A game-theoretical cooperative mechanism design for a two-echelon decentralized supply chain

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Abstract

The paper analyses non-cooperative behaviour in a two-echelon decentralized supply chain, composed of one supplier and n retailers. For sufficient supply from the supplier, we build the approximate decision model of their base stock level, in which the suppliers’ reactions are not considered, and its non-cooperative behaviour is obtained. For insufficient supply from the supplier, much more complicated non-cooperative behaviour is obtained, and we find that competition will occur between all the retailers as well as the supplier. In order to guarantee optimal cooperation in the system, several Nash equilibrium contracts are designed in echelon inventory games and local inventory games.

Keywords: Supply chain management; Stochastic inventory management; Non-cooperative game; Cooperative mechanism; Contract design

1. Introduction

The literature of supply chain inventory management mostly assumes that policies are set by a central decision-maker to optimize total supply chain performance (Cohen and Lee, 1988; Ishii et al., 1988; Pyke and Cohen, 1994; van Houtum et al., 1996; Maloni and Benton, 1997; Chopra and Meindl, 2001; Velde and Meijer, 2002). In the multi-echelon inventory domain, early in 1960, Clark and Scarf (1960) gave the exact decomposition procedure to find the optimal control parameters of a serial supply chain. Recently, Diks and de Kok (1999) gave a method to solve the general n-echelon divergent supply chain problem.

However, the central optimization approach is only an approximation of the real supply chain system, since most supply chain systems are decentralized. The central optimization models have three drawbacks:

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(1) Ignoring the independence of the supply chain members. Competitive behaviour between members may lower the supply chain efficiency.
(2) The cost of information processing may be expensive. The central decision maker must gather all the information from every supply chain member and finally issue instructions to members.
(3) The capacity of the central optimization models. If the problem is fairly large and difficult, however, it may be impossible to be modelled and calculated.

Recently the decentralized supply chains have been studied by many researchers (Monahan, 1984; Goyal and Gupta, 1989; Lee and Billington, 1993; Cachon and Zipkin, 1999; Chen, 1999; Lee and Whang, 1999). The decentralized supply chains can be categorised as two kinds: the intra-organizational-coordination one and an inter-organizational-cooperation one. In an intra-supply chain, all members are less selfish within an organization/firm, and there may exist a central power who controls the whole system to some extent. This kind of supply chain optimization is in the distributed problem solving realm, and its main problem is to design all members’ sub-goals or performance measurement schemes to mitigate the incentives problems of all members in the supply chain. Monahan (1984) first used the quantity discount as a measurement scheme in a two-echelon fixed demand system. Federgreun and Zipkin (1984) found that by constructing cost functions appropriately, a decentralized system can perform equally as a decentralized one for fixed demand two multi-echelon inventory with a single retailer, and can perform almost as well as a centralized one with multiple retailers. Lee and Whang (1999) gave a set of measurement schemes for a stochastic demand one–one supply chain inventory system.

An inter-organizational-cooperation supply chain is composed of several self-interested members/firms. In order to achieve supply chain efficiency, all members are willing to cooperate, but the cooperative mechanism, e.g. contracts, must be carefully negotiated and designed. Many studies show that competition between members lowers the system efficiency, and usually is not optimal. How we can coordinate the supply chain members? A contract-based coordination mechanism should be used to fulfill this work, and the contract must comply with the following rules:

(1) Ensure the members’ profit is not lower than the profit level before supply chain cooperation and properly share the total cooperative revenue—Participative constraint.
(2) Diminish the incentives to deviate from the system optimal solution—Incentive compatible constraint.

Cooperation in fixed demand inventory models has been studied carefully in the inventory literature as well as in the microeconomics area. Goyal and Gupta (1989) analysed the quantity discount in a two-echelon fixed demand inventory model from the joint buyer–seller perspective. The information asymmetric problem also attracted some attention, even though not many researchers (Corbett and Groote, 2000) have studied the optimal coordination contract in two-echelon fixed demand inventory system under asymmetric holding costs.

In the stochastic inventory management area, however, the coordination mechanisms have fewer studies. Cachon and Zipkin (1999) analysed the non-cooperative game behaviour and the joint optimum of a two-echelon serial supply chain, which is composed of a supplier and a retailer, and given all supply members have full information. In this paper, we extend their model to a one-supplier and \( n \)-retailers situation. If there exist multiple retailers, the supply from a supplier might not satisfy the demand of multiple retailers. The problem is how to design the distributing scheme of the supplier, and this makes models of supply chain systems more complex. We separate the sufficient supply from the supplier and insufficient supply from the supplier in our model, which is not mentioned in Cachon and Zipkin’s model (1999).

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 considers the non-cooperative game model. Section 4 deals with the system optimal policy. Section 5 gives the cooperative mechanisms based on linear contracts. Section 6 shows a numerical example.
2. Model description

The model is similar to the model presented by Cachon and Zipkin (1999), but we consider a one-product inventory system with one supplier (denoted by suffix 0) and \( n \) retailers (denoted by suffix 1 to \( n \)) as shown in Fig. 1. Time is divided into an infinite number of discrete periods, and consumer demand at each retailer is stochastic, independent across periods and stationary. The following is the sequence of events during a period: (1) shipments arrive; (2) orders are submitted and shipments are released; (3) consumer demand occurs; (4) holding and backorder penalty costs are charged.

There is a lead time for shipments from source to supplier, which is denoted as \( L_0 \) periods. There is also a lead time for shipments from the supplier to retailer \( i \), which is denoted as \( L_i \) periods. Both supplier and all retailers may order any non-negative amount in each period. There is no fixed cost for placing or processing an order (Cachon and Zipkin, 1999; Clark and Scarf, 1960). Each firm pays a constant price per unit ordered, so there are no quantity discounts.

