Capital Budgeting with Fuzzy Net Present Value Criterion

Huei-Wen Lin

Abstract

This paper proposes a more appropriate tool of incorporating uncertainties and investor’s attitude to risks into capital budgeting analysis. It has extended the classical net present value (NPV) method by developing a fuzzy logic system that takes the vague future cash flow and required rate of return into account. In order to explicitly discuss the more appropriate NPV method, the uncertain information will be fuzzified as triangular fuzzy numbers so that it would be more useful and practical for financial manager to analyze the capital budget of firms. We find that the fuzzy net present value (FNPV) method is one extension of the classical (crisp) cases.

Keywords: Fuzzy sets; Cost model; Simulation; Decision making; Risk

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1. Introduction

Capital budgeting decisions is one of the most demanding responsibilities of top financial management. Since the huge amount of capital expenditures and the results of capital budgeting decisions will continually influence the companies for many years, the financial manager losing some of their flexibility and the wrong decision will make serious financial consequences.

The evaluation of an investment or a project is often accomplished through the discounted cash flow (DCF) method. Classically, six key appraisal methods are used to rank projects and to decide whether or not they should be accepted in the capital budget, such methods are payback period method, discounted payback, net present value (NPV), internal rate of return (IRR), profitability index (PI), and real options. Indeed, most academics and professionals agree in considering the net present value rule as the most reliable criterion in ranking projects [14].

In practical, Bierman [2] surveyed the capital budgeting methods used by the Fortune 500 industrial companies and found every responding firm used some type of DCF method and most firms preferred to use IRR and NPV methods. Graham and Harvey [9] also indicated that most of the companies still used IRR and NPV methods to evaluate their investment programs and they found that the different capital budgeting practices were employed in small firms (less than $1 billion in sales) and large firms (more than $1 billion in sales). The smaller firms are more likely to rely on the payback method, while the larger firms prefer to employ IRR and/or NPV method.

Magni [14] insisted that the NPV rule is a pillar of modern finance theory and it is still so consolidated in the literature that we must admit that most financial concepts subsume it as a starting point for project’s valuation, so the financial decision maker
should not disregard the problem of the NPV. Up to now, many finance textbooks consider the NPV rule as one of the most important investment criteria [6] [16] [17] [3] [4]. And most financial concepts are based on the notions of present value and opportunity cost of capital, which are just the bricks of the NPV building [14].

In recent year, the NPV is frequently used for firm evaluations and it has been disguised as what it is generally known as Economic Value Added (EVA) [19], which is only an algebraic transformation of the NPV. Nowadays, the EVA model has gained increasing attention not only in the literature but also among firms which massively use this index, and is considered a reliable index for firms’ evaluation or as a tool for rewarding managers (see e.g., Biddle et al. [1]). Besides, Magni [14] also made a criticism: the idea is misleading that the option pricing and dynamic programming are more refined tools for evaluating investments or projects. Because it is commonly known that if an investment is not a real option (e.g., it is non-deferrable), option pricing and NPV give the same result. Dixit and Pindyck [7] clearly show the evaluation of a real option by dynamic programming boils down to a comparison between two net present values, one of which is related to investing now, the other one to waiting till the next period. So, the essence of real option evaluation can be seen as an appraising process based on the NPV criterion.

Although the classical NPV method plays a decisive role in evaluation, it does not take into account the uncertainties which may be inherent in these parameters used in it. These parameters including the vague expected net cash inflow stream and the project’s capital cost in the future. Especially in the uncertain financial environment, the companies’ capital costs should vary over time. In the classical NPV method, because the companies tend to use point input prices, implicitly assuming that these prices are predictable, the financial analysts usually incorporate the uncertainty in the field of capital budgeting analysis based on intuitive method or
probabilistic approach. However, the common methods still have the disadvantages of requiring the fulfillment of some assumptions for probabilistic distributions and relying on point estimation to obtain these uncertain parameters. Kahraman et al. [10] indicated that in an uncertain economic decision environment, an expert’s knowledge about the cash flows and the capital costs consists of a lot of vagueness instead of randomness. In order to deal with the vagueness of human thought, Zadeh [23] first introduced the fuzzy set theory, which was based on the rationality of uncertainty due to imprecision of vagueness. Afterward, fuzzy set theory becomes a powerful tool in the area of management when sufficient objective data have not been obtained. In some uncertain occasions, appropriate fuzzy numbers can indeed capture the vagueness of knowledge and overcome the difficulties in estimating these uncertain parameters.

Recently, some developments in fuzzy-financial mathematics have been well applied to deal with financial issues. For example, Buckley [5] studied the fuzzy extension of the mathematics of finance to concentrate on the compound interest law. Then, Li Calzi [13] investigated a possible general setting by considering both compact fuzzy intervals and invertible fuzzy intervals for the fuzzy mathematics of finance. Kuchta [12] also generalized fuzzy equivalents for the methods of evaluating investment projects. Several researchers have proposed a series of excellent studies about the fuzzy techniques in order to assess the investment project. For example, Dourra and Siy [8] applied fuzzy information technologies to investments through technical analysis, and used it to examine various companies to achieve a substantial investment return.

