New results on the periodic solutions for a kind of Rayleigh equation with two deviating arguments

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Abstract

With the help of the continuation theorem of the coincidence degree, a priori estimates, and differential inequalities, the authors make a further investigation of a class of Rayleigh equation with two deviating arguments of the form

$$x'' + f (x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = p(t).$$

Some new results on the existence of $T$-periodic solutions for such a system are established. Our work generalizes and improves some earlier publications.

Keywords: Rayleigh equation; Deviating argument; Periodic solution; Coincidence degree

1. Introduction

In \cite{1}, Wang and Cheng considered a kind of Raleigh equation with a deviating argument that takes the form

$$x''(t) + f (x'(t)) + g(x(t - \tau(t))) = p(t),$$

where $f$, $g$, $p$ and $\tau$ are real continuous functions defined on $\mathbb{R}$, and $f$, $p$ are periodic with period $2\pi$. It arises as a model including the delay Duffing equation and the delay Liénard equation. Under the assumptions of $f(0) = 0$ and $\int_0^{2\pi} p(s)ds = 0$, they obtained some results on the existence of periodic solutions to Eq. (1.1). Further study on...
the existence of periodic solutions to system (1.1) can be found in [2–4, 6–9]. Altogether there is only one deviating argument appearing in the equation.

Recently, Peng et al. [10] generalized system (1.1) and considered the following system:

\[ x''(t) + f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = p(t), \]  

where \( f, \tau_1, \tau_2, p : R \mapsto R \) and \( g_1, g_2 : R \times R \mapsto R \) are real continuous functions, \( f(0) = 0, \tau_1, \tau_2, p \) are \( T \)-periodic, and \( g_1, g_2 \) are \( T \)-periodic in the first argument \((T > 0)\). With the help of the following fundamental assumptions

\[ (A_1) \quad (g_1(t, u_1) - g_1(t, u_2))(u_1 - u_2) > 0, i = 1, 2, u_i \in R, \forall t \in R, u_1 \neq u_2; \]
\[ (A_2) \quad (g_2(t, u_1) - g_2(t, u_2))(u_1 - u_2) < 0, i = 1, 2, u_i \in R, \forall t \in R, u_1 \neq u_2; \]
\[ (A_3) \quad u(g_1(t, u) + g_2(t, u) - p(t)) > 0, \forall t \in R, |u| \geq d \] (\( d \) is a positive constant);
\[ (A_4) \quad u(g_1(t, u) + g_2(t, u) - p(t)) < 0, \forall t \in R, |u| \geq d \] (\( d \) is a positive constant);

they got some results about the periodic solutions, which extend and improve the previously known works. More precisely, they proved the following main result.

**Theorem A (\([10]\)).** Assume \((A_1), (A_3)\) hold, and there exist nonnegative constants \( m_1, m_2, m_3, m_4 \) such that \( 2m_1 + 4m_3 < \frac{1}{T^2} \), and one of the following conditions holds:

\[ (A_5) \quad f(u) \leq 0, |g_2(t, u)| \leq m_3|u| + m_4, \forall u \in R, t \in R \text{ and } g_1(t, u) + g_2(t, u) - p(t) \leq m_1u + m_2, \text{ for } t \in R, u \geq d; \]
\[ (A_6) \quad f(u) \geq 0, |g_2(t, u)| \leq m_3|u| + m_4, \forall u \in R, t \in R \text{ and } g_1(t, u) + g_2(t, u) - p(t) \geq -m_1u - m_2, \text{ for } t \in R, u \leq -d. \]

Then system (1.2) has at least one \( T \)-periodic solution.

**Remark 1.1.** In Theorem A, if \( m_1 = m_2 = 0 \), then \((A_3)\) contradicts \((A_5)\), and so do \((A_4)\) and \((A_5)\), \((A_3)\) and \((A_6)\), \((A_4)\) and \((A_6)\); if \( m_3 = m_4 = 0 \), then \((A_1)\) contradicts \((A_6)\), and so do \((A_2)\) and \((A_6)\), \((A_1)\) and \((A_5)\), \((A_2)\) and \((A_5)\).

Therefore, the constants \( m_1, m_2 \) should not both be zero and \( m_3, m_4 \) should not both be zero.

The main purpose of this paper is to continue to investigate the existence of \( T \)-periodic solutions to system (1.2). We shall employ an improved priori estimate to establish the existence of \( T \)-periodic solutions. The methods to estimate an a priori bound of all periodic solutions and the conditions imposed on \( f, g_1, g_2, p \) are different from the corresponding ones in [2–4, 6–10]. These conclusions generalize and improve many earlier publications.

