However, in any case we get an equivalence relation between logics given by \( l l' \) if \( s(l) \cong s(l') \) which, in the spirit of the above, could be understood to hold when two (presentations of) logics have the same essence after cleaning up the redundancies. In the same vein the sheaf property for a logic \( l \) indicates a good behavior with respect to translations from finitary logics into \( l \) and the sheafification functor yields an equivalence relation between logics. By the remarks at the end of section 2 the logics in the category \( \mathcal{L} \) we adopted here are too weakly interconnected and the two presentations of classical logic given

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13 For \( F \) a presheaf \( s(F) \) is defined by \( s(F)(X) := F(X)/\sim \) where \( a \sim b \) if there is a covering \( \{ f_i \} \) such that \( F(f_i)(a) = F(f_i)(b) \) for every \( i \).
there as an example would not be equivalent in the senses given in this para-
paragraph. For this reason we postpone the investigation of the above equivalence
relations to future work in which we intend to replace the category of signatures
by a more appropriate one.

4 Conclusion

This work responds to an increasing tendency, as mentioned in the introd-
uction and witnessed by the publications cited there, to consider logics by their
relations to other logics. This tendency arises the necessity to investigate the
properties of the categories of logics which appear in this context. We tried to
take some first steps in adopting such a global perspective as opposed to the
concrete constructions around single logics which up to now form the body of
research in logic, but much work remains to be done until the “bottom up” and
the “top down” approach to logic can be united and interact fruitfully. Among
the foreseen issues we mention the following: The considerations started here
should be extensible to other categories arising in the contexts of logic and fib-
ring as the categories of algebraizable logics (with morphisms as in Remark 8),
of institutions, π-institutions, the metafibring categories from [1] etc. Also in
choosing such categories it should ultimately be of advantage to take into ac-
count global categorial properties and not only the ad hoc requirements of the
constructions one wishes to perform. The short discussion of possible categories
of signatures at the end of section 2 can be taken as an example of what we have
in mind here.

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