Constructive Homotopy Methods for Finding All or Multiple DC Operating Points of Nonlinear Circuits and Systems

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Abstract—This paper develops a constructive homotopy-based methodology for finding all or multiple dc operating points of nonlinear circuits and systems. Several sufficient conditions for the connectivity of all the solutions along a single homotopy path are derived. These conditions offer criteria to determine an initial guess from which all the solutions can be obtained by following the single homotopy path. For the class of nonlinear circuits and systems in which all of the solutions lie on several homotopy paths, a new systematic method to explicitly construct a starting point for each homotopy path is developed. From a practical viewpoint, the constructive method developed does not require the difficult task of finding a good initial guess and is applicable to general nonlinear circuits and systems. From a methodological viewpoint, the constructive method developed is applicable to general homotopy methods with different homotopy functions. One particularly important feature of the developed constructive method is that when this method is applied to a dc problem with box constraints, it is very computationally efficient because it locates only feasible solutions satisfying the given box constraints. The proposed method is applied to several test examples with promising results.

Index Terms—DC operating points, homotopy methods, nonlinear circuits and systems.

I. INTRODUCTION

ROBUST and efficient computation of the solutions of a set of nonlinear equations is an important task in many practical applications [1], [4], [22], [25], [38]. This task, however, is a theoretically and practically difficult problem. For example, computation of one or more direct current (dc) operating points of a nonlinear circuit, such as a very large scale integration (VLSI) circuit, is one of the most important and difficult tasks in an electrical circuit simulator [4], [25], [38]. Another example is the computation of one or more operating points (i.e., the so-called power flow solutions) of an electrical power network, which is a fundamental task in the steady-state analysis of power systems.

Several factors contribute to the difficulty of obtaining an operating point of a nonlinear electronic circuit or a power network. The difficulty may stem from a physical problem, i.e., the nonexistence of operating points, indicating that the circuit or network is not properly designed and a redesign is needed. This difficulty may be mathematically linked to the fact that a set of nonlinear equations can have no solutions, a unique solution, a finite number of isolated solutions, an infinite number of isolated solutions, or a solution set composed of nonisolated solutions. On the other hand, the difficulty may simply stem from the inability of the numerical method (algorithm) used to compute an operating point due to its poor convergence property.

The Newton method and its variations are widely used to compute the solutions of a set of nonlinear equations, for example, in circuit simulation programs (such as in SPICE) and power network simulation programs. While these methods can achieve quadratic or superlinear convergence if a starting point sufficiently close to a solution is supplied, they have three shortcomings: 1) convergence is in general not guaranteed, this shortcoming is the result of the user’s inability to give a sufficiently accurate initial guess; 2) divergence does not imply the nonexistence of solutions; 3) once a solution is found, the methods provide little information regarding the location or existence of other solutions, making these methods unsuitable for computing multiple solutions. In fact, many numerical iterative methods including the Newton method and its variation for the solutions of nonlinear equations have the drawback of the heavy dependence of convergence on a good initial approximation to a desired solution [5], [8], [28].

A variety of homotopy methods have been developed to overcome the local convergence nature of many iterative methods including the Newton method and to compute multiple solutions [1], [9], [11], [22], [31]. Homotopy methods are also useful for solving difficult problems for which a good starting point close to a desired solution is hard to supply. In many cases, homotopy methods have succeeded where Newton’s method failed.

During the last two decades, a significant effort has been made to study the theoretical and algorithmic aspects of homotopy methods for finding a solution of nonlinear systems of equations. This effort has yielded important contributions toward the numerical solution of systems of equations [1], [4], [10], [11], [16], [36]. For example, for transistor circuits with passivity and no-gain properties [32], which satisfy certain “coercivity” conditions (i.e., \( F(x)^T x \geq 0 \) for all \( x \) with \( ||x|| = r \) for some \( r > 0 \)), the so-called globally convergent proba-
ibility-1 homotopy methods appear to exhibit an exceptionally wide domain of convergence [25], [32], [36]. Comparatively, much less attention has traditionally been devoted to finding all or multiple solutions of a system of nonlinear equations. However, in the last two decades this area has attracted significant interest from the applied mathematics, science, and engineering communities [4], [6], [22], [25], [32], [38].

Most homotopy methods for finding multiple solutions fall into one of two categories: the multistart approach (cf. [4] and [38]) and the extended curve following (lambda-threading) approach (cf. [22] and [38]). The multistart approach consists of choosing multiple initial points at $\lambda = 0$ and following their continuation paths to solutions at $\lambda = 1$. This approach is widely used in practice because of its simplicity and can find a number of solutions. This approach, however, cannot guarantee that all real solutions will be found and can be computationally expensive, because it frequently can lead to the same solutions from different initial conditions. In addition, since the multistart approach essentially uses different homotopy functions corresponding to different starting points, it makes its theoretical analysis and prediction for finding all or multiple solutions very difficult. The extended curve following approach follows a single homotopy path past $\lambda = 1$, hoping that it will reverse direction and pass through $\lambda = 1$ multiple times. This approach may fail to find all solutions if the path followed does not pass through all solutions, but is efficient in the sense that it can avoid revisits of the same solution.

This paper develops a constructive homotopy-based method which shares a similar spirit to the extended curve following approach. Furthermore, several analytical results are derived to address the following fundamental questions:

Question 1 Can we find an initial point from which all the solutions can be obtained by following the single homotopy path?

Question 2 If the solution set of a given homotopy function consists of several disconnected homotopy paths, can we devise a scheme to visit all the disconnected homotopy paths so as to compute all the solutions? We will conduct a comprehensive analysis of general homotopy methods related to these questions.

For the first question, which is addressed in Section III after brief definitions and notations in Section II, we derive several sufficient conditions for the connectivity of all the solutions along a single homotopy path. These conditions are an extension and generalization of Diener’s condition [6] and offer criteria to determine a starting point from which one can find all the solutions along one homotopy path.

For the second question, which is addressed in Sections IV–VI, we will develop a constructive method to systematically provide a starting point in each homotopy path past $\lambda = 1$, hoping that it will reverse direction and pass through $\lambda = 1$ multiple times. This approach may fail to find all solutions if the path followed does not pass through all solutions, but is efficient in the sense that it can avoid revisits of the same solution.

In this section, we introduce some concepts in the nonlinear dynamical systems and some definitions that are essential in the subsequent development of this paper [2], [15], [20], [21], [29]. We consider dynamical systems described by systems of ordinary differential equations of the form

$$\frac{dx}{dt} = F(x)$$

where $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be $C^2$ for the existence and uniqueness of solutions of system (1). The solution of (1) starting from $x$ at $t = 0$ is called an orbit (or trajectory), denoted by $\Phi(\cdot, x): \mathbb{R} \rightarrow \mathbb{R}^n$. A state vector $x^*$ is called a zero point or equilibrium point of system (1) if $F(x^*) = 0$. Let $E_F$ be the set of all zero points of $F$. A state vector $x$ is called a regular point if it is not a zero point. Note that 0 is called a regular value of $F$ if for all $x \in E_F := F^{-1}(0)$, the Jacobian of $F$ at $x$ has rank $m$. An equilibrium point $x^*$ is called stable, if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that:

$$||\Phi(0, x) - x^*|| < \delta \Rightarrow ||\Phi(t, x) - x^*|| < \epsilon \quad \forall t \in \mathbb{R}$$