The remainder of this article is organized as follows. In Section 2, the basic notations and preliminary lemmas are introduced. After giving the main criteria for checking the existence of periodic solutions in Section 3, two illustrative examples and remarks are given in Section 4.

### 2. Preliminaries

At first, we are ready to state the continuation theorem.

**Lemma 2.1** (Continuation Theorem \([11, P.40]\)). Let \( X \) and \( Y \) be two Banach spaces, and \( L : D(L) \subset X \mapsto Y \) be a Fredholm operator with index zero. \( \Omega \subset X \) is an open bounded set, and \( N : X \mapsto Y \) is \( L \)-compact on \( \overline{\Omega} \). If both the following conditions hold:

(a) for each \( \lambda \in (0, 1) \), every solution \( x \) of \( Lx = \lambda Nx \) is such that \( x \notin \partial \Omega \);

(b) \( QN \neq 0 \) for each \( x \in \partial \Omega \cap \text{Ker} L \) and

\[ \deg\{JQN, \Omega \cap \text{Ker} L, 0\} \neq 0, \]

then the equation \( Lx = Nx \) has at least one solution lying in \( \text{Dom} L \cap \overline{\Omega} \).

For more details about degree theory, we refer to the book by Deimling [12].

For ease of exposition, throughout this paper we will adopt the following notations:

\[ \|x\| = \|x(t)\| = \max_{s \in [0, T]} |x(s)|, \quad \|x\|_k = \|x(t)\|_k = \left( \int_0^T |x(s)|^k ds \right)^{\frac{1}{k}}. \]
Take \( X = \{x(t) | x(t) \in C^1(R, R), x(t) = x(t + T) \forall t \in R \} \) and \( Y = \{y(t) | y(t) \in C(R, R), y(t) = y(t + T) \forall t \in R \} \). Equipped with the norms \( ||x||_X = \max\{||x||_0, ||x'||_0\} \) and \( ||y||_Y = \max\{||y||_0\} \), both \( X \) and \( Y \) are Banach spaces. Define a linear operator \( L : D(L) \subset X \rightarrow Y \) by setting \( D(L) = \{x|x \in X, x'' \in C(R, R)\} \) and for \( x \in D(L), Lx = x'' \). We also define a nonlinear operator \( N : X \rightarrow Y \) by setting

\[
Nx = -f(x'(t)) - g_1(t, x(t - \tau_1(t))) - g_2(t, x(t - \tau_2(t))) + p(t).
\]

It is easy to see that

\[
\text{Ker} L = R, \quad \text{and} \quad \text{Im} L = \left\{y \in Y : \int_0^T y(s)ds = 0\right\}.
\]

It follows that \( L \) is a Fredholm mapping of index zero. Define the continuous projectors \( P : X \rightarrow \text{Ker} L \) and \( Q : Y \rightarrow Y \) by setting \( Px(t) = x(0) = x(T), Qy(t) = \frac{1}{T} \int_0^T y(s)ds \) and let \( L_p : L| : D(L) \cap \text{Ker} P \rightarrow \text{Im} L \). Furthermore, according to [3], \( L_p \) has continuous inverse \( L_p^{-1} \) on \( \text{Im} L \) defined by

\[
L_p^{-1} y(t) = -\frac{t}{T} \int_0^T (t - s)y(s)ds + \int_0^t (t - s)y(s)ds.
\]

It is easy to see that (1.2) is equivalent to the operator equation \( Lx = Nx \). Now, we consider its auxiliary equation \( Lx = \lambda Nx, \lambda \in (0, 1) \) as follows:

\[
x''(t) + \lambda \left[f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t)))\right] = \lambda p(t). \tag{2.1}
\]

**Lemma 2.2.** Assume one of the conditions (A1) and (A2) holds, and one of the conditions (A3) and (A4) holds at the same time; if \( x(t) \) is a \( T \)-periodic solution to (2.1), then

\[
||x(t)||_0 \leq d + \frac{1}{2}||x'(s)||_1. \tag{2.2}
\]

