Planarity and Edge Poset Dimension

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Different areas of discrete mathematics lead to intrinsically different characterizations of planar graphs. Planarity is expressed in terms of topology, combinatorics, algebra or search trees. More recently, Schnyder’s work has related planarity to partial order theory. Acyclic orientations and associated edge partial orders lead to a new characterization of planar graphs, which also describes all of the possible planar embeddings. We prove here that there is a bijection between bipolar plane digraphs and 2-dimensional $N$-free partial orders. We give also a characterization of planarity in terms of 2-colorability of a graph and provide a short proof of a previous result on planar lattices.

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1. INTRODUCTION

The partial order induced by an acyclic digraph on its vertex set has been studied extensively. It has led to a characterization of planar posets [1, 2, 14, 19], which seems difficult to extend to a characterization of planar graphs.

We introduce here a new partial order on the edge set of an acyclic digraph and prove that a bipolar digraph is planar if its edge partial order has dimension at most 2. We then construct a bijection between the embeddings of an $e$-bipolar planar digraph and the conjugates of its edge partial order.

We also give a characterization of planarity in terms of 2-colorability of a graph and provide a short proof of the previously mentioned result on planar lattices.

2. BACKGROUND

A digraph is $e$-bipolarly oriented if it has an acyclic orientation, with exactly one source $s$ and one sink $t$, both incident to $e$. It is easy to check that any 2-connected graph may be $e$-bipolarly oriented with respect of any edge [10]. By extension, a planar map is $e$-bipolar if the underlying graph is $e$-bipolarly oriented and if $e$ belongs to the unbounded face of the map.

Any acyclic digraph $G$ defines a partial order $\Pi_e(G)$ on its vertex set: a vertex $v$ is smaller than a vertex $v'$ if there exists a directed path from $v$ to $v'$.

Let us note that a digraph $G$ is $e$-bipolarly oriented iff $\Pi_e(G)$ is bounded by the endpoints of $e$ (i.e. $\Pi_e(G)$ has a minimum $s$, the source, and a maximum $t$, the sink, which are the endpoints of $e$) [6].

A transitive edge of an acyclic digraph $G$ is an edge incident to two vertices joined by a directed path of $G$ different from this edge.

The cover graph of a partial order is the directed graph induced by the covering relation; that is, the directed Hasse diagram of the partial order. Let us note that the cover graph of $\Pi_e(G)$ is the partial subgraph obtained from $G$ by deleting all the transitive edges; therefore, two acyclic directed graphs $G$ and $G'$ having the same vertex set have the same vertex partial order iff the partial subgraphs obtained from $G$ and $G'$ by deleting all the transitive edges are identical.

The Dushnik–Miller dimension of a partial order $\Pi$ is the minimum number of total orders the intersection of which is $\Pi$ [8].
Two partial orders $\Pi$ and $\Pi^*$ are conjugate if each pair of elements are comparable by exactly one of them. It is well known that a partial order has a conjugate partial order iff its dimension is at most 2.

Let us recall a well known result on the vertex partial order:

**Theorem 1** [14]. The partial subgraph obtained from an $e$-bipolarly oriented graph $G$ by deleting all the transitive edges different from $e$ is planar iff the partial order $\Pi_V(G)$ is a lattice of dimension at most two.

**Proof.** This theorem is a direct consequence of the following three results.

(i) Every partial order which is bounded (i.e., with a minimum and a maximum) and which admits a planar upward drawing is a lattice [2].

(ii) A lattice admits a planar upward drawing iff its dimension is at most equal to two [1].

(iii) A lattice admits a planar upward drawing iff the graph obtained from the graph defined by the covering relation (i.e., the cover graph) by adding an edge joining the minimum and the maximum elements of the lattice is planar [19].

Many acyclic digraphs may define the same vertex partial order (the partial order $\Pi_V(G)$ shown in Figure 2 is the one corresponding to the $e$-bipolar planar map shown in Figure 1, but also the one of the oriented complete graph induced by $\Pi_V(G)$).

Therefore, if one wants to use this theorem to characterize planarity for graphs, one may extend $\Pi_V(G)$ to a partial order $\Pi_{V\cup E}(G)$ on the union $V \cup E$ of the vertex set.
and the edge set of the graph (by bisecting all the edges and considering the vertex partial order on the obtained graph).

