GENERALIZED SHUFFLE-EXCHANGE NETWORKS: COMBINATORIAL AND FAULT TOLERANT PROPERTIES

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Abstract. Shuffle-exchange networks have been proposed as an attractive choice for interconnection networks. They have constant node degree and sublogarithmic diameter. Several researchers have studied various combinatorial and interconnection properties of them. In this paper, we generalize shuffle-exchange networks and define a new class of networks, called generalized shuffle-exchange network and denoted as GS(k, N), where k is the node degree and N is the number of nodes. We study various combinatorial and interconnection properties of GS(k, N) such as diameter, wide-diameter, connectivity, embedding property, and self-routing property. We also study fault tolerant properties of shuffle-exchange networks and propose a modified version of shuffle-exchange networks that improve some of the properties of shuffle-exchange networks.


1 Introduction

Experience on using parallel computers to solve various problems in the past decades indicates that the ultimate utilization of parallel computers is heavily dependent on the topology of the interconnection network that connects the processors. Due to the advent of VLSI and fiber optic technology, it is not only feasible but also practical to design large scale communication networks employing hundreds or thousands of computer nodes interconnected by routers and switches. Interconnection networks play an important role in parallel architecture, communication networks and VLSI design [15;19;23].

A number of networks have been proposed including linear array and ring [12], tree [23], hypercubes [8], de Bruijn networks [15;21], shuffle-exchange networks [15;22], butterfly network [13], cube-connected cycles [18], star networks [1], and some hypercube-based networks [8]. Architectures based on these networks have been built in industry, research, laboratories and academic institutions. We have defined a new class of networks GS(k, N), called generalized shuffle-exchange networks, which generalizes n-dimensional k-ary
shuffle-exchange network $S(k, n)$. Definitions of $S(k, n)$ and $GS(k, N)$ are provided in the following.

**Definition 1.1.** An $n$-dimensional $k$-ary shuffle-exchange network $S(k, n)$ has $N = k^n$ nodes and $k^{n+1}$ edges. Each node corresponds to a unique $n$-dimensional $k$-ary number, and there exists an edge from node $u$ to node $v$ if either

- $v$ is a left cyclic shift of $u$ (shuffle edge) or
- $u$ and $v$ differ in precisely the last position (exchange edge).

That is, if $u = u_1u_2 \cdots u_n$, there are $k-1$ exchange edges from $u$ to nodes that differ from $u$ in the last position ($i.e.$, $u_1u_2 \cdots u_{n-1}x$, where $x \in \{0, 1, 2, \cdots, k-1\}$, $x \neq u_n$). Also, there is a shuffle edge from $u$ to the node $u_2u_3 \cdots u_nu_1$.

$S(k, n)$ is a $k$-regular network with self-loop existing at any node which has the same digit at all $n$ positions.

The shuffle-exchange network $S(k, n)$ has been widely used as topologies for VLSI networks, parallel architectures, and communication systems. However, since $S(k, n)$ does not exist for the number of nodes between $k^n$ and $k^{n+1}$. We have proposed a class of digraphs $GS(k, N)$, which generalized $S(k, n)$ to any number $N$ of nodes.

Let $\lfloor x \rfloor$ be the largest integer which is smaller than or equal to $x$, and let $\lceil x \rceil$ be the smallest integer which is larger than or equal to $x$. Let $k$ and $N$ be positive integers $k < N$ and $p = \lceil N/k \rceil$.

**Definition 1.2.** The $N$ nodes generalized shuffle-exchange network $GS(k, N)$ is the digraph whose edges are the shuffle edges

$$
x \rightarrow \left\{ \begin{array}{ll}
  kx \pmod{N}, & \text{if } 0 \leq x < p \\
  \vdots & \\
  kx + i \pmod{N}, & \text{if } ip \leq x < (i+1)p \\
  \vdots & \\
  kx + k - 1 \pmod{N}, & \text{if } (k-1)p \leq x < N
\end{array} \right.
$$

and the exchange edges

$$
x \rightarrow x + i, \quad \text{where } x = b \pmod{k} \text{ and } i \in [-b, k-1-b] \setminus \{0\}.
$$

Figure 1.1 shows the 15-node 3-ary generalized shuffle-exchange network $GS(3, 15)$. When $k|N$, the digraph $GS(k, N)$ is a $k$-regular digraph that uses $k$-complete symmetric digraph as a building block. When $N = k^n$, it is easy to see that $GS(k, k^n)$ is $S(k, n)$. Obviously, $S(k-1, n)$ is a subgraph of $S(k, n)$. But the structure of $S(k, n)$ can not be constructed recursively from $S(k-1, n)$. Also $S(k, n)$ can not be constructed from $S(k, n-1)$ recursively.
Figure 1.1: The 15-node 3-ary generalized shuffle-exchange digraph $GS(3,15)$.