and is called unstable, if not stable. We say that an equilibrium point of (1) is hyperbolic if the Jacobian of $F(\cdot)$ at $x^*$, denoted by $J_F(x)$, has no eigenvalues with a zero real part. For a hyperbolic equilibrium point, it is a (asymptotically) stable equilibrium point or sink if all the eigenvalues of its corresponding Jacobian have negative real parts and a source if all the eigenvalues of its corresponding Jacobian have positive real parts. A set $M$ in $\mathbb{R}^n$ is called an invariant set of (1) if every orbit of (1) starting in $M$ remains in $M$ for all $t \in \mathbb{R}$. A point $p$ is said to be in the $\omega$-limit set (or $\alpha$-limit set) of $x$ if for any given $\epsilon > 0$ and $N > 0$, there is a $t > T$ (or $T < t$) such that $||\Phi(t, x) - p|| < \epsilon$. This is equivalent to saying that there is a sequence $t_i \in \mathbb{R}$ with $t_i \rightarrow \infty$ (or $t_i \rightarrow -\infty$) such that $p = \lim_{i \rightarrow \infty} \Phi(t_i, x)$. It can be shown that if $\{t_i, x\}$ is bounded, then its $\omega$-limit set (or $\alpha$-limit set) is a nonempty, compact, connected, and invariant set. If the Jacobian of the hyperbolic equilibrium point has exactly $k$ eigenvalues with positive real parts, we call it a type-$k$ equilibrium point. For a hyperbolic equilibrium point $x^*$, its stable and unstable manifolds $W^s_F(x^*)$, $W^u_F(x^*)$ are defined as follows:

$$W^s_F(x^*) = \{ x \in \mathbb{R}^n : \lim_{t \rightarrow \infty} \Phi(t, x) = x^* \}$$

$$W^u_F(x^*) = \{ x \in \mathbb{R}^n : \lim_{t \rightarrow -\infty} \Phi(t, x) = x^* \}.$$
1) Fixed-point homotopy
\[ H := H_{\delta}(x, \lambda) = \lambda F(x) + (1 - \lambda)(x - x_0) \]  
(2)

de-homotopy
\[ H := H_d(x, \lambda) = F(x) + (\lambda - 1)d, \quad d \neq 0 \]  
(3)

where \( x \in \mathbb{R}^n \), \( \lambda \in \mathbb{R} \), and attempt to trace an implicitly defined curve in \( H^{-1}(0) \) from an initial guess \((x_0, 0)\) to a solution point \((x^*, 1)\). If this succeeds, then a zero point \( x^* \) of \( F \) is obtained.

Note that if \( H \) denotes the set of all maps \( H \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n) \) such that 0 is a regular value for the mapping \((x, \lambda) \rightarrow (H(x, \lambda)^T, det(D_x H(x, \lambda)))\), then it was shown that \( H \) is \( C^3 \) open and dense in \( C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n) \) (cf. [20] and [16]). Also if \( H \in H_1 \), then \( H^{-1}(0) \) is one-dimensional (1-D) \( C^4 \) manifold in \( \mathbb{R}^{n+1} \) and each bounded connected component of \( H^{-1}(0) \) is homeomorphic to a circle and each unbounded connected component of \( H^{-1}(0) \) is homeomorphic with \( \mathbb{R} \). In this paper, we will assume that \( H \in H_1 \) unless otherwise specified.

For a given d-homotopy function (3),\(^{1}\) the reduced d-homotopy function \( T_d : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1} \) is defined by
\[ T_d(x) = B_d F(x) = \begin{pmatrix} -d_1 F_2(x) + d_2 F_1(x) \\ -d_1 F_3(x) + d_3 F_1(x) \\ \vdots \\ -d_1 F_n(x) + d_n F_1(x) \end{pmatrix} \]  
(4)

where
\[ B_d = \begin{bmatrix} d_2 & -d_1 & 0 & \cdots & 0 \\ d_3 & 0 & -d_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ d_n & 0 & \cdots & 0 & -d_1 \end{bmatrix}. \]

Note that \( T_d^{-1}(0) \) is the projection of \( H_d^{-1}(0) \) onto \( \mathbb{R}^{n-1} \).

III. SUFFICIENT CONDITIONS FOR THE GLOBAL CONVERGENCE OF THE HOMOTOPY METHODS

In this section, we will derive several sufficient conditions for the global convergence of the homotopy-continuation methods to find all solutions of systems of nonlinear equations. Obviously, if one can ensure the connectivity of the zero set of a homotopy function \( H : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \), then all solutions can be found by numerically tracing the single component of \( H^{-1}(0) \).

The following theorems provide several sufficient conditions for the connectivity of the zero set of a general function \( H : \mathbb{R}^n \rightarrow \mathbb{R}^k \) where \( k \leq n \). These conditions, which are an extension of Dickers’ works [6], will provide criteria to determine a homotopy function of which the zero set is connected. The basic idea behind these conditions is to construct a retraction from the connected subset of \( \mathbb{R}^n \) onto \( H^{-1}(0) \).

Theorem 1 (Global Connectedness): Suppose that \( H \in C^2(\mathbb{R}^n, \mathbb{R}^k) \), \( k \leq n \) satisfies the following conditions:

1) \( H(x) \in \text{range}(DH(x)) \) for all \( x \in \mathbb{R}^n \).
2) There exists a nonnegative continuous function \( \lambda_H : \mathbb{R}^n \rightarrow [0, 1] \) such that \( \lambda_H(x) \in H^{-1}(0) \).
3) The \( C^4 \)-vector field on \( \mathbb{R}^n \), given by
\[ V : x \mapsto -\lambda_H(x)DH(x)^+ H(x) = \frac{H(x)}{1 + ||H(x)||^2}, \quad n \geq 1 \]
is completely integrable and all of its trajectories are bounded.

Then the zero set \( H^{-1}(0) = \{ x \in \mathbb{R}^n : H(x) = 0 \} \) is connected.

Proof: See Appendix.

Remark: According to Sard’s theorem [17], \( H^{-1}(0) \) is generically a submanifold.

In general, the task of checking whether the homotopy function under study satisfies the conditions stated in Theorem 1 can be very involved. The following theorem, which presents a sufficient condition to satisfy the conditions of Theorem 1, is useful for practical applications and will be illustrated with examples.

Theorem 2 (Sufficient Conditions for Global Connectedness): Suppose that \( H \in C^2(\mathbb{R}^n, \mathbb{R}^k) \), \( k \leq n \) satisfies the following conditions:

1) \( H(x) \in \text{range}(DH(x)) \) for all \( x \in \mathbb{R}^n \);
2) \( ||H(x)|| ||DH(x)^+ H(x)\| \in C^4(\mathbb{R}^n, \mathbb{R}^k) \), \( 0 \leq c < 1 \);
3) \( \sup(||H(x)|| ||DH(x)^+ H(x)\||_{L_2} : x \in \mathbb{R}^n) \leq K < \infty \).

Then \( H^{-1}(0) \) is connected.

Proof: See Appendix.

Corollary 1 [6]: For \( k \leq n \) let \( H \) be a function in \( C^2(\mathbb{R}^n, \mathbb{R}^k) \) such that \( \text{rank} DH(x) = k \) for all \( x \in \mathbb{R}^n \), and
\[ \sup(||DH(x)^TDH(x)DH(x)^T||_{L_2} : x \in \mathbb{R}^n) \leq K < \infty \]

Then \( H^{-1}(0) \) is connected.

Proof: Let \( \lambda_H(x) = 1 \) for all \( x \in \mathbb{R}^n \). Since \( \text{rank} DH(x) = k \), for all \( x \in \mathbb{R}^n \), \( DH(x)^+ = DH(x)^TDH(x)DH(x)^T \) and \( \text{range}(DH(x)) = \mathbb{R}^k \). Hence \( H(x) \in \text{range}(DH(x)) \forall x \in \mathbb{R}^n \) and \( DH(x)^+ H(x) \) is a \( C^4 \)-function. Hence \( H \) satisfies the conditions of Theorem 2, and so \( H^{-1}(0) \) is connected.