We introduce a term, \( h_i \), to model explicitly the asymmetry in holding costs between supplier and retailer. The supplier is charged a holding cost \( h_0 \) per period for each unit in its stock or en route to the retailers. The retailer \( i \)'s holding cost is \( h_0 + h_i \) per period for each unit in its stock. Note that \( h_i \) is not the total holding cost per unit per period for retailer \( i \), but is the amount by which retailer \( i \)'s holding costs exceed those of the supplier. Generally, the management of a supplier is better than that of a retailer, and we assume that the holding cost per unit of the supplier is less than that of the retailer, i.e. \( h_i > 0 \).

Unmet demands are backlogged, and all backorders are ultimately filled. Both the retailer and the supplier may incur costs when demand is backordered. The retailers are charged \( zp \) for each backorder, and the supplier is charged \((1-z)p\). The parameter \( p \) is the total system backorder cost, and \( z \) specifies how this cost is divided among echelons. The parameter \( z \) is exogenous, and \( z \in [0, 1] \).

Each firm uses a base stock policy. At the beginning of each period, the firm orders a sufficient amount to raise its inventory position, plus outstanding orders, to that base stock level.

Other notations:

(1) \( IL_n, IT_n, IP_n \)—echelon inventory level, in-transit inventory, and echelon inventory position of retailer \( i \) in period \( t \);
(2) \( D_0, D_i \)—total random demand at all retailers or retailer \( i \) over \( \tau \) periods;
(3) \( f^*_i(x), F^*_i(x), \mu^*_i \)—the probability density distribution function, probability distribution function of demand and mean total demand at retailer \( i \) over \( \tau \) periods;
(4) \( f^*_0(x), F^*_0(x), \mu^*_0 \)—the probability density distribution function, probability distribution function of demand and mean total demand at all retailers over \( \tau \) periods;
(5) \( S_i, S_0 \)—echelon inventory level for order of retailer \( i \) and the supplier;
(6) \( S_i, S_0 \)—local inventory level for order of retailer \( i \) and the supplier;

![Fig. 1. Schematic representation of a two echelon supply chain system.](image-url)
(7) $T_i, T_0$—change of cost function under contract at retailer $i$ and the supplier;
(8) $\tilde{G}_i(t)$—actual cost function of retailer $i$ under sufficient supply;
(9) $\tilde{G}_i(t), \tilde{G}_i(t)$—expected cost function of retailer $i$ under sufficient supply for echelon inventory and local inventory;
(10) $U_i(t), \tilde{U}_i(t)$—expected cost function of retailer $i$ under insufficient supply for echelon inventory and local inventory;
(11) $U_0(t), \tilde{U}_0(t)$—expected cost function of the supplier for echelon inventory and local inventory;
(12) $\tilde{U}_i(t), \tilde{U}_i(t)$—expected cost function of retailer $i$ under the contract for echelon inventory and local inventory;
(13) $Z_i(t)$—the allocation function for retailer $i$;
(14) $[x]^+ = \max(x, 0), [x]^− = \max(−x, 0)$.

For clarity, we denote $\sum_{i=1}^{n}$ or $\sum_{k=1}^{n}$ as $\sum$.

3. Echelon inventory games and local inventory games

In a supply chain system, all members could track their stocks by echelon inventory policies or local inventory policies. A firm’s local inventory is its on-hand inventory, and its echelon inventory is its local inventory plus all inventory held lower in the supply chain. Using an echelon base stock level, each period the firm orders a sufficient amount to raise its echelon inventory position plus outstanding orders to that level. A firm’s local base stock level is similar, except the local inventory position replaces the echelon inventory position. Because all members are selfish, each member will pursue his maximum profit in spite of others. This might lead to non-optimum of the whole system without cooperation. The non-cooperative analysis is obtained as the following.

3.1. Echelon inventory games

3.1.1. Retailers’ model

(1) Sufficient supply from the supplier: If the retailers do not know the reaction of the supplier to an order because they do not have enough information, they will make their decisions as if the supplier has enough inventory to supply all orders, i.e. the inventory of the supplier is large enough to fulfill all the retailers’ orders. Below we only consider retailer $i$.

In each period the retailer $i$ is charged $h_i + h_0$ per unit held in inventory and $ap$ per unit backordered, and the sum of these costs in period $t$ can be defined if at the beginning of period $t$ retailer $i$’s inventory level is $IL_{it}$:

$$\tilde{G}_i(IL_{it} - D_i^t),$$

where

$$\tilde{G}_i(y) = (h_i + h_0)[y]^+ + ap[y]^−.$$
Its expected cost is denoted by $G_i(S_i)$, where

$$G_i(S_i) = G_i(IP_u) = E[\tilde{G}_i(S_i - D_i^{L+1})]$$

$$= \int_0^\infty (h_i + h_0)(S_i - x)^+ f_i^{L+1}(x) \, dx + \int_0^\infty xP[S_i - x]^{-} f_i^{L+1}(x) \, dx$$

$$= \int_0^{S_i} (h_i + h_0)(S_i - x) f_i^{L+1}(x) \, dx + \int_{S_i}^{\infty} xP(x - S_i) f_i^{L+1}(x) \, dx$$

$$= \int_0^{\infty} (h_i + h_0)(S_i - x) f_i^{L+1}(x) \, dx$$

$$= (h_i + h_0)(S_i - \mu_i^{L+1}) + (h_i + h_0 + xP) \int_{S_i}^{\infty} (x - S_i) f_i^{L+1}(x) \, dx.$$ 

So the base stock level that minimizes the retailer $i$’s costs is given by

$$\min_{S_i} G_i(S_i).$$

(3.1)