Shuffle-exchange networks are often studied as related to other networks, notably de Bruijn networks or butterfly networks etc [9]. The binary shuffle-exchange networks $S(2,n)$ have been studied extensively (see the excellent treatment contained in Leighton’s book [15]). Additionally, Feldmann and Mysliwietz [7] have proved that undirected binary shuffle-exchange network $US(2,n)$ contains a Hamiltonian path. In a previous paper [16], we have studied Hamiltonian property of $GS(k,N)$. In this paper, we study other combinatorial properties of $S(k,n)$ and $GS(k,N)$ including connectivity, diameter, fault tolerant and self-routing property. Section 2 provides background material for wide-diameter, fault-diameter and Rabin number of graphs. Section 3 presents our results on $S(k,n)$ and $GS(k,N)$. In Section 4 deals with the modified shuffle-exchange digraph $MS(k,n)$. We then conclude the paper with a discussion in Section 5.

2 Wide-Diameter, Fault-Diameter and Rabin Number of Graphs

The advent of VLSI circuit technology has enabled the construction of very complex and large interconnection networks for massively parallel processing system. Due to fiber optic technology, the demand for more services on local-area and wide-area networks is increasing tremendously. Consequently, the networks we face today are in the range of thousands or tens of thou-

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sands of nodes for a massively parallel computing system and communication networks. There are two issues needed to be addressed:

(a) How do these large number of processors communicate with each other;

(b) How do they communicate in the case of node or edge faults.

These issues are related to the connectivity and diameter of the network. Connectivity is widely used to measure network fault-tolerance capacities, while diameter determines routing efficiency along individual paths. In practice, we are interested in high-connectivity, small-diameter networks.

The connectivity $k(G)$ of a network $G$ is the minimum number of nodes whose removal results in a disconnected or trivial network. The line connectivity $\lambda(G)$ of a network $G$ is the minimum number of edges whose removal results in a disconnected or trivial network. According to Menger’s theorem (see [10]), there are $k$ node-disjoint paths from a node $x$ to another node $y$ in a network of connectivity $k$ and there are $\lambda$ edge-disjoint paths from a node $x$ to another node $y$ in a network of line connectivity $\lambda$.

In [10], Hsu introduced the notion of $w$-wide diameter, which unifies the concepts of diameter and connectivity. In a network $G$, a container from a node $x$ to another node $y$ is a set $C(x, y)$ of (internally) node-disjoint $x$-$y$ (di)paths. The width $w(C(x, y))$ of a container $C(x, y)$ is the number of (di)paths in $C(x, y)$ and the length $l(C(x, y))$ of $C(x, y)$ is the largest length of a (di)path in $C(x, y)$. For $1 \leq w \leq k(G)$, the $w$-wide distance $d_w(x, y)$ from $x$ to $y$ is the minimum length among all containers $C(x, y)$ of width $w$. The $w$-wide diameter $d_w(G)$ of graph $G$ is the maximum $w$-wide distance from a node to another.

Clearly, when $w = 1$, $d_w(G) = d(G)$ which is the diameter of the graph $G$. Moreover, we have

**Theorem 2.1.** $d_1(G) \leq d_2(G) \leq \cdots \leq d_k(G)$ for any network $G$ of connectivity $k$.

The ($w - 1$)-fault distance from $x$ to $y$ in a network $G$ is

$$D_w(x, y) = \max\{d_{G-S}(x, y) : |S| = w - 1 \text{ and } x, y \notin S\}.$$  

The ($w - 1$)-fault diameter of $G$ is

$$D_w(G) = \max\{D_w(x, y) : x, y \in G\}.$$  

Also,

**Theorem 2.2.** $D_1(G) \leq D_2(G) \leq \cdots \leq D_k(G)$ for any network $G$ of connectivity $k$.

Give a graph $G$, it is easy to establish the following relation between $D_w(G)$ and $d_w(G)$:
Theorem 2.3. If $1 \leq w \leq k$, then $D_w(G) \leq d_w(G)$ for any network $G$ with connectivity $k$.