Then, the previous theorem can be restated as follows:

**Corollary 2.** An e-bipolarly oriented graph G is planar iff $\Pi_{V \cup E}(G)$ is a lattice of dimension 2.

Let us note that $\Pi_{V \cup E}(G)$ is not usually a lattice (for non-planar graphs) and that testing efficiently that a partial order is a lattice is not a priori an obvious task. Due to the fact that $\Pi_{V \cup E}(G)$ is a redundant partial order, we restrict it to the edge set of the graph (excluding the edge e). This obtained partial order is the edge partial order $\Pi_E(G)$. An edge $e_1$ is smaller than an edge $e_2$ in $\Pi_E(G)$ if there exists a directed path starting with $e_1$ and ending with $e_2$. Let us note that $\Pi_E(G)$ is not usually a lattice, even if G is planar.

![Figure 3. The edge partial order $\Pi_E(G)$.](image)

Posets which are edge partial orders of graphs have been fully characterized (see, for example, [13]). They are called N-free partial orders as they do not contain any N-shaped configuration. They are also characterized by the property that any maximal chain intersects any maximal antichain. We have the following property:

**Property 3.** Each N-free partial order is the edge-partial order of an unique e-bipolarly oriented graph.

Without a proof, let us mention that the dimension of the edge partial order is related to the previously introduced partial orders by

$$\dim \Pi_v(G) \leq \dim \Pi_E(G) = \dim \Pi_{V \cup E}(G).$$

Let us note that the dimensions of $\Pi_v(G)$ and $\Pi_E(G)$ are usually different (for any e-bipolar orientation of $K_5$, $\dim \Pi_v(K_5) = 1$ and $\dim \Pi_E(K_5) = 3$).

We shall prove that planarity can be fully described by properties of this partial order: an e-bipolar oriented graph G is planar iff $\dim \Pi_E(G) \leq 2$. According to a previous remark, this implies that if $\Pi_{V \cup E}(G)$ has dimension at most 2, then $\Pi_{V \cup E}(G)$ is a lattice.

### 3. Dual Order and Left Order

Although the dual orientation of an e-bipolar planar map defines a conjugate edge partial order, we prefer to develop this property in terms of the geometry of the embedding rather than that of the dual.

In the following, $M$ denotes an $e_0$-bipolar planar map and we denote by s and t the endpoints of $e_0$. 
Let $P$ be a directed path from $s$ to $t$. An edge is to the left of $P$ if it belongs to the disk bounded by $P \cup \{e_{ij}\}$ in $M$.

**Definition 4** (left order). An edge $e$ is to the left of an edge $f$ if these edges are not comparable and $e$ is to the left of a directed path from $s$ to $t$ including $f$.

**Figure 4.** The left order.

In order to prove some properties of the left order, we introduce the following notation:

**Notation 5.** Let $P_1$ and $P_2$ be two directed paths from $s$ to $t$. Let $\gamma(P_1, P_2)$ denote the cycles, the disjoint union of which is equal to the symmetric difference $P_1 + P_2$ of $P_1$ and $P_2$. Each cycle $\gamma(P_1, P_2)$ defines a bounded disk, denoted by $\Delta(P_1, P_2)$.

**Lemma 6.** Let $P_1$ and $P_2$ be paths from $s$ to $t$. Any edge belonging to the disk $\Delta(P_1, P_2)$ is comparable to any edge belonging to $(P_1 \cup P_2) \setminus \gamma(P_1, P_2)$.

**Proof.** Let $e$ be an edge belonging to $(P_1 \cup P_2) \setminus \gamma(P_1, P_2)$ and $f$ an edge belonging to the disk $\Delta(P_1, P_2)$. The edge $e$ is either smaller or greater than any edge of $\gamma(P_1, P_2)$. Assume that the edge $e$ is smaller than any edge of $\gamma(P_1, P_2)$. Let $P$ be a directed path from $s$ to $t$ including the edge $f$. As the edge $f$ is inside $\gamma(P_1, P_2)$, the path $P$ intersects $\gamma(P_1, P_2)$ before reaching $f$ at least once at a vertex $x$. Let $P'$ be a directed path from $s$ to $x$ included in $P_1 \cup P_2$ and $P''$ the subpath of $P$ from $x$ to $t$. Hence the edges $e$ and $f$ are comparable, as $P' \cup P''$ is a directed path including both edges. If the edge $e$ is greater than any edge of $\gamma(P_1, P_2)$, a similar argument applies.

**Proposition 7** (path independence). Let $P_1$ and $P_2$ be two directed paths from $s$ to $t$ including an edge $e$ and let $f$ be an edge not comparable with $e$. Then the edge $f$ is to the left of $P_1$ if $f$ is to the left of $P_2$.