In view of the above, the opportunity can improve the classical NPV method by using the advancement of mathematics and sciences through fuzzy set theory. Therefore, the purpose of this paper is to complete the classical (crisp) NPV method
that can be fed with a fuzzy system, hoping to make it more appropriately to be
applied in capital budgeting practice. In this paper, we start by defining the problem
and list the NPV method in its classical form, and further all the uncertain parameters
will be given a fuzzified form.

The rest of this paper is organized as follows. Section 2 briefly introduces the
crisp NPV method. In Section 3, the logic of fuzzification is first described, and
then triangular fuzzy numbers and their operations will be performed to construct the
FNPV method. Finally, the $\lambda$-signed distance approach is employed to
defuzzified the fuzzy problem. In Section 4, the results obtained from Section 3 will
be compared to the crisp case with numerical simulation, and then the implications of
fuzzy capital budget are discussed in Section 5 and concluding remark in Section 6.

2. NPV method in crisp

The crisp NPV method is the most basic frameworks of capital budgeting
analysis described in detail in most financial management textbooks and it is taught in
most introductory courses in financial management. In most theoretical models, the
NPV method, which relies on DCF technique, is stated similar to Brigham and
Ehrhardt [4] and Sharpe et al. [18]. However, we note that the basic NPV method is
subject to the assumption of a constant required rate of return throughout the project.
Namely, discount cash flows with an equivalent-risk rate. From the financial point
of view, it is not legitimate to compare money having different degrees of risk. Such
as Magni’s [14] argument: the NPV rule gives rise to a partial ordering among assets
so the impossibility of comparing two assets with different risks must be coped.
Also, Turner and Morrell [20] pointed out that discount rate estimates are variable,
and clearly, companies’ capital costs vary over time. Thus, incorporating the
above-mentioned concepts with respect to the required rate of return, we further
generalize the classical NPV method as the net present value of expected future net
cash inflows for time period \( n \) which discounted at the different required rate of
return \( k_t \) that is given by

\[
N_n = C_o + \frac{C_1}{(1 + k_1)^1} + \frac{C_2}{(1 + k_2)^2} + \ldots + \frac{C_n}{(1 + k_n)^n},
\]

(1)
or shorter

\[
N_n = C_o + \sum_{t=1}^{n} \frac{C_t}{(1 + k_t)^t},
\]

(2)

where

\( N_n \): the net present value of the project for time period \( n \) (e.g. year), or an expected
present value in an finite stream of expected future net cash inflow estimated by
the decision maker.

\( C_o \): the net cash outflow at the beginning of the project, which is treated as a certain
negative value.

\( C_t \): the expected net cash inflow of the project estimated by the decision maker at
\( t \)-th time period. All future net cash inflows are expected values, so the
estimation values of \( C_t \) may differ among decision maker.

\( k_t \): the required rate of return of the project estimated by the decision maker at \( t \)-th
time period (i.e. the decision maker considers the returns available on other
investments).

Eq. (2) is a normalized capital budgeting analysis in the sense that the time
pattern of \( C_t \) should be a non-negative real number and may be rising, falling,
constant, or fluctuating randomly. As we consider that a decision maker might be
interesting to use NPV method to evaluate the investment project. At this time, the
cash flows of the project will be appraised in each period in advance, including all
inflows and outflows, and then be discounted at a certain required rate of return. To
sum these discounted net cash inflows, the project’s NPV will be obtained. If the NPV is positive, the project will be accepted; while the NPV is negative, it should be rejected. As to the mutually exclusive case, the one with the higher NPV will be chosen.

3. Fuzzy net present value (FNPV) method

3.1. Preliminary

Before presenting the FNPV method based on the λ–signed distance approach, the following definitions are provided in advance with some relevant operations [11].

Definition 1. A fuzzy set \([a, b; \alpha]\), \(a < b\) defined on \(\Re = (-\infty, \infty)\), which has the following membership function, is called a level \(\alpha\) fuzzy interval.

\[
\mu_{[a, b; \alpha]}(x) = \begin{cases} 
\alpha, & a \leq x \leq b, \\
0, & \text{otherwise}.
\end{cases}
\]

Definition 2. By Pu and Liu [15], fuzzy point \(\tilde{a}\) is a fuzzy set is defined on \(\Re\) with the following membership function:

\[
\mu_{\tilde{a}}(x) = \begin{cases} 
1, & x = a, \\
0, & x \neq a.
\end{cases}
\]

Definition 3. The triangular fuzzy number \(\tilde{D}\) is defined on \(\Re\) with the following membership function, and denoted by \(\tilde{D} = (a, b, c)\), where \(a < b < c\).

\[
\mu_{\tilde{D}}(x) = \begin{cases} 
\frac{x-a}{b-a}, & a \leq x \leq b, \\
\frac{c-x}{c-b}, & b \leq x \leq c, \\
0, & \text{otherwise}.
\end{cases}
\]

Let \(F_i\) be the family of fuzzy sets defined on \(\Re\), for each \(\tilde{D} \in F_i\), the
\(\alpha\)-cut of \(\tilde{D}\) is denoted by \(D(\alpha) = \{x | \mu_{\tilde{D}}(x) \geq \alpha \} = [\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]\) \((0 \leq \alpha \leq 1)\), and both \(\tilde{D}_L(0)\) and \(\tilde{D}_U(0)\) are finite values. For each \(\alpha \in [0,1]\), the real numbers \(\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)\) separately represent the left and right end points of \(D(\alpha)\) and satisfy the conditions that both of \(\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)\) exist in \(\alpha \in [0,1]\) and are continuous over \([0,1]\).