The $w$-Rabin number $r_w(G)$ of a network $G$ is the minimum $l$ such that for any $w + 1$ distinct nodes $x, y_1, y_2, \ldots, y_w$ there exist $w$ node-disjoint (except at $x$) paths with length of at most $l$ from $x$ to $y_1, y_2, \ldots, y_w$. This concept was first defined by Hsu [11]. The special case in which $w = k(G)$, the connectivity of the graph $G$, was studied by Rabin [18] in conjunction with the study of randomized routing algorithm.

Theorem 2.4. $r_1(G) \leq r_2(G) \leq \cdots \leq r_k(G)$ for any network $G$ with connectivity $k$.

It is easy to show that the following relation holds:

Theorem 2.5. If $1 \leq w \leq k$, then $D_w(G) \leq r_w(G)$ for any network $G$ with connectivity $k$.

3 Properties of Shuffle-exchange Networks and Generalized Shuffle-exchange Networks

Before we study the combinatorial properties of $GS(k, N)$, we recall definitions of de Bruijn networks [14] and generalized de Bruijn digraph $GD(k, N)$[4;5;6].

Definition 3.1. The $n$-dimensional $k$-ary de Bruijn network is a directed graph with $k^n$ nodes. Each node is labeled as an $n$-dimensional $k$-ary number $a_0a_1\cdots a_{n-1}$. A node $a_0a_1\cdots a_{n-1}$ is adjacent to $a_1a_2\cdots a_{n-1}a$, where $a \in [0, k - 1]$.

Definition 3.2. The generalized de Bruijn digraph $GD(k, N)$ has $N$ nodes and $k \cdot N$ edges. When we represent nodes by ordinary numbers, edges are

$$x \rightarrow kx + r \pmod{N} \quad \text{for } 0 \leq x \leq N - 1 \text{ and } 0 \leq r \leq k - 1.$$  

When $N = k^n, GD(k, k^n)$ is $D(k, n)$.

Remark 3.1. From Definition 1.2 and Definition 3.2, when $k|N$, we can see that if we map the set of nodes $\{kx + a : 0 \leq a \leq k - 1\}$ in $GS(k, N)$ to node $x$ in $GD(k, p)$, where $p = \lfloor N/k \rfloor$. Also if $0 \leq a \leq k - 1$, the shuffle edge $kx + a \rightarrow k(kx + a) + i \pmod{N}$ with $ip \leq kx + a < (i + 1)p$ in $GS(k, N)$ is mapped to an edge $x \rightarrow kx + a$ in $GD(k, p)$. All exchange edges are contracted out of $GS(k, N)$. These contracting process would transform $GS(k, N)$ to $GD(k, p)$. Reversely, we can expand $x$ in $GD(k, p)$ to $k$ nodes $\{kx + a : 0 \leq a \leq k - 1\}$ and connect these $k$ nodes with each other. If $0 \leq a \leq k - 1$, for an edge from $x$ to $kx + a$ in $GD(k, p)$, we have an edge $kx + a \rightarrow k(kx + a) + i \pmod{N}$ in $GS(k, N)$ if $ip \leq kx + a < (i + 1)p$. So we can also transform $GD(k, p)$ to $GS(k, N)$. We denote the $k$-clique that contains $\{kx + a : 0 \leq a \leq k - 1\}$ as $k$-clique $(x)$. 


Figure 3.1 shows the 5-node 3-ary generalized de Bruijn digraph GD(3,5). Together with Figure 1.1, it is easy to verify the above remark.