**Proof.** Assume that the edge $f$ is to the left of the path $P_1$. If the edge $f$ was not to the left of the path $P_2$, it would belong to a disk $\Delta(P_1, P_2)$ and hence would be comparable with the edge $e$ from Lemma 6.

**Proposition 8** (antisymmetry). If an edge $f$ is to the left of an edge $g$, then the edge $g$ cannot be to the left of the edge $f$.

**Proof.** Let $P_f$ and $P_g$ be two directed paths from $s$ to $t$ including $f$ and $g$ respectively. By Lemma 6, the edges $f$ and $g$ belong to one and the same minimal cycle $\gamma(P_f, P_g)$ as they are not comparable. As to the edge $f$ is to the left of the path $P_f$ the edge $g$ will not be to the left of the path $P_f + \gamma(P_f, P_g)$.

**Proposition 9** (transitivity). If an edge $f$ is to the left of an edge $g$ and if $g$ is to the left of an edge $h$, the edge $f$ is to the left of the edge $h$.
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**Proof.** First, we prove that the edges \( f \) and \( h \) are not comparable. Let \( P_s \) be a path from \( s \) to \( t \) including \( g \). If the edges \( f \) and \( h \) were comparable, there would exist a directed path from \( s \) to \( t \) including both edges. That path would intersect \( P_s \). \( f \) is to the left of \( P_s \) and \( h \) is not. This construction exhibits a directed path containing \( g \) and either the edge \( f \) or the edge \( h \), which contradicts the hypothesis.

Let \( P_h \) be a directed path from \( s \) to \( t \) including \( h \). As before, the edges \( g \) and \( h \) belong to one and the same minimal cycle \( \gamma'(P_g, P_h) \) in \( P_s + P_h \). The edge \( f \) is to the left of the path \( P_h + \gamma'(P_g, P_h) \).

As the left relation is defined on the pairs of non-comparable edges and is a partial order (from Propositions 8 and 9), we have the following:

**Proposition 10 (conjugate order).** The left order is conjugate to the bipolar order.

We end this section with two straightforward remarks.

**Remark (duality).** The left order is the edge partial order defined by the orientation induced on the dual graph.

**Remark (topological compatibility).** At any vertex, the left order is compatible with the clockwise order of the outgoing edges and the anti-clockwise order of the ingoing edges.

4. Planarity and Dimension

By Proposition 10, any bipolar plane digraph has dimension 2. The inductive proof of the converse needs the following topological property:

**Lemma 11.** Let \( x \) and \( y \) be two adjacent vertices of a graph \( G \), let \( G' \) be the graph obtained from \( G \) by identifying the vertices \( x \) and \( y \) to a single vertex \( z \) and let \( M' \) be a map of \( G' \) (that is, the permutations defining an embedding of \( G' \) on some surface).

If the sets of edges incident to \( x \), respectively \( y \), in \( G \) form two disjoint intervals of the circular order around \( z \) in \( M' \), then the splitting of \( z \) in \( M' \) induces a map \( M \) of \( G \) having the same genus as \( M' \).

This lemma is a corollary of Edmonds’s theorem [9].

In this section, \( \Pi_E(G) \) will denote a conjugate of the edge partial order \( \Pi_r(G) \) of a bipolar digraph \( G \). The set of incoming (respectively, outgoing) edges at a vertex \( x \) will be denoted \( I^- (x) \) (respectively, \( I^+ (x) \)).

**Theorem 12.** A 2-connected digraph \( G \) is planar iff the edge partial order defined by any \( e \)-bipolar orientation of \( G \) has dimension at most 2.

**Proof.** We prove by induction on the number of vertices of the digraph \( G \) that there exists an \( e \)-bipolar planar map \( M \) the left order of which is \( \Pi_E(G) \).

If the graph has exactly two vertices, the edges of the graph are parallel. They are not comparable and any total order on the edges defines a unique compatible embedding.

Assume that the theorem has been proved for bipolar digraphs with \( n - 1 \) vertices. Let \( G_n = (V_n, E_n) \) be a bipolar digraph with \( n \) vertices.

Let \( x \) be a minimal vertex of the set \( V_n - \{ s \} \). By the minimality of \( x \), the edges of
be the trace of the embedding of \([16]\).

A modular lattice is comparable with respect to both \(P\) and \(\bar{P}\).