Remarks:

1) The conditions of Theorem 2 are more general than Diener’s condition in Corollary 1. For example, \( H(x_1, x_2) = x_2^2 \) satisfies the conditions of Theorem 2, but not Diener’s condition.
2) The conditions of Theorem 2 and Corollary 1 are still sufficient conditions for Theorem 1. For example, \( H(x_1, x_2) = x_1^2 \) satisfies the conditions of Theorem 1, but not the conditions of Theorem 2 and Corollary 1.

For the d-homotopy function \( H_d(x, \lambda) = F(x) - \lambda d \), the following theorem offers an easy way to analyze the connectivity of \( H_d^{-1}(0) \) via a dimension reduction.

Proposition 1 (Relation Between \( H_d^{-1}(0) \) and \( T_d^{-1}(0) \)): Suppose that \( 0 \) is a regular value of \( H_d \) and \( T_d \). Then \( T_d^{-1}(0) \) is a 1-D connected manifold if and only if \( H_d^{-1}(0) \) is a 1-D connected manifold.

Proof (only if): Note \( H_d^{-1}(0) = \{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : H_d(x, \lambda) = 0 \} = \{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : T_d(x) = 0 \} \).

1Without loss of generality, we may assume \( d_1 \neq 0 \).
where $\pi$ is defined by $\pi$. Since $\pi$ is connected and $H^\tau_d(0)$ is continuous, $H^\tau_d(0)$ is connected.

**(ii)**: Let $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$ be defined by $\pi(x, \lambda) = x$. Since $\pi$ is continuous and $H^\tau_d(0)$ is connected, $T_d^{-1}(0) = \pi(H^\tau_d(0))$ is connected.

Since $T_d^{-1}(0) \subset \mathbb{R}^n$ and $H^\tau_d(0) \subset \mathbb{R}^{n+1}$, the connectivity of $T_d^{-1}(0)$ is relatively easier to analyze or visualize than the connectivity of $H^\tau_d(0)$. Next, we present some examples to illustrate the above results.

**Example 1** (Tunnel Diode Circuit [4], [38]): The tunnel diode circuit shown in Fig. 1 has operating points determined by the loop equation

$$
F_1(v_1, v_2) = E - Rg_1(v_2) - (v_1 + v_2) = 0
$$
$$
F_2(v_1, v_2) = g_1(v_1) - g_2(v_2) = 0
$$

where the tunnel diode currents are given by $i_1 = g_1(v_1)$ and $i_2 = g_2(v_2)$. Recasting these equations into the form of $d$-homotopy equation (3) and using $d = (R, -1)^T$, we obtain

$$
H_d(x_1, x_2, \lambda) = \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix} - \lambda \begin{bmatrix} R \\ -1 \end{bmatrix}.
$$

Then its reduced homotopy function $T_d$ becomes

$$
T_d(v_1, v_2) = F_1(v_1, v_2) + R \cdot F_2(v_1, v_2)
$$
$$
= -v_1 + (E - v_2 - Rg_2(v_2)).
$$

Since $DT_d(x) = (-1, -1 - Rg'_2(v_2))$, we have

$$
(DT_d(x)DT_d(x)^T)^{-1} \succeq 1.
$$

Hence the condition of Theorem 2 is satisfied and so $T_d^{-1}(0)$ (therefore, $H_d^{-1}(0)$) is connected. See Fig. 2 where the tunnel diode currents are given by

$$
g_1(v_1) = 2.5v_1^3 - 10.5v_1^2 + 11.8v_1
$$
$$
g_2(v_2) = 0.43v_2^3 - 2.69v_2 + 4.56v_2
$$

and $E = 30$ and $R = 13.3$.

This example motivates the following useful criterion in two-dimensional (2-D) cases.

**Lemma 1**: Let $F : C^1_0(\mathbb{R}^2, \mathbb{R}^2)$. If there exists a $y \in \mathbb{R}$ such that $F_2(x_1, x_2) = yF_1(x_1, x_2)$ has the form of $g(x_1) + ax_2$ or $a_2x_1 + h(x_2)$ where $a, b \neq 0$, then the homotopy paths of the $d$-homotopy method $H_d^{-1}(0)$ are globally connected, where $d = (y, 1)^T$.

**Proof**: Let $T_y = F_2(x_1, x_2) - yF_1(x_1, x_2) = g(x_1) + ax_2$. Then $(DT_y(x)DT_y(x)^T)^{-1} = (g'(x_1))^2 + a^2 \succeq a^2$. Hence the conditions in Corollary 1 are satisfied and $T_d^{-1}(0)$ is connected. Therefore, by Proposition 1, $H_d^{-1}(0)$ is connected where $d = (y, 1)^T$.

The following example is a well-known test function for which several existing stochastic methods using the Newton–Raphson scheme fail to find all the solutions, because the size of the convergent regions of some solutions is small and the computation time required to find them is huge. For this reason we will use this test function frequently in the subsequent sections.

**Example 2**: Find the solutions of the equations

$$
F_1(x_1, x_2) = 4(x_1^2 + x_2^2 - 1)x_1 + 16(2x_1^2 - 1)^2
$$
$$
+ (2x_2^2 - 1)^2 - 2/3(2x_1^2 - 1)x_1
$$
$$
F_2(x_1, x_2) = 4(x_1^2 + x_2^2 - 1)x_2 + 16(2x_1^2 - 1)^2
$$
$$
+ (2x_2^2 - 1)^2 - 2/3(2x_2^2 - 1)x_2.
$$

Recasting these equations into the form of the reduced $d$-homotopy equation (4) and using $d = (5, 1)^T$, we obtain the connected curve shown in Fig. 3, which is the projection of the homotopy path: $T_d^{-1}(0)$ is connected. However, $DT_d(x)$ does not satisfy the conditions of Theorem 1 with this value of $d$. In fact, we can show that for any $d$, there exists a vector $x$ such that $T_d(x) \not\in \text{range}(DT_d(x))$ which violates the conditions of Theorem 1, although there also exists a $d \in \mathbb{R}^2$ which makes $T_d^{-1}(0)$ (hence $H_d^{-1}(0)$) connected. This shows that the conditions in Theorem 1 could be very conservative.

Next we present another slightly general sufficient condition for the connectivity of $H^{-1}(0)$, the following theorem will be shown to overcome some of the conservativeness of Theorem 1.

**Theorem 3 (Global Connectedness II)**: Suppose that $H \in C^2(\mathbb{R}^n, \mathbb{R}^k)$, $n \geq k$ satisfies the following conditions.

1. $0$ is a regular value of $H$.
2. If $\text{Crit}(H) \subset \{x \in \mathbb{R}^n : \det(DH(x)DH(x)^T) = 0\}$ and $W^s_{\mathbb{R}^k}(x_k)$ denotes the stable manifold of $x_k$ for the $C^1$ vector field on $\mathbb{R}^n$, given by

$$
\dot{x} = G(x) := -DH(x)^T H(x) \tag{5}
$$

then the set $(\mathbb{R}^n \setminus \bigcup_{x \in \text{Crit}(H)} W^s_{\mathbb{R}^k}(x))$ is connected.
3. $\|H(x)\|$ is a proper map (i.e., the inverse image of any compact set is compact) or
3′) for any $\gamma > 0$ and for any closed subset $K$ of

$$
\{x \in \mathbb{R}^n : \|H(x)\| \leq \gamma, \|DH(x)^T H(x)\| \neq 0\}
$$

Fig. 1. Two-tunnel diode circuit.
we have
\[
\inf\{\|DH(x)^T H(x)\| : x \in K\} > 0.
\]

Then $H^{-1}(0)$ is an $(n - k)$-dimensional connected submanifold in $\mathbb{R}^n$.

**Proof:** See Appendix.

This theorem relaxes some of the restrictivity of the conditions stated in Theorem 1, as shown in the following example.