![Figure 3.1: The 5-node 3-ary general de Bruijn digraph GD(3,5).](image)

### 3.1 Combinatorial properties of $S(k, n)$ and $GS(k, N)$

**Theorem 3.1.** The diameter of $S(k, n)$ is $2n - 1$.

**Proof.** Consider any two nodes $u = u_1u_2\cdots u_n$ and $v = v_1v_2\cdots v_n$ in $S(k, n)$. We can construct a path from $u$ to $v$ with at most $2n - 1$ edges. The path begins with an exchange edge (if necessary)

$$u_1u_2\cdots u_{n-1}u_n \rightarrow u_1u_2\cdots u_{n-1}v_1.$$

Then we continue to convert $u_i$ to $v_{i+1}$ for each $i$ during the subsequent shuffle-exchange operations

$$u_i\cdots u_{n-1}v_1\cdots v_i \rightarrow u_{i+1}\cdots u_{n-1}v_1\cdots v_iu_i \rightarrow u_{i+1}\cdots u_{n-1}v_1\cdots v_{i+1}.$$

The resulting path has just $n - 1$ shuffle edges and at most $n$ exchange edges, for a total of at most $2n - 1$ edges overall. Moreover, we claim that any path from $00\cdots 0$ to $11\cdots 1$ must have at least $2n - 1$ edges. Indeed each 0 must
be changed to a 1 by an exchange edge, yields \( n \) different exchange edges; furthermore exchange edges are nonadjacent, so there must be at least \( n - 1 \) shuffle edges among them. So the diameter of \( S(k, n) \) is \( 2n - 1 \).

**Theorem 3.2.** The diameter of \( GS(k, N) \) is at most \( 2[\log_k N] + 1 \). When \( k|N \), the diameter of \( GS(k, N) \) is \( 2[\log_k N] - 1 \).

**Proof.** Let \( t = [\log_k N] \). From the definition of \( GS(k, N) \), we have the following shuffle edges:

\[
\begin{align*}
0 & \rightarrow 0 \pmod{N} \\
1 & \rightarrow k \pmod{N} \\
2 & \rightarrow 2k \pmod{N} \\
& \vdots \\
2p - 1 & \rightarrow ikp - k + i - 1 \pmod{N} \\
p & \rightarrow ikp + i \pmod{N} \\
2p & \rightarrow ikp + i \pmod{N} \\
& \vdots \\
N - 1 & \rightarrow N - 1 \pmod{N}.
\end{align*}
\]

Also every node has \( k - 1 \) exchange edges, that is

\[
ke + b \rightarrow ke + c \pmod{N}, \quad c \in \{0, 1, \ldots, k - 1\} \text{ and } c \neq b.
\]

Observe that for any node, consecutive \( t - 1 \) steps of one shuffle followed by one exchange cover all nodes except maybe \( kp - k, kp - k + 1, \ldots, N - 1 \). So after an exchange and \( i(1 \leq i \leq t - 2) \) alternative shuffles and exchanges, we visit \( \min(k^{i+1}, kp - k) \) different nodes. From node \( p - 1 \), we can visit \( kp - k, kp - k + 1, \ldots, N - 1 \) by a shuffle and then an exchange. So the diameter of \( GS(k, N) \) is at most \( 2t + 1 \).

When \( k|N \), for any node, consecutive \( t - 1 \) steps of one shuffle followed by one exchange cover all nodes. The diameter of \( GS(k, N) \) is at most \( 2t - 1 \). From node 0, after an exchange and \( i \) alternative shuffles and exchanges, we visit exactly all nodes that are less than \( k^{i+1} \), for \( i \leq t - 2 \). While none of nodes that are less than \( k^{t-1} \) are adjacent to node \( N - 1 \), we need at least \( t \) exchanges and \( t - 1 \) shuffles to connect node 0 with node \( N - 1 \). So when \( k|N \), the diameter of \( GS(k, N) \) is \( 2t - 1 \).

It is easy to show the following corollary:

**Corollary 3.3.** For any arbitrary given node \( u \) in the digraph \( GS(k, N) \) when \( k|N \), there is a balanced \( k \)-ary tree \( T_u \) of height \( 2t - 1 \) so that every node in \( GS(k, N) \) occurs as a leaf at level \( 2t - 1 \), where \( t = [\log_k N] \).

We now compute the connectivity of \( GS(k, N) \) for \( k|N \).