Next, we define \(\lambda\)-signed distance approach and provide the following properties asserted by Yao and Wu [22] and Yao et al. [21].

**Definition 4.** (a) For each \(\tilde{D} \in F_s\) and each \(\lambda \in (0,1)\), the \(\lambda\)-signed distance from \(\tilde{D}\) to \(\tilde{0}\) is defined by

\[
d(\tilde{D}, \tilde{0}; \lambda) = \int_0^1 [\lambda \tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)] d\alpha.
\] (3)

(b) When \(\tilde{D} = (a, a, a) = \tilde{a}\) is a fuzzy point at \(a\) and for all \(\alpha \in [0,1]\),

\(\tilde{D}_L(\alpha) = \tilde{D}_U(\alpha) = a\), then by (a) yields: \(d(\tilde{a}, \tilde{0}; \lambda) = a\) for all \(\lambda \in (0,1)\).

**Definition 5.** Let \(\tilde{A}, \tilde{B} \in F_s\) and for each \(\lambda \in (0,1)\), define the metric \(\rho_\lambda\) by

\[
\rho_\lambda(\tilde{A}, \tilde{B}) = \left| d(\tilde{A}, \tilde{0}; \lambda) - d(\tilde{B}, \tilde{0}; \lambda) \right|.
\]

**Definition 6.** For \(\tilde{A}, \tilde{B} \in F_s\) and each \(\lambda \in (0,1)\), relations “\(p, \approx\)” on \(F_s\) are

\(\tilde{A} \ p \ \tilde{B} \iff d(\tilde{A}, \tilde{0}; \lambda) < d(\tilde{B}, \tilde{0}; \lambda);\)

\(\tilde{A} \approx \tilde{B} \iff d(\tilde{A}, \tilde{0}; \lambda) = d(\tilde{B}, \tilde{0}; \lambda).\)

**Property 1.** For \(\tilde{A}, \tilde{B} \in F_s\) and each \(\lambda \in (0,1)\), the ordering relations \(p, \approx\) defined on \(F_s\) satisfy the law of trichotomy. Namely, one and only one of the three relations of \(\tilde{A} \ p \ \tilde{B}, \ \tilde{A} \approx \tilde{B}, \ \tilde{B} \ p \ \tilde{A}\) must be hold.
Property 2. For $\tilde{A}$, $\tilde{B}$, $\tilde{C} \in F_s$ and each $\lambda \in (0, 1)$, the ordering relations $p, \approx$ defined on $F_s$ satisfy the following axioms:

1. $\tilde{A} p \approx \tilde{A}$;
2. $\tilde{A} p \approx \tilde{B}$, and $\tilde{B} p \approx \tilde{A}$, then $\tilde{A} \approx \tilde{B}$;
3. $\tilde{A} p \approx \tilde{B}$, and $\tilde{B} p \approx \tilde{C}$, then $\tilde{A} p \approx \tilde{C}$.

From Properties 1 and 2, we know that the ordering relations $p, \approx$ on $F_s$ are linear order.

Property 3. For $\tilde{A}$, $\tilde{B} \in F_s$, $k \in \mathbb{K}$ and for each $\lambda \in (0, 1)$, the following two characteristics hold:

1. $d(\tilde{A}(+)\tilde{B},0;\lambda) = d(\tilde{A},0;\lambda) + d(\tilde{B},0;\lambda)$;
2. $d(k\tilde{A},0;\lambda) = kd(\tilde{A},0;\lambda)$.

Property 4. For $\tilde{A}$, $\tilde{B}$, $\tilde{C} \in F_s$ and each $\lambda \in (0, 1)$, metric $\rho_\lambda$ satisfies the following three metric axioms:

1. $\rho_\lambda(\tilde{A},\tilde{B}) = 0$ iff $\tilde{A} \approx \tilde{B}$;
2. $\rho_\lambda(\tilde{A},\tilde{B}) \geq 0$;
3. $\rho_\lambda(\tilde{A},\tilde{B}) + \rho_\lambda(\tilde{B},\tilde{C}) \geq \rho_\lambda(\tilde{A},\tilde{C})$.

Remark 1. By Property 4, for each $\lambda \in (0, 1)$, $(F_s, \rho_\lambda)$ is a metric space in the fuzzy sense.