**Example 3 (Example 2 Revisited):** First, recasting the equations in Example 2 into the form of reduced $d$-homotopy equation (4) and using $d_2 = (5, 1)^T$, we obtain

\[
T_{d_2}(x_1, x_2) = F_1(x_1, x_2) - 5F_2(x_1, x_2) = 0
\]
Fig. 4. The solid lines in (a) and (c) represent the projection \( T^{-1}_d(0) \) of the curve \( H^{-1}(0) \) when \( d = (5, 1)^T \) and \( d = (1, 2, 1)^T \), respectively, in Example 3. The numbers 0, 1, 2 denote the type of solutions. The mark "x" in (b) and (d) represents the points where \( D\mathcal{T}_d(x) = 0 \) when \( d = (5, 1)^T \) and \( d = (1, 2, 1)^T \), respectively, and the solid lines represent their stable manifolds. In (b), the complement of the union of these manifolds is connected, which explains the connectedness of \( T^{-1}_d(0) \) when \( d = (5, 1)^T \), whereas, in (d), the complements of the union of these manifolds and \( T^{-1}_d(0) \) are both disconnected.

Then we get

\[
G_1(x) = -T_{d_1}(x)\nabla T_{d_1}(x).
\]

Since \( T_{d_1}(x) \) is a proper function and

\[
(\mathbb{R}^n \setminus \bigcup_{x_i \in \text{Crit}(T_{d_1})} W^{s}_{G_1}(x_i))
\]

is connected as shown in Fig. 4(b), all the conditions of Theorem 3 are satisfied. Hence, \( T^{-1}_d(0) \) is connected as shown in Fig. 4(a).

Next for \( d_2 = (1, 2, 1)^T \), we obtain

\[
T_{d_2}(x_1, x_2) = F_1(x_1, x_2) - 1.2F_2(x_1, x_2) = 0
\]

and

\[
G_2(x) = -T_{d_2}(x)\nabla T_{d_2}(x).
\]

Then we get

\[
\text{Crit}(T_{d_2}) = \{(-0.5015, 0.8260), (-0.8542, 0.5169), (0.8260, -0.5015), (0.5169, -0.8542),
(0.8908, 0.8615), (0.6481, 0.2534),
(-0.2863, -0.6537), (0.2771, 0.6334),
(-0.6389, -0.2736), (-0.8954, -0.8716)\}.
\]

However, \( (\mathbb{R}^n \setminus \bigcup_{x_i \in \text{Crit}(T_{d_2})} W^{s}_{G_2}(x_i)) \) is not connected, as shown in Fig. 4(d). Note that \( T^{-1}_d(0) \) is also not connected, as shown in Fig. 4(c), and this shows that the condition of Theorem 3 is less restrictive than that of Theorem 1.

Remark: If all the points in \( \text{Crit}(H) \) are isolated and their types for the vector field \( G \) are greater than 1, then condition 2) in Theorem 1 is automatically satisfied. Although it is very difficult to check the conditions of Theorem 3 for the homotopy function of our interest unless some knowledge of the special structure of a problem is available, this theorem is useful for finding all solutions of a large class of nonlinear circuits and systems.

Now the following question naturally arises.

**Question:** For any given \( F: \mathbb{R}^n \to \mathbb{R}^n \), can we always find a \( d \in \mathbb{R}^n \) which makes \( T^{-1}_d(0) \) connected?

The next theorem shows that a homotopy path connecting all of the solution does not always exist. This theorem offers a negative answer to the above question.

**Theorem 4:** Let \( F: \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) be given by

\[
F(x) = \begin{pmatrix}
F_1(x_1, x_2) \\
F_2(x_1, x_2)
\end{pmatrix}, \quad x = (x_1, x_2), x_1, x_2 \in \mathbb{R}^n.
\]

If \( G: C^n \to C^n \), defined by

\[
G(x + iy) = F_1(x_1, x_2) + iF_2(x_1, x_2)
\]

is a proper function and
is an analytic function, then

1) all the zero points of $F$ have the positive sign of $\det(DF(x))$;
2) $T_d^{-1}(0)$ has exactly $N$ disconnected homotopy paths for any $d \in \mathbb{R}^n$, where $N$ is the number of the solutions of $F$;
3) every homotopy path in $T_d^{-1}(0)$ contains only one zero point of $F$.

Proof: Note that $z^* = x_1^* + i x_2^*$ is a zero of $G$ if and only if $(x_1^*, x_2^*)$ is a zero of $F$. Now let $(x_1^*, x_2^*)$ be any zero of $F$. Since $\det(DF(x_1^*, x_2^*)) = |\det(DG(z^*))|^2 > 0$, all the zeros of $F$ have the positive sign of $\det(DF(x))$. By the index theorem [1], any two zeros of $F$ which are adjacent along a single component of $T_d^{-1}(0)$ have different signs of $\det(DF(x))$. Hence, $T_d^{-1}(0)$ has exactly $N$ disconnected real homotopy paths for any $d \in \mathbb{R}^n$, where $N$ is the number of solutions of $F$. Moreover, every homotopy path of $T_d^{-1}(0)$ contains only one zero of $F$.

Example 4: Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by

$$F_1(x_1, x_2) = (x_1^2 - x_2^2 + 1)(x_1^2 - x_2^2 + x_1 - 6)$$
$$- 2x_1x_2(2x_1x_2 + x_2)$$
$$F_2(x_1, x_2) = (x_1^2 - x_2^2 + 1)(2x_1x_2 + x_2)$$
$$+ 2(x_1^2 - x_2^2 + x_1 - 6)x_1x_2.$$ 

Then $G(z) = F_1(x_1, x_2) + IF_2(x_1, x_2) = (z^2 + 1)(z - 1)(z + 3)$ is an analytic function. Since $G(z) = 0$ has four solutions at $z = i, -i, 1, -3$, $F = (F_1, F_2) = 0$ has four solutions at $(x_1, x_2) = (0, 1), (0, -1), (1, 0)$, and $(3, 0)$. Therefore, by Theorem 4, $T_d^{-1}(0)$ has exactly four disconnected real homotopy paths for any $d \in \mathbb{R}$ and every homotopy path contains only one zero point; see Fig. 5.

IV. A CONSTRUCTIVE METHODOLOGY FOR FINDING MULTIPLE HOMOTOPY PATHS

As we have seen in the previous section, a homotopy path which connects all the solutions of a system of nonlinear equations does not always exist. Even though such a homotopy path exists, its explicit construction can be very difficult unless some knowledge on the special structure of the underlying nonlinear equations is available. Moreover, for the problem of computing all the dc operating points restricted to box constraints, say

$$F(x) = 0$$

subject to $\alpha \leq x \leq b$, $x \in \mathbb{R}^n$ (6)

this approach can be computationally inefficient in the sense that, even though one can construct such a connected homotopy path, it is computationally demanding because too much time is spent tracing the homotopy paths outside the box constraints (see Fig. 6).

To overcome these difficulties, we will develop a constructive and systematic methodology to provide a starting point for each homotopy path, so that a continuation method can be applied to tracing each homotopy path to compute all the solutions when all the solutions lie on several homotopy paths.

The central idea of our constructive methodology to find multiple homotopy paths in $H^{-1}(0)$ is as follows: If it is possible to develop a mechanism to visit each homotopy path in a systematic way, then all of the solutions can be found this way. To this end, we develop a mechanism by finding a gradient-like induced vector field after collapsing all the points in the same components of $H^{-1}(0)$ such that all components of $H^{-1}(0)$ correspond to a stable equilibrium manifold (which is a generalized
Fig. 6. (a) represents a projection of a homotopy path on which all the solutions lie. However, when applied to a DC problem restricted to box constraints, it is computationally demanding because it requires too much time to locate solutions outside the box constraints. (b) represents a projection of a homotopy path restricted to the box constraints where the solutions lie on three separate homotopy paths.