**Theorem 3.4.** When \( k|N \), the connectivity of \( GS(k, N) \) is \( k - 1 \).
Proof. Let $p = \lceil N/k \rceil$. For any two nodes $a = ka_1 + c_1$ and $b = kb_1 + c_2$, we can construct $k - 1$ disjoint paths from $a$ to $b$ in $GS(k, N)$ from $k - 1$ disjoint paths from $a_1$ to $b_1$ in $GD(k, p)$. Assume the paths from $a_1$ to $b_1$ are $P_1', \cdots, P_{k-1}'$, where
\[
P_i' = a_1 \pmod{p} \rightarrow ka_1 + x_1 \pmod{p} \rightarrow k(ka_1 + x_1) + x_2 \pmod{p} \rightarrow \cdots \rightarrow y \pmod{p} \rightarrow b_1 \pmod{p}.
\]
For each index $i$, construct $P_i$ in $GS(k, N)$ as follows:
\[
P_i = ka_1 + c_1 \pmod{N} \rightarrow ka_1 + x_1 \pmod{N} \rightarrow k(ka_1 + x_1) + [(ka_1 + x_1)/p] \pmod{N} \rightarrow k(ka_1 + x_1) + x_2 \pmod{N} \rightarrow \cdots \rightarrow kb_1 + x \pmod{N} \rightarrow kb_1 + c_2 \pmod{N}.
\]
Since $k|N$, if $a \not\equiv b \pmod{p}$, then $ka + c \not\equiv kb + d \pmod{N}$ for $0 \leq c, d \leq k - 1$. So $P_1, P_2, \cdots, P_{k-1}$ are disjoint. $P_i'$ may be a walk, but it contains a path. So we construct $k - 1$ disjoint paths for any two nodes in $GS(k, N)$. That is, the connectivity of $GS(k, N)$ when $k|N$, is $k - 1$. \qed

Self-loops reduce the maximum possible connectivity of $GS(k, N)$ to $k - 1$ and also may hurt the diameter. We want to compute the number of loops for $GS(k, N)$.

**Theorem 3.5.** Assume $N = kp - b$, where $0 \leq b \leq k - 1, p > b + 1, k \geq 2, \gcd(k - 1, p - b - 1) = g$. The number of self loops in $GS(k, N)$ is $g + 1$.

**Proof.** Assume $x = ip + a$, where $a \in [0, p - 1]$ and $i \in [0, k - 1]$. The exchange edges of $GS(k, N)$ can not be self-loops, so we need only to examine the shuffle edges. In $GS(k, N)$, the shuffle edges are
\[
x = ip + a \rightarrow k(ip + a) + i \pmod{N}.
\]
We want to determine for how many pairs $(i, a)$, the relation
\[
i p + a = k(ip + a) + i \pmod{N}.
\]
holds, i.e.,
\[
(p - b - 1)i - (k - 1)a = 0 \pmod{N}.
\]
Since $p \geq b + 1$ and $i \leq k - 1$, we have
\[
(p - b - 1)i - (k - 1)a < (k - 1)p - (k - 1)a < kp - b = N,
\]
and
\[
(p - b - 1)i - (k - 1)a > -(k - 1)a > -kp + b.
\]
So we have
\[
(p - b - 1)i - (k - 1)a = 0,
\]
that is,
\[
(k - 1)a = i(p - b - 1).
\]
When $\gcd(k - 1, p - b - 1) = g$, we have totally $g + 1$ loops. \qed
A network is said to have the self-routing property if for any pair of nodes \( u \) and \( v \), a path from \( u \) to \( v \) can be determined by knowing \( u \) and \( v \) only. Consider \( GS(k, N) \) where \( k \mid N \), and let \( t = \lceil \log_k N \rceil \). For given \( u \) and \( v \), let \((r_0, \cdots, r_{t-2})\) be the \( k \)-ary number representative of the residue \( v_0 - u_0 \cdot k^{t-1} \) (mod \( p \)), where \( v_0 = \lceil v/k \rceil \), \( u_0 = \lceil u/k \rceil \) and \( p = N/k \). There exists a walk whose length is at most \( 2t - 1 \) from \( u \) and \( v \), say,

\[
u = x_0 \rightarrow y_0 \rightarrow x_1 \rightarrow y_1 \rightarrow \cdots \rightarrow x_{t-2} \rightarrow y_{t-2} \rightarrow x_{t-1} \rightarrow v
\]
such that \( y_0 = k \cdot \lceil x_0/k \rceil + r_0 \), \( x_i = ky_{i-1} + y_{i-1}/p \) (mod \( N \)), \( y_i = ky_{i-1} + r_i \) (mod \( N \)).