3.2. Capital budgeting with FNPV method

In real economic environment, the companies often go through life cycle. Such like some operation evidences of high-tech companies in Taiwan, their operations usually have the following pattern with regard to the economic cycle:
during the early part of their lives, the companies’ growth rates are higher than that of the economy; then match the economy’s growth; and finally maintain a steady growth or lower than the economy’s growth. Similarly, for their investment projects, the cash inflows and the required rates of returns will vary with the shifting economy. Based on this viewpoint, as the decision makers use the classical (crisp) NPV method to evaluate their investment projects, it is necessary to make several assumptions for the net cash inflows and the required rates of returns and further to estimate them by using educated guesses or other statistical skills because of the difficulties of precisely predicting these parameters in the future operation periods. Nevertheless, the real values of these two parameters will be not necessarily equal to the former estimations exactly. Especially regarding the estimations for the required rates of returns \( (k_t) \) in different time periods, they usually could be derived from using the capital asset pricing model (CAPM) in which the risk factor, the market expected rate of return, and the expectations about the risk-free rate are embodied \([18]\). Since either \( k_t \) or the other financial data in CAPM is uncertain, these magnitudes should be more suitable to be considered as fuzzy numbers. Thus it will more fit in with real situation to predict the future cash flows and required rates of returns in different time periods by taking possible intervals such as \( [C_t - \varepsilon_t, C_t + \beta_t] \) and \( [k_t - \theta_t, k_t + \omega_t] \) in each period \( t \) (e.g. \( t = 1, 2, 3 \), L , \( n \)) instead of point estimation. In such the closed intervals, \( \varepsilon_t, \beta_t, \theta_t \) and \( \omega_t \) may be appropriately determined by the decision maker satisfying that \( 0 < \varepsilon_t < C_t, \ 0 < \theta_t < k_t, \ 0 < \beta_t, \) and \( 0 < \omega_t \).

Furthermore, since both the intervals \( [C_t - \varepsilon_t, C_t + \beta_t] \) and \( [k_t - \theta_t, k_t + \omega_t] \) are not definite values, the decision maker must respectively estimate a certain value from such the intervals for the calculation of the project’s NPV. When the decision maker respectively takes the estimation values of net cash inflow and required rate of return by \( C_t \) and \( k_t \) as the same as the former expected \( C_t \) and \( k_t \), the estimation
errors would be zero. Hence we can link the statistical concept of confidence level with membership grade in fuzzy theory and hereby set the maximum confidence level as “1.” According to the concept of confidence level, if the estimation values of $CF_i$ and $k_i$ respectively determined during the intervals $[C_i - \varepsilon_i, C_i + \beta_i]$ or $[k_i - \theta_i, k_i + \omega_i]$ are more far away from the expected $C_i$ and $k_i$, then the confidence level would be smaller. Similarly, the right and left end points (i.e. $C_i - \varepsilon_i, C_i + \beta_i, k_i - \theta_i$ and $k_i + \omega_i$) have the same minimum confidence levels set to be “0.”

Therefore, corresponding to the intervals of $C_i$ and $k_i$ (i.e. $[C_i - \varepsilon_i, C_i + \beta_i]$ and $[k_i - \theta_i, k_i + \omega_i]$), the fuzzy intervals of $C_i$ and $k_i$ can be expressed as the following triangular fuzzy numbers:

$$\tilde{C}_i = (C_i - \varepsilon_i, C_i + \beta_i), \quad t = 1, 2, 3, L, n;$$  

(4)

$$\tilde{k}_i = (k_i - \theta_i, k_i + \omega_i), \quad t = 1, 2, 3, L, n.$$  

(5)

where $\varepsilon_i, \beta_i, \theta_i$ and $\omega_i$ may be appropriately determined by the decision maker satisfying the following conditions:

$$0 < \varepsilon_i < C_i, \quad 0 < \theta_i < k_i, \quad 0 < \beta_i, \quad \text{and} \quad 0 < \omega_i.$$  

(6)

Note that if we set the membership grade of $\tilde{C}_i$ at $C_i$ (or $\tilde{k}_i$ at $k_i$) as “1,” then the larger distance from the right and left end points, $C_i - \varepsilon_i$ and $C_i + \beta_i$, (or $k_i - \theta_i$ and $k_i + \omega_i$) to $C_i$ (or $k_i$) is, the smaller membership grade would be. Namely, both the membership grades on the end points are “0.” Obviously, there are similar characteristics between the membership grade and confidence level.

The $\alpha$-cut of $\tilde{C}_i$ and $\tilde{k}_i$ can be denoted by $C_i(\alpha) = \left[\tilde{C}_{\alpha l}(\alpha), \tilde{C}_{\alpha u}(\alpha)\right]$ and $k_i(\alpha) = \left[\tilde{k}_{\alpha l}(\alpha), \tilde{k}_{\alpha u}(\alpha)\right]$, respectively, $\alpha \in [0,1]$, where

$$\tilde{C}_{\alpha l}(\alpha) = C_i - (1 - \alpha)\varepsilon_i = (C_i - \varepsilon_i) + \alpha \varepsilon_i > 0;$$
\[ \tilde{C}_{t|l}(\alpha) = C_i + (1 - \alpha)\tilde{\beta}_i > 0. \]  

(7)