Prove that the edge \(G_1\) is maximal edge of \(G_1\) smaller than \(P\), comparable with respect to \(P\) and let \(E\) be an edge of \(G_1\) and \(E\) is conjugate. \(\bar{P}\) and \(\bar{P}\) are also conjugate.

By induction, \(G_{n-1}\) is a plane digraph and \(\bar{P}(G_{n-1})\) is the left order defined by an embedding of \(G_{n-1}\).

To achieve the proof of the theorem, we have to prove that \(G_n\) is planar and that \(\bar{P}(G_n)\) is the left order defined by an embedding of \(G_n\).

According to Lemma 11 and as \(G_n\) is obtained by splitting the vertex \(s\) of \(G_{n-1}\) into \(s\) and \(x\) and adding parallel edges from \(s\) to \(x\), the planarity of \(G_n\) is a consequence of the two followings properties. The edges of \(\Gamma^-(\chi)\) form an interval of the circular order around \(s\) in the embedding of \(G_{n-1}\), and the edges of \(\Gamma^-(\chi)\) are not interlaced with the edges of \(\Gamma^-(\chi)\).

Let us first prove that the edges of \(\Gamma^-(\chi)\) form an interval of the circular order around \(s\) in the embedding of \(G_{n-1}\). From Remark 3, it is sufficient to prove that the edges of \(\Gamma^-(\chi)\) are consecutive in the left order \(\bar{P}(G_{n-1})\).

Let \(g_1\) and \(g_2\) be the minimal and maximal edges of \(\Gamma^-(\chi)\), let \(h\) be an edge of \(\Gamma^-(\chi)\), and let \(f\) be an edge of \(\Gamma^-(\chi)\) not belonging to \(\Gamma^-\).

As the edges \(f\) and \(h\) are both incident to \(s\), they are not comparable with respect to \(\bar{P}(G_n)\). As the partial orders \(\bar{P}(G_n)\) and \(\bar{P}(G_n)\) are conjugate, \(f\) and \(h\) are comparable with respect to \(\bar{P}(G_n)\). By hypothesis, the edge \(g\) is greater than \(g_1\) and smaller than \(g_2\) with respect to \(\bar{P}(G_n)\). Thus, the edge \(h\) is greater than \(g_1\) or smaller than \(g_2\) with respect to \(\bar{P}(G_n)\) and hence not comparable with \(g_1\) or \(g_2\) with respect to \(\bar{P}(G_n)\). This gives rise to a contradiction, as the edge \(h\) is smaller than \(g_1\) and \(g_2\) by hypothesis.

Let us now prove that the edges of \(\Gamma^-(\chi)\) are not interlaced with the edges of \(\Gamma^-(\chi)\). Let \(g_1\) and \(g_2\) be the minimal and maximal edges of \(\Gamma^-(\chi)\), let \(f_1\) be the maximal edge of \(\Gamma^-(\chi)\) smaller than \(g_1\), let \(f_2\) be the minimal edge of \(\Gamma^-(\chi)\) greater than \(g_2\), and let \(h\) be an edge belonging to \(\Gamma^-\). We have to prove that the edge \(h\) is between the edges \(f_1\) and \(f_2\) with respect to \(\bar{P}(G_n)\).

As the edges \(f_1\) and \(h\) are both incident to \(s\), they are not comparable with respect to \(\bar{P}(G_n)\) and thus are comparable with respect to \(\bar{P}(G_n)\). Moreover, \(h\) cannot be smaller than \(f_1\) with respect to \(\bar{P}(G_n)\), as \(f_1 <^{\bar{P}} g_1\) would imply that \(h\) and \(g_1\) would be comparable with respect to \(\bar{P}(G_n)\) and \(\bar{P}(G_n)\). Thus, \(f_1\) is smaller than \(h\). In the same way, one proves that \(h\) is smaller than \(f_2\).

Corollary 13. An e-bipolar oriented graph \(G\) is a series-parallel graph iff \(\bar{P}(G)\) is a modular lattice.

Proof. This follows from the characterization of N-free modular lattices given in [16].

Theorem 14. The embeddings of a bipolar plane digraph are in bijection with the conjugates of its edge partial order.

Proof. Two different embeddings of \(G\) define two different left orders; that is, two different conjugates of the edge partial order.

Conversely, given an arbitrary partial order \(\bar{P}(G)\) conjugate to the edge partial order.
order of a bipolar digraph, the induction used in the proof of Theorem 12 exhibits the only embedding of $G$ having $\Pi_E(G)$ as left order.