The derivatives of the cost function can be obtained as follows:

$$G_i'(S_i) = (h_i + h_0) - (h_i + h_0 + xP) \int_{S_i}^{\infty} f_i^{L+1}(x) \, dx$$

$$= (h_i + h_0) - (h_i + h_0 + xP)(1 - F_i^{L+1}(S_i))$$

$$= (h_i + h_0 + xP)F_i^{L+1}(S_i) - xP,$$

$$G_i''(S_i) = (h_i + h_0 + xP) f_i^{L+1}(S_i) \geq 0.$$ 

So, $G_i(S_i)$ is convex with respect to $S_i$. According to first order condition, the best $S_i$ can be obtained as follows:

$$F_i^{L+1}(S_i) = \frac{xP}{h_i + h_0 + xP}.$$ 

(3.2)

(2) Insufficient supply from the supplier: Suppose at the beginning of a period, the supplier has an echelon inventory stock $x$ of the product, and each retailer $i$ wants to raise their echelon inventory position to $S_i$. If $x \geq \sum S_i$, then the echelon inventory position of retailer $i$ yields $S_i$, and the remainder $x - \sum S_i$ is retained at the supplier. However, if $x < \sum S_i$ we have to deal with one of the main difficulties of distributed systems: How should the supplier ratio the available stock over its successors? Here we use the allocation function defined by (Diks and de Kok, 1999)

$$Z_i(x) = S_i - q_i \left[ \sum S_i - x \right]^+.$$ 

We refer to $Z_i$ as the echelon stock level of retailer $i$ just after the allocation, $q_i$ is the allocation fraction belonging to each, $\sum q_i = 1$, and $S_i$ denotes the desired order-up-to-level of retailer $i$, and $\sum S_i$ is the sum of all retailers’ order-up-to-levels. Below we denote $\sum S_i$ as $S$.

After all firms place their orders in period $t - L_0$, the supplier’s echelon inventory position equals $S_0$. After inventory arrives in period $t$, but before period $t$ demand is met, the supplier’s echelon inventory level
equals \( S_0 - D_0^{L_0} \). Hence \( E[\delta_0 - D_0^{L_0}] \) is the supply chain’s expected inventory level (average supply chain inventory minus mean demand). When \( S_0 - D_0^{L_0} > S \), the supplier can completely fill all the retailers’ orders in period \( t \). When \( S_0 - D_0^{L_0} < S \), the supplier will use the above mentioned rationing policy.

So, retailer \( i \)'s expected cost can be denoted by

\[
U_i(S_1, \ldots, S_n, S_0) = E\left\{ G_i[S_i - q_i(S - S_0 + D_0^{L_0})^+] \right\} = G_i(S_i) F_0^{L_0}(S_0 - S) + \int_{S_0 - S}^{\infty} G_i(S_i - q_i(S - S_0 + x)) f_0^{L_0}(x) \, dx.
\]

So retailer \( i \)'s problem is to decide its own order-up-to-level \( S_i \), that minimizes the expected cost:

\[
\min_{S_i} U_i. \tag{3.3}
\]

The derivative of the cost function can be obtained:

\[
\frac{\partial U_i}{\partial S_i} = G_i(S_i) F_0^{L_0}(S_0 - S) + (1 - q_i) \int_{S_0 - S}^{\infty} G_i(S_i - q_i(S - S_0 + x)) f_0^{L_0}(x) \, dx. \tag{3.4}
\]

However, we could not prove that the cost function \( U_i \) of retailer \( i \) is convex with respect to \( S_i \). It is only convex under some special conditions. Retailer \( i \)'s reaction is very complicated in non-cooperative games or cooperative games, and it is more difficult therefore to design the contract in the echelon inventory games.

### 3.1.2. Supplier’s model

The expected cost of the supplier, which is composed of on-hand and in-transit inventory holding cost, and a part of the total backorder penalty cost at the lower echelon, can be written as

\[
U_0(S_1, \ldots, S_n, S_0) = \sum h_0 \mu_i^{L_0} + h_0 E[\delta_0 - D_0^{L_0}]^+ + \sum E\left\{ B_i[S_i - q_i(S - S_0 + D_0^{L_0})^+] \right\}.
\]

The first term above is the expected holding cost for the units in-transit to the retailers (from Little's Law) (Cachon and Zipkin, 1999). The second term is the expected holding cost for inventory held at the supplier. The final term is the expected backorder cost charged to the supplier, where

\[
B_i(y) = (1 - z) p E[y - D_i^{L_1 + 1}]^+ = (1 - z) p \int_y^{\infty} (x - y) f_i^{L_1 + 1}(x) \, dx
\]

is retailer \( i \)'s backorder cost charged to the supplier at period \( t + L_1 \).

We can obtain

\[
U_0(S_1, \ldots, S_n, S_0) = \sum h_0 \mu_i^{L_0} + h_0 \int_{S_0 - S}^{\infty} (S_0 - S - x) f_0^{L_0}(x) \, dx \\
+ \sum \left\{ \int_0^{S_0 - S} B_i(S_i) f_0^{L_0}(x) \, dx + \int_{S_0 - S}^{\infty} B_i[S_i - q_i(S - S_0 + x)] f_0^{L_0}(x) \, dx \right\} \\
= \sum h_0 \mu_i^{L_0} + h_0 \int_{S_0 - S}^{\infty} (S_0 - S - x) f_0^{L_0}(x) \, dx \\
+ \sum \left\{ B_i(S_i) F_0^{L_0}(S_0 - S) + \int_{S_0 - S}^{\infty} B_i[S_i - q_i(S - S_0 + x)] f_0^{L_0}(x) \, dx \right\}. \tag{3.5}
\]

The supplier’s problem is to minimize his total cost:

\[
\min_{S_0} U_0(S_1, \ldots, S_n, S_0).
\]
Theorem 1. $U_0$ is convex with respect to $S_0$.