Next, employing \( \lambda \)-signed distance approach to defuzzify \( \tilde{C}_i \) and \( \tilde{k}_i \), then for each \( \lambda \in (0,1) \), we have

\[ \begin{align*}
C_{i,\lambda}^* & \equiv d(\tilde{C}_i, \tilde{0}; \lambda) = C_i + \frac{1}{2}[(1 - \lambda)\beta_i - \lambda\varepsilon_i] > 0; \\
k_{i,\lambda}^* & \equiv d(\tilde{k}_i, \tilde{0}; \lambda) = k_i + \frac{1}{2}[(1 - \lambda)\theta_i - \lambda\omega_i] > 0.
\end{align*} \]

(8) \hspace{1cm} (9)

By (8) and (9), we denote \( C_{i,\lambda}^* \) and \( k_{i,\lambda}^* \) as the estimation values of net cash inflow and required rate of return in the fuzzy sense based on \( \lambda \)-signed distance, where \( C_{i,\lambda}^* > 0 \), \( C_{i,\lambda}^* \in \left[ C_i - \varepsilon_i, C_i + \beta_i \right] \) and \( k_{i,\lambda}^* > 0 \), \( k_{i,\lambda}^* \in \left[ k_i - \theta_i, C_i + \omega_i \right] \).

The relation between \( \lambda \) and \( C_{i,\lambda}^*, k_{i,\lambda}^* \) will be further discussed in Section 5.2.

According to the fuzzy operations, let \( \sum_{j=1}^{n} \tilde{A}_j \) be represented as

\[ \tilde{A}_1(+\tilde{A}_2(+\cdots+(+\tilde{A}_n \right) and \ \tilde{M}^t = \tilde{M}(\times\tilde{M}(\times\cdots(\times)\tilde{M} \ (t \ times)). \]

Using (4) and (5) to fuzzify (2), then we have the project’s NPV for \( n \) period in the fuzzy sense expressed by

\[ \tilde{N}_n = \tilde{C}_o(+)\sum_{j=1}^{n} \left\{ \tilde{C}_i(+)\left[ \tilde{I} (+\tilde{k}_j \right]\right\}. \]

(10)

where both \( \tilde{C}_o \) and \( \tilde{I} \) are fuzzy points at \( C_o \) and \( I \), respectively. In (10), the left and right end points of \( \alpha-cut \) of \( \left[ \tilde{I} (+\tilde{k}_j \right]\) are

\[ \begin{align*}
\left( \left[ \tilde{I} (+\tilde{k}_j \right]\right)_L(\alpha) & = [1 + k_i - (1 - \alpha)\theta_i] (> 0); \\
\left( \left[ \tilde{I} (+\tilde{k}_j \right]\right)_R(\alpha) & = [1 + k_i + (1 - \alpha)\omega_i] (> 0).
\end{align*} \]

(11) \hspace{1cm} (12)

Meanwhile, we can also derive the right and left end points of \( \alpha-cut \) of \( \tilde{C}_i(+)\tilde{I} (+\tilde{k}_j \right]\) shown below.
According to the decomposition theory, we can obtain the following Theorem 1.

**Theorem 1.** By (4), (5) and fuzzify \( C_o, C_i \) and \( k_i \) shown in (2), the project’s FNPV can be expressed as

\[
\tilde{N}_n = \tilde{C}_o (\alpha) \sum_{i=1}^{n} \left( \tilde{C}_i \left[ (\alpha) \tilde{k}_i \right] \right) + C_o + \sum_{i=1}^{n} \left( C_i \left[ (\alpha) \tilde{k}_i \right] \right) (15)
\]

### 3.3. Defuzzification by using the \( \lambda \)-signed distance approach

Concerning the consideration of defuzzification for fuzzy number, there are some different defuzzification approaches have been proposed, for example, mean-of-maxima (MOM) approach, center-of-area (COA) approach, and fuzzy mean (FM) approach, etc. (see e.g., Zhao and Govind [24]) But, none of these methods can dominate over the other methods for the defuzzification of fuzzy intervals. From a technical point of view, especially in dealing with some complex financial models, when the fuzzy logic is introduced to deal with uncertainty, the results of defuzzification often depend on the type of the membership function. However, it is difficult for the decision maker to apply the classical defuzzification approach such as the max-min extension principle to defuzzify the complicated fuzzy models.

Therefore, the major advantage of the proposed defuzzification approach (\( \lambda \)-signed distance approach) is that it is not necessary to find the membership function for fuzzy sets and the typical financial decision makers can easily apply it in evaluating the capital budgeting project. In this proposed FNPV method, due to the difficulty of
deriving the membership functions of fuzzy sets, we cannot use Zhao and Govind’s approaches (e.g. MOM, COA and FM) to defuzzify our fuzzy sets. But rather, we may employ the \( \lambda \) – signed distance approach without considering membership function to defuzzify the fuzzy sets throughout the paper.