**Proof.** \( \forall t \in R \), there exist \( t_0 \in [t, t + T] \) such that \( |x(t_0)| = \max_{s \in [t, t + T]} |x(s)| \). Then

\[
|x(t_0)| = |x(t) + \int_t^{t_0} x'(s)ds| \leq |x(t)| + \int_t^{t_0} |x'(s)|ds, \tag{2.3}
\]

and

\[
|x(t_0)| = |x(t_0 - T)| = |x(t) - \int_{t_0-T}^t x'(s)ds| \leq |x(t)| + \int_{t_0-T}^t |x'(s)|ds. \tag{2.4}
\]

Combining the above two inequalities, for any \( T \)-periodic solution \( x(t) \), we have

\[
||x(t)||_0 \leq |x(t)| + \frac{1}{2} \int_{t_0-T}^{t_0} |x'(s)|ds = |x(t)| + \frac{1}{2}||x'(s)||_1. \tag{2.5}
\]

As \( x(t) \) is a \( T \)-periodic solution to (2.1), then there exist \( t_1, t_2 \in [0, T] \) such that

\[
x(t_1) = \min_{s \in [0, T]} x(s), \quad x(t_2) = \max_{s \in [0, T]} x(s). \tag{2.6}
\]

It follows that

\[
x'(t_1) = x'(t_2) = 0, \quad x''(t_1) \geq 0, \quad \text{and} \quad x''(t_2) \leq 0. \tag{2.7}
\]

This, together with (2.1), implies that

\[
g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) \leq 0,
\]

\[
g_1(t_2, x(t_2 - \tau_1(t_2))) + g_2(t_2, x(t_2 - \tau_2(t_2))) - p(t_2) \geq 0. \tag{2.8}
\]
As \( g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - p(t) \) is a continuous function on \( R \), from inequalities (2.8), there exists a constant \( t_3 \in R \) such that
\[
0 = g_1(t_3, x(t_3 - \tau_1(t_3))) + g_2(t_3, x(t_3 - \tau_2(t_3))) - p(t_3) > 0.
\]
(2.9)
If \((A_1)\) and \((A_3)\) hold, we claim that there exists a constant \( \tilde{t} \in R \) such that
\[
|x(\tilde{t})| \leq d.
\]
(2.10)
By way of contradiction, \( \forall t \in R, \)
\[
|x(t)| > d.
\]
(2.11)
This implies that one of the following relations holds:
\[
x(t) > d, \quad \text{or} \quad x(t) < -d.
\]
(2.12)
According to \((A_1), (A_3), (2.9)\) and \((2.11)\), we have \( x(t_3 - \tau_1(t_3)) \neq x(t_3 - \tau_2(t_3)) \). We shall consider four cases as follows.

Case i: If \( x(t_3 - \tau_1(t_3)) > x(t_3 - \tau_2(t_3)) > d, \) we have
\[
0 = g_1(t_3, x(t_3 - \tau_1(t_3))) + g_2(t_3, x(t_3 - \tau_2(t_3))) - p(t_3) > 0.
\]
(2.13)
Case ii: If \( x(t_3 - \tau_2(t_3)) > x(t_3 - \tau_1(t_3)) > d, \) we have
\[
0 = g_1(t_3, x(t_3 - \tau_1(t_3))) + g_2(t_3, x(t_3 - \tau_2(t_3))) - p(t_3) > 0.
\]
(2.14)
Case iii: If \( x(t_3 - \tau_2(t_3)) < x(t_3 - \tau_1(t_3)) < -d, \) we have
\[
0 = g_1(t_3, x(t_3 - \tau_1(t_3))) + g_2(t_3, x(t_3 - \tau_2(t_3))) - p(t_3) < 0.
\]
(2.15)
Case iv: If \( x(t_3 - \tau_1(t_3)) < x(t_3 - \tau_2(t_3)) < -d, \) we have
\[
0 = g_1(t_3, x(t_3 - \tau_1(t_3))) + g_2(t_3, x(t_3 - \tau_2(t_3))) - p(t_3) < 0.
\]
(2.16)
In view of (2.13)–(2.16), they all are contradictions to themselves. Then (2.11) does not hold and (2.10) is true. From (2.5) and (2.10), if \((A_1), (A_3)\) hold and \( x(t) \) is a \( T \)-periodic solution to (2.1), then \( \|x(t)\| \leq d + \frac{1}{2} \|x'(t)\|_{1} \). Using the methods similar to those used in \((A_1)(A_3)\), we can prove the remaining three cases: \((A_1)(A_4), (A_2)(A_3), (A_2)(A_4)\) are also true. This completes the proof. \( \Box \)