**Theorem 3.6.** When \( k \mid N \), \( GS(k, N) \) has the self-routing property.

**Example 3.1.** In \( GS(3, 30) \), we construct a walk from node \( u = 5 \) to node \( v = 26 \). Here \( t = 4 \) and \( p = 10 \).

\[
v_0 = \lceil 26/3 \rceil = 9, \quad u_0 = \lceil 5/3 \rceil = 1.
\]

\[
v_0 - 1 \cdot k^{t-1} = 9 - 1 \cdot 3^3 = -19 = 1 \pmod{10}.
\]

So

\[
(r_0, r_1, r_2) = (0, 0, 1),
\]

and the walk is

\[
u = x_0 = 5 \rightarrow y_0 = k \lceil x_0/k \rceil = 3 \rightarrow x_1 = ky_0 + \lceil y_0/p \rceil = 9
\]

\[
\rightarrow y_1 = k^2 y_0 + r_1 = 9 \rightarrow x_2 = ky_1 + \lceil y_1/p \rceil = 27
\]

\[
\rightarrow y_2 = k^3 y_1 + r_2 = 28 \rightarrow x_3 = ky_2 + \lceil y_2/p \rceil = 24 \rightarrow v = 26.
\]

### 3.2 Fault Tolerant Properties of \( S(k, n) \)

We first prove the following theorem about the fault tolerant properties of the de Bruijn network \( D(k, n) \).

**Theorem 3.7.** For any \( s+1 \) nodes \( v_0, v_1, \cdots, v_s \) in \( D(k, n) \), where \( v_1, v_2, \cdots, v_s \) are distinct, but one of \( v_1, \cdots, v_s \) may be the same as \( v_0 \), and for any set \((k_1, \cdots, k_s)\) that satisfies \( k_i > 0 \) and \( \sum_{i=1}^s k_i = k - 1 \), there exist \( k - 1 \) node-disjoint paths (except at origin and terminus), each of them length at most \( n + 1 \), and with \( k_i \) paths going from \( v_0 \) to \( v_i \) (for \( 1 \leq i \leq s \)).

**Proof.** Induction: For \( n = 1 \), \( D(k, 1) \) is a complete directed graph with self-loops. For any \( s + 1 \) nodes, there are two cases:

**Case I.** The nodes are totally distinct. Assume they are 0, 1, 2, \cdots, \( s \). For any set \((k_1, \cdots, k_s)\) that satisfies \( k_i > 0 \) and \( \sum_{i=1}^s k_i = k - 1 \), we construct
\(k - 1\) node-disjoint paths as following:

\[
\begin{array}{c}
0 \to 1, \quad 0 \to s + j \quad \to 1 \quad (\text{for } 1 \leq j \leq k_1 - 1) \\
0 \to 2, \quad 0 \to s + k_1 - 1 + j \quad \to 2 \quad (\text{for } 1 \leq j \leq k_2 - 1) \\
\vdots \quad \vdots \quad \vdots \\
0 \to i, \quad 0 \to s + \sum_{l=1}^{i-1} (k_l - 1) + j \quad \to i \quad (\text{for } 1 \leq j \leq k_i - 1) \\
\vdots \quad \vdots \quad \vdots \\
0 \to s, \quad 0 \to s + \sum_{l=1}^{s-1} (k_l - 1) + j \quad \to s \quad (\text{for } 1 \leq j \leq k_s - 1)
\end{array}
\]

If \(k_i = 1\), there is no path like

\[0 \to s + \sum_{l=1}^{i-1} (k_l - 1) + j \to i, \quad 1 \leq j \leq k_i - 1.\]

These \(k - 1\) paths are node-disjoint, and the longest one has length \(n + 1 = 2\).

**Case II.** There is an index \(i \in \{1, \cdots, s\}\) such that \(v_i = v_0\). Assume the nodes are 0, 1, \(\cdots\), \(s - 1\). For any set \((k_1, \cdots, k_s)\) that satisfies \(k_i > 0\) and \(\sum_{i=1}^{s} k_i = k - 1\), we construct \(k - 1\) node-disjoint paths as following:

\[
\begin{array}{c}
0 \to 0, \quad 0 \to s - 1 + j \quad \to 0 \quad (\text{for } 1 \leq j \leq k_1 - 1) \\
0 \to 1, \quad 0 \to s - 1 + k_1 - 1 + j \quad \to 1 \quad (\text{for } 1 \leq j \leq k_2 - 1) \\
\vdots \quad \vdots \quad \vdots \\
0 \to i, \quad 0 \to s - 1 + \sum_{l=1}^{i-1} (k_l - 1) + j \quad \to i \quad (\text{for } 1 \leq j \leq k_i - 1) \\
\vdots \quad \vdots \quad \vdots \\
0 \to s - 1, \quad 0 \to s - 1 + \sum_{l=1}^{s-1} (k_l - 1) + j \quad \to s - 1 \quad (\text{for } 1 \leq j \leq k_{s-1} - 1)
\end{array}
\]