Proof. According to the supplier’s expected cost function, its derivative can be obtained as follows:

$$
\frac{\partial U_0}{\partial S_0} = h_0 f_0^{L_0}(S_0 - S) + \sum B_i(S_i) f_0^{L_0}(S_0 - S) + \sum q_i \int_{S_0 - S}^{\infty} B'_i(S_i - q_i(S - S_0 + x)) f_0^{L_0}(x) \, dx
$$

$$
= h_0 f_0^{L_0}(S_0 - S) + \sum q_i \int_{S_0 - S}^{\infty} B'_i(S_i - q_i(S - S_0 + x)) f_0^{L_0}(x) \, dx,
$$

(3.6)

$$
\frac{\partial^2 U_0}{\partial S_0^2} = h_0 f_0^{L_0}(S_0 - S) - \sum q_i B'_i(S_i) f_0^{L_0}(S_0 - S) + \sum q_i^2 \int_{S_0 - S}^{\infty} B''_i(S_i - q_i(S - S_0 + x)) f_0^{L_0}(x) \, dx,
$$

:** $B'_i(y) = -(1 - \alpha)p \int_{y}^{\infty} f_i^{L_{i+1}}(x) dx = -(1 - \alpha)p[1 - F_i^{L_{i+1}}(y)] \leq 0,$

and

$$
B''_i(y) = (1 - \alpha)p f_i^{L_{i+1}}(y) \geq 0,
$$

:** \[ \frac{\partial^2 U_0}{\partial S_0^2} \geq 0. \]

So given all the retailers’ actions, the supplier’s optimal echelon inventory level $S_0$ satisfies

$$
\frac{\partial U_0}{\partial S_0} = 0.
$$

Lemma 1. If the supply from the supplier is sufficient, the system has a unique non-cooperative solution.

Proof. The result can be obtained easily according to the Existence Theorem of Nash Equilibria (Myerson, 1990; Fudenberg and Tirole, 1991). □

3.2. Local inventory games

If supply chain members track their inventory by local stock policy, the system behaviour will be slightly different. A firm’s inventory level is equal to its inventory position plus its in-transit inventory, and its echelon inventory level is its local inventory level plus all inventory level held lower in the supply chain. We suppose the customer demands are stationary. All retailers have no lower members, so the following conditions hold:

$$
S_i = \bar{S}_i, \quad i = 1, 2, \ldots, n, \quad \text{and} \quad S_0 = \bar{S}_0 + \sum S_i = \bar{S}_0 + \sum \bar{S}_i.
$$

Under this rationing policy, we can derive the following results from Section 3.1.

3.2.1. Retailers’ model

(1) Sufficient supply from the supplier: If the supply from the supplier is sufficient, the retailers’ decision functions can be written as
\[ G_i(\bar{S}_i) = G_i(S_i). \]

That has the same result as in (3.2).

2. Insufficient supply from the supplier: We can easily see that retailer \( i \)'s decision function is

\[
U_i(S_1, \ldots, S_i, \ldots, S_n, \bar{S}_0) = U_i(\bar{S}_1, \ldots, \bar{S}_i, \ldots, \bar{S}_n, \bar{S}_0 + \sum \bar{S}_i)
\]

\[
= G_i(\bar{S}_i) F_0^l(\bar{S}_0) + \int_{\bar{S}_0}^{\infty} G_i(\bar{S}_i - q_i(x - \bar{S}_0)) f_0^l(x) \, dx.
\]

**Theorem 2.** \( U_i \) is convex with respect to \( \bar{S}_i \).

**Proof**

\[
\frac{\partial U_i}{\partial \bar{S}_i} = G_i'(\bar{S}_i) F_0^l(\bar{S}_0) + \int_{\bar{S}_0}^{\infty} G_i'(\bar{S}_i - q_i(x - \bar{S}_0)) f_0^l(x) \, dx,
\]

\[
\frac{\partial^2 U_i}{\partial \bar{S}_i^2} = G_i''(\bar{S}_i) F_0^l(\bar{S}_0) + \int_{\bar{S}_0}^{\infty} G_i''(\bar{S}_i - q_i(x - \bar{S}_0)) f_0^l(x) \, dx,
\]

\[ \therefore G_i''(\bar{S}_i) = (h_i + h_0 + zp) f_0^l(\bar{S}_i) \geq 0, \]

\[ \therefore \frac{\partial^2 U_i}{\partial \bar{S}_i^2} \geq 0. \quad \square \]

3.2.2. Supplier’s model

From (3.5) we can easily derive:

\[
U_0(S_1, \ldots, S_i, \ldots, S_n, \bar{S}_0) = U_0(\bar{S}_1, \ldots, \bar{S}_i, \ldots, \bar{S}_n, \bar{S}_0)
\]

\[
= U_0(\bar{S}_1, \ldots, \bar{S}_i, \ldots, \bar{S}_n, \bar{S}_0 + \sum \bar{S}_i)
\]

\[
= \sum h_0 \mu_i^l + h_0 \int_{0}^{\bar{S}_0} (\bar{S}_0 - x) f_0^l(x) \, dx
\]

\[
+ \sum \left[ B_i(\bar{S}_i) F_0^l(\bar{S}_0) + \int_{\bar{S}_0}^{\infty} B_i(\bar{S}_i - q_i(x - \bar{S}_0)) f_0^l(x) \, dx \right].
\]

**Lemma 2.** \( U_0 \) is convex with respect to \( \bar{S}_0 \).

**Proof.** The proof is similar to Theorem 1. \( \square \)

**Theorem 3.** If \( f_i^l(x) > 0, f_0^l(x) > 0 \) when \( x > 0 \) and \( q_i > 0, i = 1, 2, \ldots, n \), the system has a unique Nash equilibrium point.