3. Main results

**Theorem 3.1.** Assume that \((A_1)\) and \((A_3)\) hold, and there exist nonnegative constants \( m_1, m_2, m_3, m_4 \) \( (m_1 + m_2 \neq 0 \) and \( m_3 + m_4 \neq 0) \) such that \( \frac{1}{2} m_1 + m_3 < \frac{1}{27 \pi^2} \), and one of the following conditions holds:
\[
(A_5) \quad f(u) \leq 0, \quad |g_2(t, u)| \leq m_3|u| + m_4, \forall u \in R, t \in R \quad \text{and} \quad g_1(t, u) + g_2(t, u) - p(t) \leq m_1 u + m_2, \quad \text{for} \quad t \in R, u \geq d;
\]
(3.1)
\[
(A_6) \quad f(u) \geq 0, \quad |g_2(t, u)| \leq m_3|u| + m_4, \forall u \in R, t \in R \quad \text{and} \quad g_1(t, u) + g_2(t, u) - p(t) \geq -m_1 u - m_2, \quad \text{for} \quad t \in R, u \leq -d.
\]
Then system (1.2) has at least one \( T \)-periodic solution.

**Proof.** In order to apply Lemma 2.1, it suffices to show that the set of all possible \( T \)-periodic solutions of (2.1) are bounded. Let \( x(t) \) be a \( T \)-periodic solution of (2.1). Integrating (2.1) from 0 to \( T \), we have
\[
\int_0^T f(x'(s))ds + \int_0^T \left[ g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_2(s)) - p(s) \right] ds = 0.
\]
(3.1)
Set \( \Omega_1 = \{ t : t \in [0, T], x(t - \tau_1(t)) < -d \}, \quad \Omega_2 = \{ t : t \in [0, T], x(t - \tau_1(t)) \geq -d \}, \quad \Omega_3 = \{ t : t \in [0, T], x(t - \tau_1(t)) > d \}, \quad \Omega_4 = \{ t : t \in [0, T], |x(t - \tau_1(t))| \leq d \}. \) If \((A_1), (A_3), (A_5)\) and \( \frac{1}{2} m_1 + \frac{3}{4} m_3 < \frac{1}{27 \pi^2} \) hold,
from (3.1), we have
\[
\int_{\Omega_1} |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
= - \int_{\Omega_1} [g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)] \, ds \\
= \int_{\Omega_2} [g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)] \, ds \\
- \int_0^T [g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)] \, ds \\
= \int_{\Omega_2} [g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)] \, ds \\
- \int_0^T [g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)] \, ds \\
- \int_0^T g_2(s, x(s - \tau_1(s))) \, ds + \int_0^T g_2(s, x(s - \tau_2(s))) \, ds \\
= \int_{\Omega_2} [g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)] \, ds \\
- \int_0^T g_2(s, x(s - \tau_1(s))) \, ds + \int_0^T g_2(s, x(s - \tau_2(s))) \, ds + \int_0^T f(x'(s)) \, ds \\
\leq \int_{\Omega_2} |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
- \int_0^T |g_2(s, x(s - \tau_1(s)))| \, ds + \int_0^T |g_2(s, x(s - \tau_2(s)))| \, ds \\
= \int_{\Omega_3} |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
\quad + \int_{\Omega_4} |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
- \int_0^T |g_2(s, x(s - \tau_1(s)))| \, ds + \int_0^T |g_2(s, x(s - \tau_2(s)))| \, ds \\
\leq \int_0^T (|m_1||x||_0 + m_2) \, ds + GT + 2 \int_0^T (|m_3||x||_0 + m_4) \, ds \\
= T(G + m_2 + 2m_4) + T(m_1 + 2m_3)||x||_0. 
\]
(3.2)

where \( G = \max_{t \in [0, T]} |g(t, x)| \). Thus, we have
\[
\int_0^T |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
= \int_{\Omega_1} |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
\quad + \int_{\Omega_2} |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
\quad + \int_{\Omega_3} |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
\quad + \int_{\Omega_4} |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)| \, ds \\
\leq T(G + m_2 + 2m_4) + T(m_1 + 2m_3)||x||_0 + T(m_1||x||_0 + m_2) + GT \\
= 2T(G + m_2 + m_4) + 2T(m_1 + m_3)||x||_0. 
\]
(3.3)
Therefore,
\[
\int_0^T |f(x'(s))|ds = -\int_0^T f(x'(s))ds
\]
\[
= \int_0^T [g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_2(s))) - p(s)]ds
\]
\[
= \int_0^T [g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_1(s))) - p(s)]ds
\]
\[
- \int_0^T g_2(s, x(s - \tau_1(s)))ds + \int_0^T g_2(s, x(s - \tau_2(s)))ds
\]
\[
\leq 2T(G + m_2 + m_4) + 2T(m_1 + m_3)\|x\|_0 + 2T(m_3\|x\|_0 + m_4)
\]
\[
= 2T(G + m_2 + m_4 + 2T(m_1 + 2m_3)\|x\|_0).
\]