5. Planarity and Bipartition

With any bipolar digraph $G$ there is associated another graph $\mathcal{J}(G)$ on the pairs of non-comparable edges which characterizes the planarity of $G$. Two pairs $(e, e')$ and $(f, f')$ are adjacent in $\mathcal{J}(G)$ iff $e' = f$ and $e$ is comparable with $f'$.

**Theorem 15.** A bipolar digraph $G$ is planar iff $\mathcal{J}(G)$ is bipartite.

![Figure 5. A conjugate of $\Pi_E(G)$ and the corresponding embedding.](image)

If $G$ is plane, the left order defines the bipartition of $\mathcal{J}(G)$: $(e, f)$ belongs to the left-right class if the edge $e$ is to the left of $f$ and to the right-left class otherwise. If $(e, f)$ and $(f, g)$ belong to the same class, the edge $e$ is comparable to the edge $g$ with respect to the left order by transitivity. Hence the edges $e$ and $g$ are not comparable with respect to the bipolar order and, by definition, are not adjacent in $\mathcal{J}(G)$.

The main tool to prove the converse is the theorem of Ghouila-Houri [11] and Gilmore and Hoffman [12] on the characterization of comparability graphs:

**Theorem 16 (Ghouila-Houri, Gilmore and Hoffman).** A graph $G = (V, E)$ is a comparability graph if and only if each odd pseudo-cycle has a chord of length 2.

**Proof.** Let $I(G)$ be the complement of the comparability graph of $\Pi_E(G)$, the vertices of which are the edges of $G$, two vertices of $I(G)$ being adjacent if the corresponding edges of $G$ are not comparable with respect to the bipolar order. If the graph $I(G)$ is a comparability graph, any compatible partial order is a conjugate of the bipolar order of $G$ and, from Theorem 12, $G$ is planar. From Theorem 16 it is
sufficient to prove that every odd pseudo-cycle of $I(G)$ has a chord of length 2. In the authors’ terminology, a pseudo-cycle is a closed walk, i.e., a sequence of vertices $(x_1, \ldots, x_k)$, such that $x_i$ is adjacent to $x_{i+1}$ and $x_k = x_1$.

The bipartition of the vertices of $H(G)$ defines a bipartition of the edges of $I(G)$. Let $\gamma$ be an odd pseudo-cycle of $I(G)$. There exist two consecutive edges $(e, f)$ and $(f, g)$ of $\gamma$ belonging to the same class (which implies, in particular, that $e$ is different from $g$), as $\gamma$ is odd. The corresponding vertices of $H(G)$ are not adjacent, which means that $e$ and $g$ are not comparable. Hence $(e, g)$ is an edge of $I(G)$ and a chord of length 2 of $\gamma$.

For another characterization of planarity in terms of a bipartition, see [4].

6. GEOMETRIC INTERPRETATION

Any bipolar plane graph can be represented by a visibility graph of segments. The vertices are represented by horizontal segments, the $y$ co-ordinates of which are compatible with the vertex bipolar order [20]. The edges are represented by vertical segments, the $x$ co-ordinates of which are compatible with the left order. This construction gives a visual interpretation of the 2-dimensionality of the edge partial order of a plane bipolar digraph.

Conversely, given a partial order conjugate to the bipolar order of a bipolar digraph, the arguments used in [5] can be adapted to provide a drawing of $G$ as a visibility graph and hence to prove its planarity.

7. DIMENSION OF THE VERTEX POSET

Basically, we shall only prove the difficult part; that is, that a lattice of dimension 2 is the vertex partial order of a plane digraph free of transitive edges.

Let $H$ be the cover graph of the lattice. As the lattice is of dimension 2, there exists a partial order $\Pi^*_v(H)$ conjugate to the vertex partial order $\Pi_v(H)$ of $H$. As the lattice is bounded, $H$ is bipolar oriented. The bipolar orientation of $H$ defines a bipolar order $\Pi^*_e(H)$ on the edge partial order of $H$. Let us define a binary relation on the edges of $H$.

DEFINITION 17. If $e$ and $f$ are two edges of $H$, then $e \not\leq f$ if one vertex incident to $e$ is smaller to one vertex incident to $f$ with respect to $\Pi^*_v(H)$.

This definition implies that two edges are related iff they are not compared by $\Pi^*_e(H)$ as $H$ has no transitive edge. We shall prove that this relation is a partial order conjugate to $\Pi^*_e(H)$.