Next, by Definition 4, for each \( \lambda \in (0,1) \), we defuzzify (15) by the \( \lambda \) – signed distance approach to yield

\[
N_{n,\lambda}^* = d(\tilde{N}_n, \tilde{o}; \lambda)
\]

\[
= C_o + \sum_{i=1}^{n} \int_0^1 \left\{ \frac{\lambda [C_i - (1-\alpha)\varepsilon]}{[1+k_i + (1-\alpha)\omega_i]} + \frac{(1-\lambda)[C_i + (1-\alpha)\beta]}{[1+k_i - (1-\alpha)\theta_i]} \right\} d\alpha . \tag{16}
\]

**Theorem 2.** In Theorem 1, for each \( \lambda \in (0,1) \), using \( \lambda \) – signed distance approach to defuzzify the fuzzy sets \( \tilde{N}_n = \tilde{C}_o(+) \sum_{i=1}^{n} \left\{ \tilde{C}_i(+) \left[ \tilde{I}(+) \tilde{k}_i \right]^n \right\} \), the estimate value of the project’s NPV for \( n \) period (\( N_{n,\lambda}^* \)) in the fuzzy sense corresponding to (2) can be expressed as the following three general forms:

(a) For \( n = 1 \),

\[
N_{1,\lambda}^* = C_o + \lambda \left[ \frac{C_i \omega_i + \varepsilon_i (1+k_i)}{\omega_i^2} \left( \ln \frac{1+k_i + \omega_i}{1+k_i} \right) - \frac{\varepsilon_i}{\omega_i} \right] \\
+ (1-\lambda) \left[ \frac{C_i \theta_i + \beta_i (1+k_i)}{\theta_i^2} \left( \ln \frac{1+k_i + \theta_i}{1+k_i - \theta_i} \right) - \frac{\beta_i}{\theta_i} \right].
\]

(b) For \( n = 2 \),

\[
N_{2,\lambda}^* = N_{1,\lambda}^* + \lambda \left[ \frac{\varepsilon_2}{\omega_2^2} \left( \ln \frac{1+k_2 + \omega_2}{1+k_2 + \omega_2} \right) + \frac{\varepsilon_2 (1+k_2) + C_2 \omega_2}{\omega_2 (1+k_2) (1+k_2 + \omega_2)} \right] \\
+ (1-\lambda) \left[ \frac{\beta_2}{\theta_2^2} \left( \ln \frac{1+k_2 - \theta_2}{1+k_2} \right) + \frac{\beta_2 (1+k_2) + C_2 \theta_2}{\theta_2 (1+k_2) (1+k_2 - \theta_2)} \right].
\]

(c) For \( n \geq 3 \),
4. Case study and numerical simulations

In this section, we will illustrate the methodology given in the preceding sections to evaluate a realized project’s FNPV with the following example under different $\lambda$ levels ($\lambda = 0.2, 0.5, 0.9$).

4.1. The application of FNPV method: a project for constructing the students’ dormitory in Nan Hwa University

In Taiwan, the college students have numbered in the millions, but the current supply of students’ dormitory is unable to meet the demand. Nan Hwa University, a nonprofit organization in Taiwan, is one of the private universities located in the remote district in midland of Taiwan. Because their students come from many counties, in order to solve the problems with the shortage of non-local students’ accommodation, the school plans to build a dormitory through using a small build-operate-transfer (BOT) scheme. This BOT project is launched by Nan Hwa University, after coordinating with other two partners (constructor and bank), the resolution is that the constructor presides over building and borrows from bank at a lending rate 6%. During the building period, all of the construction risks including completion delay risk and cost overruns risk are undertaken by the constructor. When the construction is completed, the school leases such the construction from the
constructor and operates for a certain period estimated about 10 years. Every year
the net cash inflow from dormitory operation will be paid to constructor as the rent, so
as to reimburse the principal and interest to the bank. In addition, during the leasing
term, the ownership of dormitory including the building and land will be transferred
to the school by means of donation year after year up to the expiry date.

The basic conditions of the BOT project are briefly described as follows.
1. Total building input (i.e. the selling price including fully-equipped room): about
   NT$ 200 million.
2. Required rate of return: about 6%.
3. Estimated net cash inflow for constructor: about NT$30 million/per year.

Now we use the FNPV method to illustrate an appraisal example for the small
BOT project which is under proceeding in Nan Hwa University. For further
comparison, NPV and FNPV methods are simultaneously used to assess the
investment feasibility of the BOT project.

4.2. Simulation results
To specify the capital budgeting analysis, we employ (2) and the fuzzy case of
Theorem 2 to compute the project’s NPV and FNPV with the above-mentioned data:
i.e. the estimation value of \( C_i \) amounts to about NT$ 30 million per year and will be
increases year by year as well as that of \( k_i \) (about 6%). As to the variations such
as \( \varepsilon, \beta, \theta, \) and \( \omega, \) the decision makers can appropriately determine them
beforehand. Furthermore, in order to more clearly compare the fuzzy cases \( N_{10,\lambda}^\ast \)
with crisp case \( N_{10} \), we show the simulation results in Figs. 1 to 4.

[Insert Fig. 1 about here]

[Insert Fig. 2 about here]
From the numerical simulations shown in Figs. 1 to 4, one may conclude that if \( \lambda \) is closer to 0, then the estimation values of FNPV \( N_{n,\lambda}^{+} \) is greater. Contrarily, if \( \lambda \) is closer to 1, then the estimation values of FNPV \( N_{n,\lambda}^{+} \) is smaller.