The concept of stable equilibrium points ([23]) of the corresponding vector field. In this manner, it is feasible to find all or multiple components of $H^{-1}(0)$ via stable solutions of that vector field. One way to implement the above idea is to build the following quotient gradient system

$$
\begin{bmatrix}
\dot{x} \\
\dot{\lambda}
\end{bmatrix} = Q_H(x, \lambda) := \begin{bmatrix}
-D_x H(x, \lambda)^T H(x, \lambda) \\
-D_\lambda H(x, \lambda)^T H(x, \lambda)
\end{bmatrix},
$$

(7)

Note that the zero set, $Q_H^{-1}(0)$, is not of zero dimension but generally of higher dimensional manifold. We call a connected component of $Q_H^{-1}(0)$ an equilibrium manifold of system (7) and call an equilibrium manifold, say $\Sigma$, (asymptotically) stable if there is a $\delta = \delta(\epsilon) > 0$ such that

$$(x, \lambda) \in B_\delta(\Sigma) \rightarrow \Phi(t, (x, \lambda)) \rightarrow \Sigma, \text{ as } t \rightarrow \infty$$

and call it unstable, if not stable. (Note that $\Phi$ is a trajectory of system (7).) The stability region of a stable equilibrium manifold $\Sigma_{\delta}$ is then defined as

$$A_F(\Sigma_{\delta}) := \left\{ x \in \mathbb{R}^n : \lim_{t \to \infty} \Phi(x(t), \lambda(t)) \in \Sigma_{\delta} \right\}.$$

A quite comprehensive analysis for the topological and dynamical characterization of stability regions of stable equilibrium manifolds for the quotient gradient system is described in [23] and [24] and some results needed for the subsequent development are summarized below.

**Result 1)** Each homotopy path in $H^{-1}(0)$ corresponds to a stable equilibrium manifold of system (7).

**Result 2)** The quotient gradient system (7) is completely stable for a fairly large class of homotopy functions.

**Result 3)** For this class of homotopy functions, the stability boundary of a stable equilibrium manifold is the union of the stable manifolds of the equilibrium manifolds on the stability boundary.

One key issue in this mechanism is how to escape from a homotopy path in $H^{-1}(0)$ and move on toward another homotopy path. If one can systematically move from one homotopy path to another homotopy path, then it is feasible to visit all of the homotopy paths. The above results shed light on this issue. In terms of our formulation, we translate the problem of how to escape from a homotopy path and move on toward another homotopy path into the problem of how to escape from the stability region of a stable equilibrium manifold and enter the stability region of another stable equilibrium manifold.

We next present a constructive method to systematically find all or multiple homotopy paths in $H^{-1}(0)$ as follows.

**Step 1)** Approach a stable equilibrium manifold of system (7), which is a homotopy path of $H^{-1}(0)$.

**Step 2)** Find all of the solutions on this homotopy path (satisfying box constraints) by applying an efficient continuation method.

**Step 3)** Escape from a stable equilibrium manifold and move toward another stable equilibrium manifold and go to Step 2).

The first step can be implemented by numerically integrating system (7) forward from any initial starting point. The second
step can be accomplished by any suitable continuation methods (such as in HOMPACK) [1], [16], [25], [36]. The third step is the most challenging task. A simple approach that we adopted in this paper (see Algorithm 1), is to numerically integrate system (7) backward starting from an initial point near the homotopy path computed in Step 2) until it approaches a point in an unstable equilibrium manifold on the stability boundary. Then, by numerically integrating system (7) from that point, one approaches another stable equilibrium manifold.

V. ALGORITHM AND IMPLEMENTATION

A. Algorithm

From a theoretical viewpoint, the method proposed in the previous section builds a corresponding graph $G$ describing the connections between the stable equilibrium manifolds and unstable equilibrium manifolds with the following elements:

1) The vertices of $G$ consist of three partitions of vertices $N$, $V_s$, and $V_u$ where the vertex $N$ records the location of homotopy paths in $H^{-1}(0)$, i.e., $\Sigma^1_k$, $\cdots$, $\Sigma^p_k$, the vertex $V_s$ records the location of stable manifolds other than homotopy paths, i.e., $\Sigma^1_u$, $\cdots$, $\Sigma^q_u$ and the vertex $V_u$ records the location of unstable equilibrium manifolds, i.e., $\Sigma^1_u$, $\cdots$, $\Sigma^r_u$ of (7), where $p$, $q$, and $r$ are the total number of homotopy paths, stable equilibrium manifolds other than homotopy paths, and unstable equilibrium manifolds, respectively.

2) The edge of $G$ can only connect vertices of different types; $\Sigma^i_k$ is connected with $\Sigma^j_u$ if, and only if, a trajectory of (7) emanates from $\Sigma^i_k$ and approaches $\Sigma^j_u$.

Proposition 2 [23]: If system (7) is completely stable, then graph $G$ is connected.

The above analytical result theoretically guarantees that all the solutions can be found by the proposed methodology if system (7) is completely stable. We are now in a position to present a constructive homotopy method for finding all or multiple solutions of $F = 0$. The following algorithm can construct a graph $G$ which contains all or multiple solutions.

Algorithm 1 (Constructive Homotopy Methods for Finding all or Multiple Solutions):

Initialization 1) First, choose a homotopy function $H$ with a starting point, say $x_I$. Let $\Sigma^i_k$ be a homotopy path containing $x_I$. Set $V_s = V_u = \emptyset$, $N = V = \{\Sigma^i_k\}$, and a tolerance $\epsilon > 0$.

Main Step:

2) (Continuation Method) Find all the solutions on this homotopy path satisfying the box constraints.

3) (Building a Graph)

while $V \neq \emptyset$ do

begin

choose $\Sigma \in V$, set $V = V \setminus \{\Sigma\}$;

if $\Sigma \in V_s$ then find all or multiple adjacent stable equilibrium manifolds, say, $\Sigma^1_k$, $\cdots$, $\Sigma^k_k$; set $V = V \cup \{\Sigma^1_k, \cdots, \Sigma^k_k\}\setminus \{V_u\}$;

set $V_u = V_u \cup \{\Sigma^1_u, \cdots, \Sigma^k_u\}$;

set $E = E \cup \{\Sigma^1_u, \cdots, \Sigma^k_u\}$;

else find all or multiple adjacent stable equilibrium manifolds, say, $\Sigma^1_k$, $\cdots$, $\Sigma^k_k$; set $V = V \cup \{\Sigma^1_k, \cdots, \Sigma^k_k\\setminus \{V_u\}$;

for $i = 0$ to $l$

if $H(\Sigma^i_k) = 0$

set $N = N \cup \{\Sigma^i_k\}$

/*Continuation method*/

find all solutions on this homotopy path satisfying the box constraints.

end

end

end

At the end of the algorithm, three partitions of vertices $N$, $V_s$, and $V_u$, which correspond to the set of homotopy paths, stable equilibrium manifolds, and unstable equilibrium manifolds, respectively, and an edge-list $E$ will be obtained. For systems with some constraints other than box constraints, the above algorithm can be easily modified by adding some stopping criteria. There still exist many other possibilities for implementing the proposed constructive homotopy-based methodology, and this remains an open area of development.