**Proof.** See Appendix A. \( \square \)
4. System optimal solution

In Section 3, we have analysed the non-cooperative solution. Because the supplier and retailers are selfish profit maximizing centers, the sum of their own maximum profits is not the optimal solution of the whole system. If the whole system has an optimal solution, which minimizes the total average cost per period, and this will be the objective of cooperation. Diks and de Kok (1999) demonstrated an echelon based stock policy \( (S'_1, \ldots, S'_i, \ldots, S'_n, S'_0) \) is optimal in this setting:

\[
h_i - (h_i + h_0 + p) \left[1 - x_i'(S'_i)\right] = 0,
\]

and

\[
h_0 + \sum_i q_i [h_i - (h_0 + h_i + p)(1 - x_i'(S'_i))] = 0,
\]

where

\[
x_i'(S'_i) = F_i^{L+1}(S'_i) \quad \text{and} \quad x_i'(S'_0) = \begin{cases} \int_0^\infty F_i^{L+1}[Z_i(S'_0 - x)] \, dF_0^{L+1}(x) & S_0 < S, \\ \int_0^\infty F_i^{L+1}[Z_i(S - x)] \, d(F_0^{L+1})_{S=S_0}^{S_0} \, dF_0^{L+1}(x) & S_0 \geq S, \end{cases}
\]

Further, the retailer \( i \)'s optimal echelon base stock level \( S'_i \) satisfies

\[
F_i^{L+1}(S'_i) = \frac{h_0 + p}{h_i + h_0 + p}.
\]

At the same time, the supplier's optimal echelon base stock level \( S'_0 \) satisfies

\[
h_0 - (h_0 + p) \left[1 - F_0^{L+1}(S'_0 - \sum_i S'_i)\right] + \sum_i q_i (h_i + h_0 + p) \int_{S'_i - \sum_i S'_i}^{\infty} F_i^{L+1} \left( S'_i - q_i \left( \sum_i S'_i - S'_0 + x \right) \right) \, dF_0^{L+1}(x) = 0.
\]

If supply chain members are tracking their stock by local inventory policy, the optimal solution \( (S'_1, \ldots, S'_i, \ldots, S'_n, S'_0) \) satisfies: \( S'_i = S'_i, i = 1, 2, \ldots, n \) and \( S'_0 = S'_0 + \sum S'_i \).

5. Cooperative mechanism design

In order to achieve the optimal cooperative solution, supply chain members must sign some contracts to diminish the incentives of the selfish profit maximizing motivation of partners. Below we consider a kind of contract that is based on transferable subsidy.

In echelon inventory games, the members' decision functions under contract are:

For the retailers, \( \bar{U}_i = U_i + T_i \), where \( i = 1, 2, \ldots, n \); for the supplier, \( \bar{U}_0 = U_0 + T_0 \).

In local inventory games, the members' decision functions under contracts are:

For the retailers, \( \bar{U}_i = \bar{U}_i + T_i \), where \( i = 1, 2, \ldots, n \); for the supplier: \( \bar{U}_0 = \bar{U}_0 + T_0 \).

In both situations \( \sum T_i + T_0 = 0 \) holds.
5.1. Contracts in echelon inventory games

One of the objectives of contract design is to obtain the Nash solution. For the rational supplier and retailers, if there exists a Nash solution, they could get to the consistent solution. Because the cost function of retailer \( i \) is not convex with \( S_i \) in echelon inventory games, we have to choose a subsidy function of retailer \( i \) that is non-linear in its decision variable in order to guarantee that a Nash solution can be obtained.

(1) Each retailer has to pay an extra subsidy, and retailer \( i \) will pay \( q_i r_i(S_i)^2 \), where \( r_i \geq \max\left[f_0^{\ell_0}(x) \cdot (h_i + h_0)/2, x \geq 0\right] \).

(2) Each member has to bear a subsidy which is linear in his decision variable, for example supplier will pay \( k_0 S_0 \), retailer \( i \) pay \( k_i S_i \), where \( k_0 \) and \( k_i \) satisfy \( \partial \tilde{U}_i / \partial S_i \mid_{S_i = S'_i, S_0 = S'_0} = 0 \) and \( \partial \tilde{U}_0 / \partial S_0 \mid_{S_i = S'_i, S_0 = S'_0} = 0 \), where \( i = 1, 2, \ldots, n \).

(3) Constants \( C_i, i = 1, 2, \ldots, n \), assign the member’s shares of the jointly optimal solution-cooperation profits. The \( C_i \) can be decided by negotiation of all supply chain members, until \( \tilde{U}_0(S'_1, \ldots, S'_i, \ldots, S'_n, S'_0) \) and \( \tilde{U}_i(S'_1, \ldots, S'_i, \ldots, S'_n, S'_0) \) are accepted by all members.

So

\[
T_i = k_i S_i + q_i r_i(S_i)^2 - q_i k_0 S_0 + C_i, \quad i = 1, 2, \ldots, n, \quad \text{and} \quad T_0 = -\sum [k_i S_i + r_i q_i(S_i)^2] + k_0 S_0 - \sum C_i.
\]

**Theorem 4.** Under the above contract, retailer \( i \)'s cost function is convex with respect to \( S_i \).

**Proof.** Under the contract, retailer \( i \)'s cost function becomes

\[
\tilde{U}_i = U_i + T_i = U_i + k_i S_i + q_i r_i(S_i)^2 - q_i k_0 S_0 + C_i.
\]

So

\[
\begin{align*}
\frac{\partial^2 \tilde{U}_i}{\partial S_i^2} &= G'_i(S_i)F_0^{\ell_0}(S_0 - S) + (1 - q_i) \int_{S_0 - S}^{\infty} G''_i(S_i - q_i(S - S_0 + x))f_0^{\ell_0}(x) \, dx + 2q_i \frac{r_i}{S_i}, \\
&= G''_i(S_i)F_0^{\ell_0}(S_0 - S) + 2q_i \frac{r_i}{S_i} + \left[2r_i - G'_i(S_i)F_0^{\ell_0}(S_0 - S)\right] \geq 0.
\end{align*}
\]

Note

\[
G'_i(S_i) = (h_i + h_0 + x)F_i^{\ell_0 + 1}(S_i) - ax \leq h_i + h_0 \quad \text{and} \quad G''_i(S_i) = (h_i + h_0 + x)F_i^{\ell_0 + 1}(S_i) \geq 0, \quad r_i \geq \max \left[f_0^{\ell_0}(x) \cdot (h_i + h_0)/2\right].
\]

It’s easy to verify the supplier’s cost function remains convex under the contract because of its linearity:

\[
\frac{\partial^2 \tilde{U}_0}{\partial S_0^2} = \frac{\partial U_0}{\partial S_0^2} + k_0,
\]

and \( \frac{\partial^2 \tilde{U}_0}{\partial S_0 S_i} = \frac{\partial^2 U_0}{\partial S_i S_0} \), where \( \tilde{U}_0 \) is supplier’s cost function under the contract.
We can also obtain the following result:

*Under the above contract, all the members’ decisions will make a unique Nash solution, which is exactly equal to \((S_1', \ldots, S_n', S_0')\).*

### 5.2. Local inventory games

Because the cost functions of retailers and supplier are convex with their decision variables, we could choose a linear subsidy function.