The antisymmetry and the transitivity of the relation are straightforward consequences of the following lemma:

LEMMA 18. Let $f$ be an edge, and let $x$ and $y$ be any vertices of $H$. If, with respect to $\Pi^*_v(H)$, $x$ is smaller than an endpoint of $f$ and $y$ is greater than an endpoint of $f$, then $x$ is smaller than $y$ with respect to $\Pi^*_v(H)$.

Proof. If the vertices $x$ and $y$ are compared using the same endpoint of $f$, the result follows from the transitivity of $\Pi^*_v(H)$. Otherwise, denote $v$ and $w$ the endpoints of $f$ in such a way that $x <^* v$ and $w <^* y$. Assume that the vertices $x$ and $y$ are comparable with respect to $\Pi_v(H)$. Then, $v$ and $y$ cannot be compared with respect to $\Pi^*_v(H)$, since this would imply that either $x$ and $y$ or $v$ and $w$ are comparable with respect to $\Pi^*_v(H)$. In the same way, $x$ and $w$ cannot be compared with respect to $\Pi^*_v(H)$. If $v$
is smaller than \( w \), we have \( v < w, v < y, x < y \) and \( x < w \). As \( f \) is not a transitive edge and as \( \Pi_r(H) \) is a lattice, \( v \) is the meet of \( w \) and \( y \). Then, the vertex \( x \) would be smaller than (and so comparable with) \( v \), which contradicts the hypothesis. Similarly, if \( v \) was greater than \( w \), it would be the join of \( w \) and \( y \) and a contradiction would arise. Thus, the vertices \( x \) and \( y \) are not comparable with \( \Pi_r(H) \) and are comparable with respect to \( \Pi^*_r(H) \). In order to avoid a comparability in \( \Pi^*_r(H) \) of \( v \) and \( w \) by transitivity, the only choice is \( x <^* y \).

It is straightforward to prove that the conjugates of the vertex partial order are in bijection with the embeddings of the cover graph.

8. Left Trees and Right Trees

Let \( G \) be a bipolar plane digraph. The total orders defined by the bipolar order and its conjugate left order may be exhibited using the so-called left tree.

A leftmost edge is an edge the next clockwise edge of which is outgoing. As the incoming edges define an interval in the clockwise order, there is exactly one leftmost edge incident to each vertex of \( G \).

As the graph is acyclic, the set of the leftmost edges of \( G \) without the edge \( e_0 \) defines a tree rooted in \( s \), the left tree of \( G \).

Let \( \mathcal{D} \) be a tubular neighbourhood of the left tree, i.e., a neighbourhood homeomorphic to a bounded disk. Going along the boundary of \( \mathcal{D} \), each edge is met twice. An anti-clockwise walk along the boundary of \( \mathcal{D} \) defines a double occurrence circular sequence in which the two occurrences of \( e_0 \) are consecutive. Starting at \( e_0 \), the list of the first occurrences of the other edges defines the left order, a total order on the edges of \( G \). It is easy to check that this total order is compatible with both the bipolar order and the left order.

Similarly, we define the right tree and the right order of \( G \), by exchanging anti-clockwise order and clockwise order.

Assume that \( G \) is a maximal planar graph. For some bipolar orientation, the left and right trees may overlap. However, the vertex packing algorithm [7] generates all the bipolar orientations, such that the left tree and the right tree are edge disjoint. Such a property arises because these special orientations imply that each vertex (except the source) has at least two distinct incoming edges. The edges not belonging to these trees (except the exterior edges) define a third tree. This decomposition into three trees is the main tool used by Schnyder to prove that the incidence partial order of a planar graph has a dimension less or equal to 3 [21]. With the decomposition of the incidence partial order of a graph \( G \) into three total orders, Schnyder associates three edge disjoint trees of a graph \( \tilde{G} \) having \( G \) as a subgraph. These trees define a bipolar orientation of \( \tilde{G} \) and a conjugate partial order on the edge set. Hence, the Theorem 12 may be applied to prove the planarity of \( \tilde{G} \).

9. Final Remarks

Theorems 12 and 14 may be expressed in the following way:

**Theorem 19.** There is a bijection between \( N \)-free partial orders of dimension at most two and bipolar plane digraphs.

The use of the edge partial order instead of the vertex partial order has proved to be very effective. With somewhat different techniques, one can extend various results to produce a characterization of graphic oriented matroids [18, 3]:
Theorem 20. An e-bipolar oriented regular matroid is graphic iff its positive cocircuits are the antichains of some partial order.

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