If \(k_i = 1\), there is no path like

\[0 \to s + \sum_{l=1}^{i-1} (k_l - 1) + j \to i, \quad \text{for } 1 \leq j \leq k_i - 1.\]

These \(k - 1\) paths are node-disjoint, and the longest one has length \(n + 1 = 2\).

Assume when the dimension is less than \(n\), the proposition is true. Now we will show that when the dimension is \(n\), the theorem is also true.

For \(s \geq 1\) and any \(s + 1\) nodes \(v_0, v_1, \cdots v_s\) in \(D(k, n)\), and for any set \((k_1, \cdots, k_s)\) that satisfies \(k_i > 0\) and \(\sum_{i=1}^{s} k_i = k - 1\), we now construct the \(k - 1\) node-disjoint paths of \(D(k, n)\) from the paths of \(D(k, n - 1)\). Assume \(v_i = v_{i1} v_{i2} \cdots v_{in}\), for \(0 \leq i \leq s\). From induction, we know that for the
probably-non-distinguish nodes $v_0' = v_{i_2}v_{i_3} \cdots v_{i_m}, v_i' = v_{i_1}v_{i_2} \cdots v_{i(n-1)}$, with $1 \leq i \leq s$, there exist $k-1$ node-disjoint paths (except at the origin and terminus), each of them of length at most $n - 1 + 1 = n$ and $k_i$ paths from $v_0'$ to each $v_i'$.

For the path $P_{ij}'$ that is the $j$th path from $v_0'$ to $v_i'$,

$$P_{ij}' = v_{02}v_{03} \cdots v_{0n} \rightarrow v_{03} \cdots v_{0n}x \rightarrow \cdots \rightarrow yv_{i1} \cdots v_{i(n-2)} \rightarrow v_{i1}v_{i2} \cdots v_{i(n-1)},$$

we extend the following walk

$$v_{01}v_{02} \cdots v_{0(n-1)} \rightarrow P_{ij}' \rightarrow v_{i2}v_{i3} \cdots v_{in}$$

to $P_{ij}$ which is the $j$th path from $v_0$ to $v_i$ as following:

$$P_{ij} = v_{01}v_{02} \cdots v_{0n} \rightarrow v_{02} \cdots v_{0n} x \rightarrow \cdots \rightarrow yv_{i1} \cdots v_{i(n-1)} \rightarrow v_{i1}v_{i2} \cdots v_{in}.$$

Since in $D(k, n-1)$, those $k - 1$ paths are node-disjoint with length at most $n$. When we extend $v_{01}v_{02} \cdots v_{0(n-1)} \rightarrow P_{ij}' \rightarrow v_{i2}v_{i3} \cdots v_{in}$ to $P_{ij}$, length increases by 1. So in $D(k, n)$, the longest length of these $k - 1$ paths is at most $n + 1$. Also since each node of $P_{ij}$ is an edge of $v_{01}v_{02} \cdots v_{0(n-1)} \rightarrow P_{ij}' \rightarrow v_{i2}v_{i3} \cdots v_{in}$, except the origin and terminal edges, those edges are different. So these $k - 1$ paths are node-disjoint (except at origin and terminus).

The following theorem shows that similar property holds for $S(k, n)$.

**Theorem 3.8.** For any $s+1$ nodes $v_0, v_1, \ldots, v_s$ in $S(k, n)$, where $v_1, v_2, \ldots, v_s$ are distinct, but one of $v_1, \ldots, v_s$ may be the same as $v_0$, and for any set $(k_1, \ldots, k_s)$ that satisfies $k_i > 0$ and $\sum_{i=1}^s k_i = k - 1$, there exist $k - 1$ node-disjoint paths (except at the origin and terminus), each of them length at most $2n + 1$, and with $k_i$ paths going from $v_0$ to $v_i$ (for $1 \leq i \leq s$).