1. Each member has to bear a subsidy which is linear in his decision variable, e.g., supplier will pay \(k_0S_0\), retailer \(i\) pays \(k_iS_i\), where \(k_0\) and \(k_i\) satisfy \(\frac{\partial \bar{U}_i}{\partial S_i}|_{S_i=S'_i, S_0=S_0}=0\) and \(\frac{\partial \bar{U}_0}{\partial S_0}|_{S_i=S'_i, S_0=S_0}=0\), where \(i=1,2,\ldots,n\).

2. Constants \(C_i\), \(i=1,2,\ldots,n\), assign the member’s shares of the jointly optimal solution—cooperation profits. The \(C_i\) can be decided by negotiation of all supply chain members, until \(\bar{U}_0(S'_1, \ldots, S'_i, \ldots, S'_n, S'_0)\) and \(\bar{U}_i(S'_1, \ldots, S'_i, \ldots, S'_n, S'_0)\) are accepted by all members.

So

\[
T_i = k_iS_i \quad q_ik_0S_0 + C_i, \quad i=1,2,\ldots,n, \quad \text{and} \quad T_0 = -\sum k_iS_i + k_0S_0 - \sum C_i.
\]

Because of the linearity of the contract, we can obtain the following result:

*Under the above local inventory contract, all the members’ decisions will make a unique Nash solution, which is exactly equal to \((S_1', \ldots, S_n', S_0')\).*

### 6. Numerical study

We consider a supply chain system of one supplier and two retailers. The supplier and two retailers review their inventory level by a period of one week. One period demand of each retailer is normally distributed with mean 100 units and standard deviation 30. The supplier’s holding cost is \(h_0 = 1\) per week per unit, and \(h_0 + h_1 = 3(h_1 = 2)\) and \(h_0 + h_2 = 4(h_2 = 3)\) are the unit holding costs for retailer 1 and retailer 2 respectively. The backorder penalty is \(p = 4\) per week per unit, and \(z = 0.2\). The lead time of the supplier is \(L_0 = 1\) period, the retailers 1’s lead time is \(L_1 = 1\) period, and and retailer 2’s lead time is \(L_2 = 1\) period.

We can obtain system optimal solutions according to Section 4:

\[
q_1 = 0.5362, \quad q_2 = 0.4608, \quad S'_1 = 224.0, \quad S'_2 = 213.5, \quad S'_0 = 825.0.
\]

While in local inventory games:

\[
\bar{S}_1 = 224.0, \quad \bar{S}_2 = 213.5, \quad \bar{S}_0 = 387.5.
\]

To fulfill the system optimal solution, design the contracts as mentioned above respectively:

1. **For echelon inventory game**, and the supply from the supplier is insufficient,

\[
r_1 = 0.01, \quad r_2 = 0.014 \quad \text{and} \quad k_1 = -3.5867, \quad k_2 = -4.0548 \quad k_0 = 0.6244.
\]

For the supplier, he will make his decision to select \(S_0 = S'_0 = 855.0\) before contracting according to the first order condition \(q_0S_0 = 0\) (Eq. (3.6)) and select \(S_0 = S'_0 = 825.0\) after contracting according to the first order condition \(\bar{q}_0S_0 = 0\) (Eq. (5.2)). Fig. 2 shows the first order derivative curves of decision functions \(\frac{\partial \bar{U}_0}{\partial S_0}\) and \(\bar{U}_0/\partial S_0\). We assume the decisions of two retailers are \(\bar{S}_1 = S'_1 = 224, \quad S_2 = S'_2 = 213.5\). Before contracting, i.e. under \((S'_1, S'_2, S'_0)\), the costs of the supplier and whole system
are 317.79 and 502.47 respectively. After contracting, i.e. under $(S^1_J, S^2_J, S^0_J)$, the costs of the supplier and whole system are 327.12 and 494.84 respectively. Obviously, the cost of the supplier increases and the cost of the whole system decreases after contracting.

For the retailer 1, he will make his decision to select $S_1 = S^1_G = 167.5$ before contracting according to the first order condition $\partial U_i / \partial S_1 = 0$ (Eq. (3.4), note in this numerical example, the cost function $U_i$ is convex with respect to $S_1$, so its minimum point satisfies the first order condition.) and select $S_1 = S^1_J = 224$ after contracting according to the first order condition $\partial \tilde{U}_i / \partial S_1 = 0$ (Eq. (5.1)).

![Fig. 2. Supplier’s first order derivative curves of decision functions.](image)

![Fig. 3. Retailer 1’s first order derivative curves of decision functions.](image)
shows the first order derivative curves of decision functions $\partial U_1/\partial S_1$ and $\partial U_1/\partial S_1$. We assume that the decisions of another retailer and the supplier are $S_2 = S_G^2 = 213.5$, $S_0 = S_G^0 = 825.0$. Before contracting, i.e. under $(S_1^G, S_2^G, S_0^G)$, the costs of the retailer 1 and whole system are 47.72 and 575.12 respectively. After contracting, i.e. under $(S_1^J, S_2^J, S_0^J)$, the costs of the retailer 1 and whole system are 80.88 and 494.84 respectively. Obviously, the cost of the retailer 1 increases and the cost of the whole system decreases after contracting.