Noting that \(x(t)\) is \(T\)-periodic, we obtain that \(x'(t)\) is also \(T\)-periodic. Just from Lemma 2.2, (2.5) and (2.7), we have \(\|x'\|_0 \leq \frac{1}{2T}\|x''\|_1\). Thus, together with (3.1), we have

\[
\|x''\|_0 \leq \frac{1}{2}\|x''\|_1 = \frac{1}{2} \left[ \int_0^T |f(x'(s))|ds + \int_0^T |g_1(s, x(s - \tau_1(s))) + g_2(s, x(s - \tau_2(s))) - p(s)|ds \right]
\]
\[
\leq 2T(G + m_2 + 2m_4) + 2T(m_1 + 2m_3)\|x\|_0
\]
\[
\leq 2T(G + m_2 + 2m_4) + 2T(m_1 + 2m_3)\left(d + \frac{1}{2}\|x'(s)\|_1\right)
\]
\[
\leq 2T(G + m_2 + 2m_4) + 2T(m_1 + 2m_3)\left(d + \frac{T}{2}\|x'\|_0\right)
\]
\[
= 2T(G + m_1d + m_2 + 2m_3d + 2m_4) + (m_1 + 2m_3)T^2\|x'\|_0.
\]

Since \(\frac{1}{2}m_1 + m_3 < \frac{1}{2T}\),
\[
\|x''\|_0 \leq \frac{2T(G + m_1d + m_2 + 2m_3d + 2m_4)}{1 - (m_1 + 2m_3)T^2} \triangleq M_0(> 0).
\]

Using Lemma 2.2 again, we have

\[
\|x\|_0 \leq d + \frac{T}{2}M_0 \triangleq M_1(> 0).
\]

It follows from (3.6) and (3.7) that
\[
\|x\|_X \leq M_0 + M_1 \triangleq M_3(> 0).
\]

Let \(M = M_3 + d + 1\), and set \(\Omega = \{x| x \in X, \|x\|_X \leq M\}\). Clearly, \(M\) is independent of \(\lambda\) and there are no \(\lambda \in (0, 1)\), \(u \in \partial \Omega\) such that \(Lu = \lambda Nu\). On the other hand, when \(u \in \partial \Omega \cap \text{Ker} L = \partial \Omega \cap R\), \(u\) is a constant vector in \(R\) with \(|u| = M\). If

\[
\int_0^T [g_1(s, v) + g_2(s, v) - p(s)]ds = 0
\]

then from (A3), we have \(|v| < d < M\). Therefore, \(v \notin \partial \Omega \cap \text{Ker} L\). Consider the homotopy \(F : (\Omega \cap \text{Ker} L) \times [0, 1] \rightarrow \Omega \cap \text{Ker} L\), defined by

\[
F(x, \mu) = -(1 - \mu)x - \frac{\mu}{T} \int_0^T [g_1(s, x) + g_2(s, x) - p(s)]ds.
\]

In view of (A3), we have

\[
xF(x, \mu) \neq 0 \text{ for } (x, \mu) \in (\partial \Omega \cap \text{Ker} L) \times [0, 1].
\]
It follows from the property of invariance under a homotopy that
\[ \deg\{ JQN, \Omega \cap \text{Ker } L, 0 \} = \deg\{ F(\cdot, 1), \Omega \cap \text{Ker } L, 0 \} \]
\[ = \deg\{ F(\cdot, 0), \Omega \cap \text{Ker } L, 0 \} \]
\[ = \deg\{ -x, \Omega \cap \text{Ker } L, 0 \} \neq 0. \] (3.12)
We have shown that \( \Omega \) satisfies all the assumptions of Lemma 2.1. Hence, \( Lu = Nu \) has at least one \( \omega \)-periodic solution on \( \text{Dom } L \cap \mathcal{D} \). If \((A_1), (A_3), (A_6)\) and \( \frac{1}{2}m_1 + m_3 < \frac{1}{2T^2} \) hold, the proof is similar and we omit it. This completes the proof. \( \square \)

A similar argument leads to:

**Theorem 3.2.** Assume that \((A_2)\) and \((A_4)\) hold, and there exist nonnegative constants \(m_1, m_2, m_3, m_4\) \((m_1 + m_2 \neq 0\) and \(m_3 + m_4 \neq 0\) such that \( \frac{1}{2}m_1 + m_3 < \frac{1}{2T^2} \), and one of the conditions \((A_3)\) and \((A_6)\) holds, then system (1.2) has at least one \( T \)-periodic solution.