B. Implementation

Next, we will address the issue of the implementation of the developed constructive homotopy-based method. In the initialization step, any well-defined and well-behaved (or bifurcation-free) homotopy function could be used. Regarding the computational efficiency of the proposed method, however, homotopy functions which can guarantee a bifurcation-free path with a finite arc length to a simple solution of $F(x) = 0$ are preferred. This choice of homotopy functions can facilitate the computational regard in the main step. For example, for transistor circuits with passivity and no-gain properties [32], which satisfy certain “coercivity” conditions (i.e., $F(x)^T x \geq 0$ for all $x$ with $\|x\| = r$ for some $r > 0$), a fixed homotopy function may be preferable because numerical implementations of globally convergent probability-one homotopy methods (or artificial parameter homotopy methods) appear to exhibit an exceptionally wide domain of convergence [25], [32], [36]. In addition, if one is concerned with solving nonlinear circuits which satisfy certain conditions, as in [12] and [14], the algorithm suggested there will be a good choice for this step.

The continuation method step poses little numerical difficulty. In the past, a significant effort has been spent in studying theoretical and algorithmic aspects of homotopy methods for finding solutions along a homotopy path. A number of efficient algorithms are now available such as in HOMPACK, PITCON (cf. [1], [7], [16], [25], [34], and [36]). So one can use any available algorithm that can trace all of the solutions on the homotopy path. We have, for instance, programmed the proposed method using commercially available MATLAB tools. The implementation proved to be adequate for solving a number of nonlinear
systems of equations including benchmark circuit equations. In
the next section, we show some numerical examples.

In the main step, any reliable numerical integrator of
the system (7) can be used, such as the variable step pre-
dictor–corrector method or the Runge–Kutta methods. In
our implementation, we adopted a simple trust-region-based
numerical integrator and observed that it can solve a number
of nonlinear systems and circuits. Other numerical integrators
can, of course, be more efficient than this, depending on the
problem, and the choice is left to the user.

It is clear that the ability of the developed constructive method
to find all or multiple solutions depends significantly on its
ability to find all adjacent stable or unstable equilibrium man-
ifolds. Moreover, the number of homotopy paths plays a cen-
tral role in the performance of the method. This issue is by no
means easily addressed. It is closely related to the structure of
the given homotopy function. One way to address this issue will
be to choose a homotopy function which has as small a number
of homotopy paths as possible. Also, further development of ef-
fective computational methods for computing stable or unstable
equilibrium manifolds will greatly enhance the developed con-
structive method. In many examples we studied, we observed a
small number of homotopy paths containing all solutions.

VI. NUMERICAL STUDIES

In this section, for the purpose of illustrating the proposed al-
gorithms in Section IV, several numerical examples are studied.

Example 5: To illustrate how constructive homotopy
methods work, we first consider the following system of
box-constrained nonlinear equations:

\[ F_1(x_1, x_2) = (4x_1^2 + 8) e^{-(x_1^2 + 2x_2^2)} + 24x_1 e^{-3x_1^2 - (x_2 - 2)^2} \]

\[ + (12x_1 - 36)e^{-3x_1^2 - 3x_2^2} + 8x_1 e^{-2x_1^2 - 2(x_2 + 1)^2} \]

\[ + (6x_1 + 6)e^{-3(x_1 + 1)^2 - 3(x_2 - 2)^2} = 0 \]

\[ F_2(x_1, x_2) = (8x_2 + 8) e^{-2x_1^2 - 2(x_2 + 1)^2} \]

\[ + 6x_2 e^{-3x_1^2 - x_2^2} + 4x_2 e^{-x_1^2 - 2x_2^2} \]

\[ + (8x_2 - 16)e^{-3x_1^2 - (x_2 - 2)^2} \]

\[ + (6x_2 - 12)e^{-3(x_1 + 1)^2 - 3(x_2 - 2)^2} = 0 \]

where the circuit trajectory is confined within the hypercube

\[-4 \leq x_1 \leq 4\]

\[-4 \leq x_2 \leq 4.\]

Now we choose \(d = (0.3, 1)^T\) as an initial guess. If we recast
these equations into the form of reduced \(d\)-homotopy equation
(4), we obtain

\[ h(x) := T_d(x_1, x_2) = F_1(x_1, x_2) - 0.3F_2(x_1, x_2) = 0 \quad x = (x_1, x_2). \]

The corresponding quotient gradient system (7) then becomes

\[ \dot{x} = -h(x) \nabla h(x). \tag{8} \]

At Step 1), we choose an arbitrary initial guess, say \(x_I =
(4, 0)\) which is a boundary point of the constraint hypercube
and set \(c = 0.001\). Note that this point, \(x_I\), does not satisfy
\(h(x) = 0\).

At Step 2), by integrating (8), we find a starting point \(x_F =
(2.89, 0)\) on the path \(P_1\). Then, by using a continuation method,
we begin the process to find all the solutions on this path until it
Fig. 8. The projection $T_d^{-1}(0)$ of the curve $H_d^{-1}(0)$ for Example 6, where $d = (1, 2, 1)^T$ consists of three separate reduced homotopy paths $P_1$, $P_2$ and $P_3$ which were highlighted by solid lines. The dotted line represents the trajectory generated by our algorithm 1 with a starting point $x_0 = (0.5119, -0.5431)$ near a solution in path $P_1$.

hits the boundary of the constraint hypercube in both directions. During the process, one solution $(2.9036, -0.0216)$ on path $P_1$ is obtained.

At Step 3), we begin to build a graph. See Fig. 7. By integrating (8) backward starting from a point $x_0 = x_F + \epsilon(x_F - x_I)$, we obtain an unstable singular point at $(2.4955, 0.1199)$, denoted by $u$ in Fig. 7. Next, numerically integrating (8) forward starting from a point $x_1 = u + \epsilon(u - x_0)$ where $\epsilon = 0.001$, we find a feasible point $(1.3088, -0.2339)$ in the second path $P_2$. On the path $P_2$, we obtain three solutions at $(1.4809, -0.6807), (1.2261, 0.4095)$, and $(1.3094, 1.0129)$. After this, we apply Step 3) again. We then obtain a feasible point in the third path $P_3$ and obtain two solutions at $(-0.8917, -0.5388), (0, -1.0345)$. Next, applying Step 3) again after finding all the solutions on this path $P_3$, we obtain a feasible point in the same path. By looking at the recorded solution set, we stop the computation for the solutions in this path. Applying Step 3) with a different starting point near a solution at $(-0.8917, -0.5388)$, we can obtain a different homotopy path $P_4$ and get five solutions at $(-0.8599, 0.8405), (-1.3376, 1.2234), (-0.0159, 1.9628), (-2.0068, 0.0216)$, and $(-1.0350, 1.0345)$. In this way, we found all separate homotopy paths in the constraint hypercube and thus all the solutions of the box-constrained nonlinear equations.

Example 6 (Example 3 Revisited): We next consider Example 3 again, this time for $d = (1, 2, 1)^T$, which was shown in Fig. 4 and redrawn in Fig. 8. The projection $T_d^{-1}(0)$ of the curve $H_d^{-1}(0)$ consists of three separate homotopy paths $P_1$, $P_2$ and $P_3$. Suppose that we have found all the solutions on the path, for example, $P_1$ by performing Step 1) and Step 2). At Step 3), by integrating (7) backward with a starting point near a solution in path $P_1$, we obtain a different homotopy path $P_4$ and get five solutions at $(-0.8599, 0.8405), (-1.3376, 1.2234), (-0.0159, 1.9628), (-2.0068, 0.0216)$, and $(-1.0350, 1.0345)$. In this way, we found all separate homotopy paths in the constraint hypercube and thus all the solutions of the box-constrained nonlinear equations.