(2) For local inventory game,

$k_1 = -1.9142$, $k_2 = -2.2001$, $k_0 = 0.6244$.

Fig. 4 shows the reaction curves of the supplier and one of the two retailers before contracting and after contracting under local inventory game. Note in this numeric study, we obtain the non-cooperative Nash equilibrium $(S_1^G, S_2^G, S_0^G)$, where $S_1^G = 169.7$, $S_2^G = 162.3$, $S_0^G = 438.1$, before the contract is designed. The system optimal solution $(S_1^J, S_2^J, S_0^J)$, where $S_1^J = 224.0$, $S_2^J = 213.5$, $S_0^J = 387.5$ is a new Nash equilibrium under the contract.

The supply chain cooperation will save up to 129.66. If this total saving is distributed to the supplier as 65%, retailer 1 as 20% and retailer 2 as 15%, then $C_1 = 151.52$, $C_2 = 170.34$. The contract will be as follows:

<table>
<thead>
<tr>
<th></th>
<th>Costs at non-cooperative Nash equilibrium $(S_1^G, S_2^G, S_0^G)$</th>
<th>Costs at system optimal solution $(S_1^J, S_2^J, S_0^J)$</th>
<th>Costs if at system optimal solution $(S_1^J, S_2^J, S_0^J)$ after contracting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retailer 1</td>
<td>47.92</td>
<td>80.88</td>
<td>21.99</td>
</tr>
<tr>
<td>Retailer 2</td>
<td>52.03</td>
<td>86.84</td>
<td>32.58</td>
</tr>
<tr>
<td>Supplier</td>
<td>524.55</td>
<td>327.12</td>
<td>440.27</td>
</tr>
<tr>
<td>Whole system</td>
<td>624.50</td>
<td>494.84</td>
<td>494.84</td>
</tr>
</tbody>
</table>
\[ T_1 = k_1S_1 - q_1k_0S_0 + C_1 = -1.3714S_1 - 0.3348S_0 + 151.52, \]
\[ T_2 = k_2S_2 - q_2k_0S_0 + C_2 = -1.5967S_2 - 0.2896S_0 + 170.34, \]
\[ T_0 = -\sum k_3S_3 + k_0S_0 - \sum C_k = 1.3714S_1 + 1.5967S_2 + 0.6244S_0 - 321.86. \]

The costs of supplier, retailers and whole system before contracting and after contracting can be shown in Table 1.

### 7. Conclusion and suggestions for future research

In this paper, we have analysed the non-cooperative behaviour in a two-echelon decentralized supply chain. In a decentralized supply chain, all partners’ decision must be carefully coordinated to achieve the system optimal solution. When all partners are independent profit centers, their behaviour must be carefully studied and formal contracts must be assigned between all partners to diminish the derivation from system optimal solution to selfish motivations. Therefore, the partners’ decision model must be studied carefully, including their information, the attitude to risk (risk averse or neutral etc). Generally, all these things are very complicated to be analysed.

We try to analyze the system under two conditions, sufficient supply and insufficient supply from the supplier. If the supply from the supplier is sufficient, and the retailers make their base stock level decision using the approximate model that does not consider the suppliers’ reaction, we can obtain the similar results with that of Cachon and Zipkin’s model. If the supply from the supplier is insufficient, we can obtain the much more complicated non-cooperative behaviour of the system, competition will occur between all the retailers as well as the supplier. The retailers may exaggerate their base stock level in order to get more rational stocks especially when the supplier has scarce sources. In local inventory games, partners will always have several Nash equilibrium contracts that are designed to guarantee the system optimal cooperation according to all models respectively. In general, the contracts may be of many kinds, for example, the well-known discount contracts etc. However, we do not use discount contract in our paper, for they are too sensitive to parameters derivatives i.e. are not robust contracts. So we use continuous incentive contracts, which mostly are linear to decision variables.

We have supposed that the system has full information. However, if there exists asymmetric information, the behaviour will be much more complicated (Gavirneni et al., 1999). There are two kinds of private information problems: moral hazard and inverse selection. If supply chain partners’ actions are very hard to observe, partners will hide their actions, which may not be system optimal. This situation is usually called ‘moral hazard’. While partners’ some properties, for example, holding cost etc., are private information, and these information are important to calculate the system optimal solution, partners maybe lying about the information to get extra profits. This situation is called ‘inverse selection’. In both situations, the owner of private information will get more profits sharing (information value). If the contracts are designed by one of the supply chain members, he will maximize his own profit but not system optimal profit, and will tradeoff between the system optimal profit gain and the information incentive cost. In our paper, the contracts are based on profit sharing scheme, which has been decided by all members (probably through negotiation process) and have several advantages than the former contract designed by one powerful member. We show the effect on costs of optimal negotiated contracts. Will our profit-sharing-based contracts still be effective under asymmetric information? The validity analysis of our contracts is our next study direction.

Our contracts are designed according to Nash equilibrium. However all partners may make their decision according to other equilibria, for example, mixed strategy Nash equilibrium or Bayesian Nash equilibrium etc. In these situations, the cooperative contracts must be redesigned.
Our models should be extended to more general supply chain networks, for example, multi-echelon divergent or assembly supply chains. However, these will be a great challenge. More partners, non-cooperative reactions, as well as coalition formation problems will make the system much more complicated.

Acknowledgements

The research is jointly supported by the National Natural Science Foundation of China (no. 70171015), the Excellent Young Teachers Fund, Foundation for University Key Teacher by the Ministry of Education, and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of MOE, PR China.

Appendix A. Proof of Theorem 3

From $\partial U_i / \partial S_i$, we know retailer $i$’s decision $S_i$ only depends on $S_0$ and does not depend on other $S_k$, $k \neq i$, so his reaction function may be written as $S_i = r_i(S_0)$. $S_i$ is obtained from $y_i(S_i, S_0) = \partial U_i / \partial S_i = 0$.