According to the outcome computed by EXCEL software, we find that the FNPV and crisp NPV of the BOT project amounts to “about” NT$ 20.8 million. Based on the evidences, a significant positive rate of return for this project can be received by the constructor. It means that the BOT project is worthy to invest for the constructor; meanwhile, it is also profitable for the bank and Nan Hwa University.

5. Discussions

As we are interested in applying the FNPV method to solve the problem of capital budgeting in which the net cash inflows and the required rates of returns are uncertainty, it allows us to employ triangular fuzzy numbers to explicitly analyze and provide insights into how the NPV in the fuzzy sense could be impacted by the variations of these two vague parameters. In conclusion, our work has provided the following aspects for the proposed FNPV method.

5.1. The relations among Theorem 2 and crisp case

(a) In Theorem 2, let \( \varepsilon_{i} = \beta_{i} = \theta_{i} = \omega_{i} = 0 \), then both (4) and (5) become fuzzy points \( \tilde{C}_{i} = (C_{i}, C_{i}, C_{i}) \) at \( C_{i} \) and \( \tilde{k}_{i} = (k_{i}, k_{i}, k_{i}) \) at \( k_{i} \), respectively.

Hence (15) becomes
\[ \tilde{N}_n = \tilde{C}_o(+) \sum_{i=1}^n \left\{ \tilde{C}_i(+) \left[ \tilde{I} (+) \tilde{k}_i \right] \right\} \]

\[ = \bigcup_{0 \text{ satisfied}} \left[ C_o + \sum_{i=1}^n \frac{C_i}{1+k_i} \right], C_o + \sum_{i=1}^n \frac{C_i}{1+k_i}; \alpha \right\} \]

Using the \( \lambda \)-signed distance approach to defuzzify (15), we have

\[ d(\tilde{N}_n, 0; \lambda) = N_n \quad \text{that is the same as (2).} \]

\((b)\) Let \( \varepsilon_i = \beta_i = \theta_i = \omega_i = \Delta_i \), by Theorem 2, for each \( \lambda \in (0,1) \), we have

\[ N_{n, \lambda}^* = C_o + \sum_{i=1}^n \int_0^1 \left\{ \lambda \left[ C_i - (1-\alpha)\Delta_i \right] + \frac{(1-\lambda)\left[ C_i + (1-\alpha)\Delta_i \right]}{1+k_i} \right\} d\alpha. \]

For \( n = 1 \),

\[ N_{1, \lambda}^* = C_o + \lambda \left[ \left( \frac{C_1\Delta_1 + \Delta_1(1+k_1)}{\Delta_1^2} \right) \ln \frac{1+k_1 + \Delta_1}{1+k_1} - \frac{\Delta_1}{\Delta_1} \right] \]

\[ + (1-\lambda) \left[ \left( \frac{C_1\Delta_1 + \Delta_1(1+k_1)}{\Delta_1^2} \right) \ln \frac{1+k_1 - \Delta_1}{1+k_1} - \frac{\Delta_1}{\Delta_1} \right]. \]

For \( n = 2 \),

\[ N_{2, \lambda}^* = N_{1, \lambda}^* + \lambda \left[ \left( \frac{\Delta_2}{\Delta_2^2} \right) \ln \frac{1+k_2}{1+k_2 + \Delta_2} - \frac{\Delta_2(1+k_2) + C_2\Delta_2}{\Delta_2(1+k_2)(1+k_2 + \Delta_2)} \right] \]

\[ + (1-\lambda) \left[ \left( \frac{\Delta_2}{\Delta_2^2} \right) \ln \frac{1+k_2 - \Delta_2}{1+k_2} + \frac{\Delta_2(1+k_2) + C_2\Delta_2}{\Delta_2(1+k_2)(1+k_2 - \Delta_2)} \right]. \]

For \( n \geq 3 \),

\[ N_{n, \lambda}^* = N_{1, \lambda}^* + N_{2, \lambda}^* + \sum_{i=3}^n \left\{ \lambda \left[ (-1)^i \left( \frac{\Delta_i(1+k_i) - C_i\Delta_i(t-2)}{\Delta_i^2(t-1)(t-2)(-1-k_i)^{i-1}} \right) \right] \right. \]

\( - (-1)^i \left( \frac{\Delta_i(1+k_i) - C_i\Delta_i(t-2) + \Delta_i^2(t-1)}{\Delta_i^2(t-1)(t-2)(-1-k_i - \Delta_i)^{i-1}} \right) \]

\[ + (1-\lambda) \left[ \left( \frac{\Delta_i(1+k_i) - C_i\Delta_i(t-2) - \Delta_i^2(t-1)}{\Delta_i^2(t-1)(t-2)(1+k_i - \Delta_i)^{i-1}} \right) \right]. \]
When $\Delta_t \rightarrow 0$, 

$$
\lim_{\Delta_t \to 0} N^*_n = \lim_{\Delta_t \to 0} \left\{ C_o + \sum_{r=1}^{n} \int_{0}^{1} \left\{ \frac{\lambda \left[ C_r - (1-\alpha)\Delta_r \right]}{[1+k_r + (1-\alpha)\Delta_r]} + \frac{(1-\lambda)\left[ C_r + (1-\alpha)\Delta_r \right]}{[1+k_r - (1-\alpha)\Delta_r]} \right\} d\alpha \right\} = N^*_n,
$$

the result is the same as (2) (cf. Table 1).