**Table I**

<table>
<thead>
<tr>
<th>Path</th>
<th>Solutions on the path</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$(0.7110, -0.7035), (0.5228, -0.5212)$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$(-0.0040, 0.7933), (0.5175, 0.5186), (0.7836, 0.0026), (-0.0013, -0.7908), (-0.5175, -0.5237), (-0.7836, 0.0026)$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$(-0.8481, 0.8575), (0.8562, 0.8524), (0.8508, 0.8549), (-0.8562, -0.8549), (-0.4556, 0.8858), (-0.8844, 0.4570), (-0.8938, -0.4724), (-0.4530, -0.8832), (0.4610 - 0.8883), (0.8884 - 0.4698), (0.8885, 0.4621), (0.4610, 0.8883), (0, 0), (0.7110, 0.7035), (-0.7030, -0.7112)$</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$(-0.7110, 0.7035), (0.5228, 0.5212)$</td>
</tr>
</tbody>
</table>

Fig. 9. A four-transistor multistate circuit.
TABLE II

SOLUTIONS OF EXAMPLE 7

<table>
<thead>
<tr>
<th>Path</th>
<th>No.</th>
<th>(v_1)</th>
<th>(v_2)</th>
<th>(v_3)</th>
<th>(v_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_1)</td>
<td>1</td>
<td>-1.0510</td>
<td>0.3775</td>
<td>0.3846</td>
<td>-3.942</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.3366</td>
<td>0.3644</td>
<td>0.3829</td>
<td>-3.5125</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.3840</td>
<td>-3.7950</td>
<td>0.3800</td>
<td>-2.8301</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.3859</td>
<td>-4.3186</td>
<td>0.3439</td>
<td>0.3577</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.3879</td>
<td>-4.9101</td>
<td>-1.5114</td>
<td>0.3775</td>
</tr>
<tr>
<td>(P_2)</td>
<td>6</td>
<td>0.3301</td>
<td>0.3681</td>
<td>0.3381</td>
<td>0.3633</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>-0.7143</td>
<td>0.3775</td>
<td>0.3351</td>
<td>0.3653</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>-0.5158</td>
<td>0.3775</td>
<td>-0.9691</td>
<td>0.3775</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>0.3240</td>
<td>0.3704</td>
<td>-1.0696</td>
<td>0.3775</td>
</tr>
</tbody>
</table>

**Example 7 (Four-Transistor Multistate Circuit [4], [7], [39], [40]):**

We next consider the four-transistor multistate benchmark circuit shown in Fig. 9. Replacing each bipolar-junction transistor by the simplified Ebers–Moll model and simplifying the circuit equations, we obtain the following system of four nonlinear equations involving the four unknown base-to-emitter voltages \(v_1, v_2, v_3, v_4\):

\[
F_1 = 6.103168I_e(e^{40v_1} - 1) + 4.36634v_2 + 2.863168I_e(e^{40v_2} - 1) - 12
\]
\[
F_2 = 5.4v_1 + 3.58I_e(e^{40v_1} - 1) + 6.62I_e(e^{40v_2} - 1) + v_3 + (0.7)I_e(e^{40v_3} - 1) + (0.5)I_e(e^{40v_4} - 1) - 22
\]
\[
F_3 = 6.103168I_e(e^{40v_3} - 1) + 2.863168I_e(e^{40v_4} - 1) + 4.36634v_4 - 12
\]
\[
F_4 = 1v_1 + (0.7)I_e(e^{40v_1} - 1) + (0.5)I_e(e^{40v_2} - 1) + 5.4v_3 + 3.58I_e(e^{40v_3} - 1) + 6.62I_e(e^{40v_4} - 1) - 20
\]

\(x_0 = (0.5119, -0.5431)\) near a solution in path \(P_1\), we obtain an unstable singular point at \((0.2258, -0.2509)\), denoted by \(u\) in Fig. 8. Next, by numerically integrating (7) forward from a point \(x_0 = u + e(u - x_0)\), we find a feasible point in the second path \(P_2\). Starting from the point and applying the continuation process, we obtain all of the solutions on this path. Applying Step 3), we then obtain a feasible point in the third path \(P_3\) via an unstable singular point at \((-0.2295, 0.2529)\). In this way, we could find all the separate homotopy paths and thus could find all the solutions; see Table I. This example shows that our algorithm is efficient, and that it does not require the difficult task of finding a good initial guess \(d\). In other words, our algorithm is self-starting. In addition, these two examples show that our proposed method has the potential for finding all the solutions.
where \( I_s = 10^{-6} \). In this example, we first chose the following \( d \)-homotopy function

\[
H(x, \lambda) = F(x) + (\lambda - 1) d
\]

where \( d = (1, 0, -1, 0) \). We next applied the proposed algorithm with a randomly chosen initial guess. After the simulation, we found two homotopy paths \( P_1 \) and \( P_2 \) which contain all of the solutions. For the detailed results after each iteration, see Fig. 10 and Table II. In this example, we have successively found all the solutions.

Example 8 (Schmitt-Trigger Circuit [18], [35]): We consider the Schmitt-trigger benchmark circuit discussed in [18] and [35] (see Fig. 11). Replacing each transistor by the simplified Ebers–Moll model and simplifying the circuit equations, based on the modified nodal formulation [18], we obtain the following system of four nonlinear equations involving the four unknown base-to-emitter voltages \( v_{13}, v_2, v_3, v_4 \).

\[
\begin{align*}
F_1 &= 0.6790v_2 - 0.7315v_1 - 0.1527v_3 + 0.0526v_4 + 5.6838 \\
&
+ 1.01 \cdot I_s(e^{-38.78v_1} - 1) - 0.52 \cdot I_s(e^{-38.78v_2} - 1) \\
F_2 &= 0.6789v_2 - 1.5474v_1 + 0.2317v_3 - 0.1316v_4 - 6.2009 \\
&
- 0.99 \cdot 1.01 \cdot I_s(e^{-38.78v_1} - 1) + 2 \cdot I_s(e^{-38.78v_2} - 1) \\
F_3 &= 0.9210v_3 - 0.1316v_1 + 0.0526v_2 - 0.9211v_4 + 8.5261 \\
&
+ 1.01 \cdot I_s(e^{-38.78v_1} - 1) - 0.52 \cdot I_s(e^{-38.78v_2} - 1) \\
F_4 &= 0.9210v_4 + 0.2317v_1 - 0.1527v_2 - 1.0219v_3 - 8.5287 \\
&
- 0.99 \cdot 1.01 \cdot I_s(e^{-38.78v_1} - 1) + 2 \cdot I_s(e^{-38.78v_2} - 1)
\end{align*}
\]

where \( I_s = 10^{-13} \). In this example, we applied the proposed algorithm to the same \( d \)-homotopy function as Example 7 and found one homotopy path \( P_1 \) which contains all the solutions. For the detailed results after each iteration, see Fig. 12 and Table III.

**Table III**

<table>
<thead>
<tr>
<th>Path</th>
<th>No.</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>1</td>
<td>-0.7970</td>
<td>3.5277</td>
<td>-0.8292</td>
<td>-0.7375</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.8130</td>
<td>0.5605</td>
<td>-0.8194</td>
<td>1.7094</td>
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<td></td>
<td>3</td>
<td>-0.8223</td>
<td>-0.7506</td>
<td>-0.1596</td>
<td>9.1606</td>
</tr>
</tbody>
</table>

**VII. Conclusion**

In this paper, we have conducted a comprehensive analysis of homotopy methods on the computation of all dc solutions of nonlinear circuits and systems. We have derived several sufficient conditions for which a continuation method is guaranteed to find all of the solutions along one homotopy path. In addition, we have developed a constructive homotopy method to find a starting point of each homotopy path for general nonlinear circuits and systems. This method is applicable to all kinds of homotopies and can in addition be computationally efficient for a box constrained dc problem. We believe that our systematic method is potentially practical in certain problems and will be widely accepted as soon as more efficient implementation schemes, which work well on many challenging applications in the sciences and engineering, are available. In addition, we note that the application field of the theorems and algorithms developed in this paper is not limited to circuit analysis; it can be applied to any kind of nonlinear system of equations.