Supplier’s decision $S_0$ is associated with all of the retailers’ reaction $(S_1, S_2, \ldots, S_i, \ldots, S_n)$ and his reaction function can be written as $S_0 = r_0(S_1, S_2, \ldots, S_i, \ldots, S_n)$, where $S_0$ is obtained from $y_0(S_0, S_1, S_2, \ldots, S_i, \ldots, S_n) = \partial U_0 / \partial S_0 = 0$.

Suppose there exist two Nash points A: $(S_0^A, S_1^A, S_2^A, \ldots, S_i^A, \ldots, S_n^A)$, and B: $(S_0^B, S_1^B, S_2^B, \ldots, S_i^B, \ldots, S_n^B)$. In $S_1, S_2, \ldots, S_i, \ldots, S_n$, at least one parameter is not equal. For if all $S_1, S_2, \ldots, S_i, \ldots, S_n$ are equal, we can see $S_i^d = S_i^f$ from $S_0 = r_0(S_1, S_2, \ldots, S_i, \ldots, S_n)$, then A and B are the same point.

Without losing generality, we suppose $S_i^d < S_i^f$. Consider curve $S_i = r_i(S_0)$ in plane $(S_i, S_0)$. From the Implicit Theorem:

$$\frac{dr_i}{dS_0} = \frac{\partial y_i}{\partial S_0} - \frac{\partial^2 U_i}{\partial S_i \partial S_0} = -q_i \int_{S_0}^{\infty} G_i'() f_i^{L_0}(x) dx + \int_{S_0}^{\infty} G_i'() f_i^{L_0}(x) dx,$$

where $(\cdot)$ denote for $(S_i - q_i(x - S_0))$ or $(S_k - q_k(x - S_0))$ for simplicity.

For $f_i^{L_0}(x) > 0$ and $G_i'(x) > 0$, $d r_i / d S_0 \in (-q_i, 0)$, curve $r_i$ is strictly decreasing with $S_0$, so its inverse function can be written as $S_0 = r_i^{-1}(S_i)$.

We denote $m_i(S_1, S_2, \ldots, S_i, \ldots, S_n) = r_0(S_1, S_2, \ldots, S_i, \ldots, S_n) - r_i^{-1}(S_i)$, then

$$\frac{\partial m_i}{\partial S_i} = \frac{\partial y_i}{\partial S_0} + \frac{\partial y_i}{\partial S_0} = \left( -\frac{\partial y_i}{\partial S_0} \frac{\partial y_i}{\partial S_0} + \frac{\partial y_i}{\partial S_i} \frac{\partial y_i}{\partial S_0} \right) \left( \frac{\partial y_i}{\partial S_0} \frac{\partial y_i}{\partial S_0} \right),$$

where

$$\frac{\partial y_i}{\partial S_0} = \frac{\partial^2 U_i}{\partial S_i \partial S_0} = q_i \int_{S_0}^{\infty} G_i'() f_i^{L_0}(x) dx > 0, \quad \frac{\partial y_i}{\partial S_0} = \frac{\partial^2 U_i}{\partial S_0^2} > 0,$$

so

$$\frac{\partial y_i}{\partial S_0} \frac{\partial y_i}{\partial S_0} > 0.$$
From

\[
\frac{\partial y_0}{\partial S_i} = \frac{\partial^2 U_0}{\partial S_0 \partial S_i} = q_i \int_{S_0}^{\infty} B''_i(y) f_{0,0}^L(y) \, dy > 0,
\]

\[
\frac{\partial y_i}{\partial S_i} = \frac{\partial^2 U_i}{\partial S_i^2},
\]

and \( B'_k(y) < 0 \) or \( B'_k(y) > 0 \) or \( B''_k(y) > 0 \),

\[
- \frac{\partial y_0}{\partial S_i} \frac{\partial y_i}{\partial S_j} + \frac{\partial y_j}{\partial S_i} \frac{\partial y_i}{\partial S_j} = -q_i^2 \int_{S_0}^{\infty} B''_i(y) f_{0,0}^L(y) \, dy \cdot \int_{S_0}^{\infty} G''_i(y) f_{0,0}^L(y) \, dy + \left[ G''_i(S_i) F_{0,0}^L(S_0) + \int_{S_0}^{\infty} G''_i(y) f_{0,0}^L(y) \, dy \right]
\]

\[
\quad \cdot \left[ h_0 f_{0,0}^L(S_0) - \sum q_k B'_k(S_k) f_{0,0}^L(S_0) + \int_{S_0}^{\infty} B'_k(y) f_{0,0}^L(y) \, dy \right]
\]

\[
= \left[ G''_i(S_i) F_{0,0}^L(S_0) + \int_{S_0}^{\infty} G''_i(y) f_{0,0}^L(y) \, dy \right] \cdot \left[ h_0 f_{0,0}^L(S_0) - \sum q_k B'_k(S_k) f_{0,0}^L(S_0) \right]
\]

\[
+ G''_i(S_i) F_{0,0}^L(S_0) \cdot \sum q_k^2 \int_{S_0}^{\infty} B''_k(y) f_{0,0}^L(y) \, dy + \int_{S_0}^{\infty} G''_i(y) f_{0,0}^L(y) \, dy \cdot \sum_{k \neq i} q_k^2 \int_{S_0}^{\infty} B''_k(y) f_{0,0}^L(y) \, dy > 0.
\]

So \( m_i \) is strictly increasing with \( S_i \).

Since \( A \) is a Nash point, \( m_i(S_i^d, S_2^d, \ldots, S_n^d) = 0 \) will hold, but \( S_i^d < S_i^g \), then \( m_i(S_i^d, S_2^g, \ldots, S_n^g) > 0 \), this shows that \( B \) is not a Nash point. \( \square \)

References