Obviously, according to the above discussions, we can verify that the FNPV method is one extension of the crisp NPV methods.

5.2. **The relations of the estimated net cash inflow $C^*_{r,\lambda}$ [cf. (8)] and required rate of return $k^*_{r,\lambda}$ [cf. (9)] in the fuzzy sense under different $\lambda$ levels**

(a) When $\lambda < 0.5$, $\lambda < 0.5 < (1-\lambda)$, for each $\alpha \in [0,1]$, the point

$$
\lambda \tilde{C}_{ul}(\alpha) + (1-\lambda)\tilde{C}_{ul}(\alpha) \in [\tilde{C}_{ul}(\alpha), \tilde{C}_{ul}(\alpha)]
$$

will be closer to the right-end point $\tilde{C}_{ul}(\alpha)$. Obviously, because $0 < \tilde{C}_{ul} < \tilde{C}_{ul}(\alpha)$ for all $\alpha \in [0,1]$, we have

$$
\lambda \tilde{C}_{ul}(\alpha) + (1-\lambda)\tilde{C}_{ul}(\alpha) > \tilde{C}_{ul}(\alpha) - 0.5[\tilde{C}_{ul}(\alpha) - \tilde{C}_{ul}(\alpha)] = 0.5 \tilde{D}_L(\alpha) + 0.5 \tilde{D}_U(\alpha).
$$

By (4.4), accordingly we have

$$
C^*_{r,\lambda} = d(\tilde{C}_{r,\lambda}; 0, \lambda) = \int_{0}^{1} [\lambda \tilde{C}_{ul}(\alpha) + (1-\lambda)\tilde{C}_{ul}(\alpha)] d\alpha > \int_{0}^{1} [0.5 \tilde{C}_{ul}(\alpha) + 0.5 \tilde{C}_{ul}(\alpha)] d\alpha.
$$

Similarly, when $\lambda > 0.5$, for each $\alpha \in [0,1]$, then $C^*_{r,\lambda} < C^*_{r,0.5}$. Based on the derivation, we can also obtain the same relations with respect to $\tilde{k}_r$. That is, when $\lambda < 0.5$, then $k^*_{r,\lambda} > k^*_{r,0.5}$; when $\lambda > 0.5$, then $k^*_{r,\lambda} < k^*_{r,0.5}$. The above-mentioned relations may refer to Figs. 1 to 4.

From the analytic results, we may conclude that the use of $\lambda$ level can be regarded as a simple concept of describing the decision maker’s attitude to risk.
That is, if $\lambda < 0.5$, then we may denote that such a decision maker is an optimist in estimating the values of fuzzy net cash inflow ($\tilde{C}_i$) and fuzzy required rate of return ($\tilde{k}_i$); if $\lambda > 0.5$, then such a decision maker is a pessimist in estimating them. Also, if $\lambda = 0.5$, then such a decision maker is a neutral to risk.

6. Concluding remarks

We propose in this paper the notion that the NPV technique offers the potential for flexibility beyond its classical interpretation. Our extension of FNPV method demonstrates the variability of the project’s overall returns from a vague perspective and provides insight into investment decisions unavailable through classic NPV analysis. In order to benefit from the technique, decision maker needs very clear definitions of the elements of analysis-capital costs and net cash flows to reveal the potential of the FNVP method. Practitioners can reason in terms of uncertain financial variables to yield a complete picture of the investment project. The FNPV method requires further research and elucidation before it will be widely applied in practice.
References


Fig. 1. Graph of the FNPV ($N_{10,\lambda}$) and crisp NPV ($N_{10}$) with different $\lambda$ levels and left-hand-side variations $\varepsilon_r$. 
Fig. 2. Graph of the FNPV ($N_{10,\lambda}^*$) and crisp NPV ($N_{10}$) with different $\lambda$ levels and right-hand-side variations of net cash inflow $\beta_t$.
Fig. 3. Graph of the FNPV ($N^*_{10,\lambda}$) and crisp NPV ($N_{10}$) with different $\lambda$ levels and left-hand-side variations of required rate of return $\theta_r$. 

$NPV \ (million)$
Fig. 4. Graph of the FNPV ($N_{10,\lambda}^*$) and crisp NPV ($N_{10}$) with different $\lambda$ levels and right-hand-side variations of required rate of return $\omega_j$.

Table 1. The comparison of FNPV ($N_{10,\lambda}^*$) and crisp NPV ($N_{10}$) with different $\lambda$ levels (scenario: for $\varepsilon_i = \beta_i = 0.01; \theta_i = \omega_i = 0.0001$)

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