**Appendix**

Proof of Theorem 1: Consider the \( C^1 \)-vector field on \( \mathbb{R}^n \), given by

\[
V(x) = -\lambda H(x) DH(x)^T \frac{H(x)}{1 + ||H(x)||^2_2}.
\]
Since $V$ is completely integrable, $V$ determines a global $C^1$ flow $\Phi(t, x)$ such that
\[
\frac{\partial \Phi(t, x)}{\partial t} = V(\Phi(t, x)), \quad \Phi(0, x) = x.
\]

Therefore, we have
\[
DH(\Phi(t, x)) \frac{\partial \Phi(t, x)}{\partial t} = -\lambda_H(\Phi(t, x)) \frac{H(\Phi(t, x))}{1 + ||H(\Phi(t, x))||^2}
\]
or, equivalently,
\[
\frac{\partial}{\partial t} H(\Phi(t, x)) = -\lambda_H(\Phi(t, x)) \frac{H(\Phi(t, x))}{1 + ||H(\Phi(t, x))||^2}
\]
and
\[
\frac{\partial}{\partial t} ||H(\Phi(t, x))||^2 = -2\lambda_H(\Phi(t, x)) \frac{||H(\Phi(t, x))||^2}{1 + ||H(\Phi(t, x))||^2} \leq 0.
\]

Now define a $C^1$ function $N_H: \mathbb{R}^n \to \mathbb{R}$ by $N_H(x) = ||H(x)||^2 \geq 0 \forall x \in \mathbb{R}^n$. Then $N_H(0) = H^{-1}(0)$ and
\[
\frac{\partial}{\partial t} N_H(\Phi(t, x)) = -2\lambda_H(\Phi(t, x)) \frac{N_H(\Phi(t, x))}{1 + N_H(\Phi(t, x))^{n/2}} \leq 0,
\]
which is a decreasing function of $t$. Since $N_H$ is bounded from below (i.e., $N_H(x) = ||H(x)||^2 \geq 0 \forall x \in \mathbb{R}^n$), $g(t) = N_H(\Phi(t, x))$ has a limit $\alpha$ as $t \to \infty$. Let $\omega(x_0)$ be the $\omega$ limit set of $x_0$. Then for any $p \in \omega(x_0)$, $\exists$ a sequence $\{t_n\}$ with $t_n \to \infty$ and $\Phi(t_n, x_0) \to p$ as $n \to \infty$. By the continuity of $N_H$, $N_H(p) = \lim_{n \to \infty} N_H(\Phi(t_n, x_0)) = \alpha$. Hence, $N_H(p) = \alpha \forall p \in \omega(x_0)$. Since $\omega(x_0)$ is an invariant set,
\[
N_H(x) = 0 \text{ on } \omega(x_0)
\]
or, equivalently, $N_H(\omega(x_0)) = ||H(\omega(x_0))||^2 = 0$. Since $\Phi(t, x_0)$ is bounded, $\Phi(t, x_0)$ approaches $\omega(x_0)$ as $t \to \infty$. Hence it follows that
\[
\lim_{t \to \infty} ||H(\Phi(t, x_0))||^2 = 0
\]
since the trajectories are bounded and reach $H^{-1}(0)$ for $t \to \infty$. So we can define a continuous, surjective mapping $\Phi: \mathbb{R}^n \to H^{-1}(0)$ which leaves $H^{-1}(0)$ fixed. Since $\mathbb{R}^n$ is connected, it follows that $H^{-1}(0)$ is connected.

Proof of Theorem 2: Since $V$ is bounded, it is complete [21], [29]. Next, we shall show that all trajectories are bounded. Let $x \in \mathbb{R}^n$ and $T \geq 0$. Since
\[
\frac{\partial}{\partial t} ||H(\Phi(t, x))||^2 = -2||H(\Phi(t, x))||^2 \leq 0
\]
we obtain
\[
1) \text{ For } 0 < c < 1
\]
\[
||H(\Phi(T, x))||^c = ||H(x)||^c \left(1 + \int_0^T \frac{||H(x)||^c}{c(1 + ||H(\Phi(t, x))||^{n/2})} dt \right)^{-1/c}
\]
\[
\leq ||H(x)|| \left(1 + \frac{||H(x)||^c}{c(1 + ||H(x)||^{n/2})} \right)^{-1/c}.
\]
\[
2) \text{ For } c = 0
\]
\[
||H(\Phi(T, x))||^c \leq ||H(x)|| \exp \left(-\int_0^T \frac{dt}{1 + ||H(\Phi(t, x))||^n} \right)
\]
\[
\leq ||H(x)|| \exp \left(-\frac{\beta c}{1 + ||H(x)||^{n/2}} \right)
\]
From this we get
\[
||\Phi(T, x) - x || \leq \left| \int_0^T V(\Phi(t, x)) dt \right|
\]
\[
\leq \int_0^T ||V(\Phi(t, x))|| dt
\]
\[
\leq K \int_0^T ||H(\Phi(t, x))|| dt
\]
\[
\leq K \int_0^T ||H(\Phi(t, x))|| dt.
\]
Hence,
\[
1) \text{ For } 0 < c < 1
\]
\[
||\Phi(T, x) - x || \leq K ||H(x)|| \left(1 + \frac{||H(x)||^c}{c(1 + ||H(x)||^{n/2})} \right)^{-1/c} dt
\]
\[
\leq K \frac{c^2}{1 - c} ||H(x)|| \frac{||H(x)||^n}{||H(x)||^c}.
\]
\[
2) \text{ For } c = 0
\]
\[
||\Phi(T, x) - x || \leq K ||H(x)|| \int_0^T \exp \left(\frac{-\beta c}{1 + ||H(x)||^n} \right) dt
\]
\[
\leq K ||H(x)|| \left(1 + ||H(x)||^n \right).
\]
Therefore, all trajectories are bounded. Now all the conditions of Theorem 1 are satisfied, so \( H^{-1}(0) \) is connected.

**Proof of Theorem 3:** If \( H \) satisfies condition 1), then the \( C^4 \)-vector field \( G \) is completely integrable and every trajectory for the vector field \( G \) is bounded [23]. Since \( G \) determines a global \( C^4 \) flow \( \Phi(t, x) \) such that

\[
\frac{\partial \Phi(t, x)}{\partial t} = G(\Phi(t, x)) \quad \Phi(0, x) = x
\]

we have

\[
\begin{align*}
\frac{\partial}{\partial t} \|H(\Phi(t, x))\|_2^2 &= 2H(\Phi(t, x))^T DH(\Phi(t, x)) \frac{\partial \Phi(t, x)}{\partial t} \\
&= -2\|H(\Phi(t, x))^T H(\Phi(t, x))\|_2^2 \leq 0.
\end{align*}
\]

Let \( x_0 \in \mathbb{R}^n \) and \( \Phi(t, x_0) \) be a bounded trajectory starting at \( x_0 \in \mathbb{R}^n \). Then by the same argument as in the proof of Theorem 1, we can show that

\[
\lim_{t \to \infty} \|H(\Phi(t, x))^T H(\Phi(t, x))\|_2 = 0
\]

since the trajectories are bounded and reach \( \{ x \in \mathbb{R}^n : DH(x)^T H(x) = 0 \} \) for \( t \to \infty \). So we can define a continuous, subjective mapping \( \Phi : (\mathbb{R}^n \setminus \bigcup_{x \in C(t)}(H)) \to H^{-1}(0) \) which leaves \( H^{-1}(0) \) fixed. (Note that since \( H \in \mathcal{H} \), Crit(\( H \)) and \( H^{-1}(0) \) are disjoint generically [23]). Since \( (\mathbb{R}^n \setminus \bigcup_{x \in C(t)}(H)) W_{\epsilon}(x^*) \) is connected, it follows that \( H^{-1}(0) \) is connected.

**REFERENCES**


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