**Remark 3.1.** Note that the condition \( \frac{1}{2}m_1 + m_3 < \frac{1}{2T^2} \) is weaker than \( 2m_1 + 4m_3 < \frac{1}{2T^2} \); therefore, Theorems 3.1 and 3.2 improve the corresponding theorem in [10].

**4. Examples and remarks**

**Example 4.1.** Consider the following Rayleigh equation:
\[ x''(t) - (x'(t))^2 + g_1(t, x(t - (1 + \sin t))) + g_2(t, x(t - (1 + \sin t))) = e^{\sin^2 t}, \] (4.1)
where \( g_1(t, x) = \begin{cases} x^9 + \frac{3}{32\pi^2} x, & x \leq 0, t \in R, \\ \frac{1}{32\pi^2} x, & x > 0, t \in R, \end{cases} \) and \( g_2(t, x) = \frac{1}{32\pi^2} x \). Obviously, \( f(x) = -x^{12} \leq 0 \); by some simple calculations, we obtain \( m_1 = \frac{1}{8\pi^2}, m_3 = \frac{1}{32\pi^2} \) and \( \frac{1}{2}m_1 + m_3 = \frac{3}{32\pi^2} < \frac{1}{8\pi^2} = \frac{1}{2T^2} \). It is easy to verify that all the conditions of Theorem 3.1 are satisfied. Therefore, system (4.1) has at least one \( 2\pi \)-periodic solution.

**Example 4.2.** Consider the following Rayleigh equation:
\[ x''(t) + f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e^{\sin^2 t}, \] (4.2)
where \( f(x) = 6|x| \geq 0 \), \( g_2(t, x) = -\frac{x}{32\pi^2} \), \( g_1(t, x) = \begin{cases} -x^9 - \frac{3}{32\pi^2} x, & x \leq 0, t \in R, \\ -\frac{1}{32\pi^2} x, & x > 0, t \in R, \end{cases} \) and \( \tau_1, \tau_2 \) are any non-negative \( 2\pi \)-periodic functions. By some simple calculations, we obtain \( m_1 = \frac{1}{8\pi^2}, m_3 = \frac{1}{32\pi^2} \) and \( \frac{1}{2}m_1 + m_3 = \frac{3}{32\pi^2} < \frac{1}{8\pi^2} = \frac{1}{2T^2} \). It is easy to verify that all the conditions of Theorem 3.2 are satisfied. Therefore, system (4.2) has at least one \( 2\pi \)-periodic solution. Using the method of numerical simulation in [5], we get the following systems for system (4.2) (See Fig. 1).

**Remark 4.1.** System (4.1) is a very simple version of the Rayleigh equation. As \( 2m_1 + 4m_3 = \frac{3}{8\pi^2} > \frac{1}{8\pi^2} = \frac{1}{2T^2} \), then Theorem A in [10] is not applicable to (4.1). On the other hand, let \( g(t, x) = \begin{cases} x^9 + \frac{3}{8\pi^2} x, & x \leq 0, t \in R, \\ \frac{x}{8\pi^2}, & x > 0, t \in R, \end{cases} \); then \( g(t, x) = g_1(t, x) + g_1(t, x) \) and the equivalent Eq. (4.1) is
\[ x''(t) - (x'(t))^2 + g(t, x(t - (1 + \sin t))) = e^{\sin^2 t}. \] (4.3)
It is easy to check that all the results in [1–4,6–9] and the references cited therein cannot be applicable to (4.3). Similarly, the results in [1–4,6–10] and the references cited therein cannot be applicable to (4.2). This implies that the results of this paper extend and improve the earlier publications.
Fig. 1. Numerical solution $x(t)$ of system (4.2), where $\tau_1(t) = 0.2$, $\tau_2(t) = 0.8$, $x(s) = 4.5$ for $s \in [-0.8, 0]$.

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References