**Proof.** Let $s \geq 1$. For any $s+1$ nodes $v_0, v_1, \ldots, v_s$ in $S(k, n)$ and for any set $(k_1, \ldots, k_s)$ that satisfies $k_i > 0$ and $\sum_{i=1}^s k_i = k - 1$, we now construct the $k - 1$ node disjoint paths from the paths of $D(k, n-1)$.

Assume $v_i = v_1v_2 \cdots v_{n-1}$ for $0 \leq i \leq s$. We know that for the probably-non-distinguish nodes $v_i' = v_1v_2 \cdots v_{i(n-1)}$ in $D(k, n-1)$, where $0 \leq i \leq s$, there exist $k - 1$ node-disjoint paths (except at the origin and terminus), each of them of length at most $n - 1 + 1 = n$, $k_i$ paths from $v_0'$ to each $v_i'$. For the path $P_{ij}'$ that is the $j$th path from $v_0'$ to $v_i'$,

$$P_{ij}' = v_{01}v_{02} \cdots v_{0(n-1)} \rightarrow v_{02} \cdots v_{0(n-1)} x \rightarrow v_{03} \cdots v_{0(n-1)} x z \rightarrow \cdots \rightarrow yv_{i1} \cdots v_{i(n-2)} \rightarrow v_{i1}v_{i2} \cdots v_{i(n-1)};$$

we extend $P_{ij}'$ to $P_{ij}$ as follows:

$$P_{ij} = v_{01}v_{02} \cdots v_{0n} \rightarrow v_{01} \cdots v_{0(n-1)} x \rightarrow v_{02} \cdots v_{0(n-1)} x v_{01} \rightarrow \cdots \rightarrow v_{02} \cdots v_{0(n-1)} x z \rightarrow \cdots \rightarrow yv_{i1} \cdots v_{i(n-1)} w \rightarrow v_{i1}v_{i2} \cdots v_{in}.$$
Although $P_{ij}$ may not be a path, it is a walk and it contains a path. It is easy to see that those $P_{ij}$ are disjoint. And since an edge in path $P_i'$ transforms to two edges of $P_i$, the length of these $(k - 1)$ disjoint paths are at most $2n + 1$.

It is straightforward to prove the following corollaries.

**Corollary 3.9.** For any $k, n, w$, where $1 < w < k$, the $w$-wide diameter $d_w(S(k,n))$ is $2n + 1$.

**Proof.** From Theorem 3.8, we know that for any two distinct nodes $u, v$, there exist $k - 1$ disjoint paths from $u$ to $v$, with the longest one having length at most $2n + 1$. Since from $000 \cdots 0$ to $111 \cdots 1$, we know the length of shortest path is $2n - 1$. It only changes 0 to 1 with any exchange edge. For finding $w$ disjoint paths, some path must change 0 to some other digit and then change back, so we need at least one more exchange edge and one more shuffle edge. Hence the longest length of $w$ disjoint paths is at least $2n + 1$. So for any $w \in \{1, \cdots, k - 1\}$, the $w$-wide diameter $d_w(S(k,n))$ is $2n + 1$.

**Corollary 3.10.** For any $k, n, w$, where $1 < w < k$, the $w$-fault diameter $D_w(S(k,n))$ is $2n + 1$.

**Proof.** From Theorem 2.3, we know that

$$D_w(S(k,n)) \leq d_w(S(k,n)) = 2n + 1,$$

for $1 \leq w \leq k - 1$.

Since for $00 \cdots 0$ and $11 \cdots 1$, if the node $00 \cdots 01$ is fault, the shortest path from $00 \cdots 0$ and $11 \cdots 1$ must take $n + 1$ exchange edges and at least $r$ shuffle edges, for a total of $2n + 1$ edges. So for any $w \in \{2, \cdots, k - 1\}$, the $w$-fault diameter $D_w(S(k,n))$ is $2n + 1$.

**Corollary 3.11.** For any $k, n, w$, where $1 < w < k$, the $w$-Rabin number of $S(k,n)$ is $2n + 1$.

**Proof.** This follows easily from Theorem 2.5 and Theorem 3.8.

**Corollary 3.12.** For any $k$, the line connectivity of $S(k,n)$ is $k - 1$.

**Proof.** $D(k,n)$ is $(k-1)$-line-connected. From Remark 1.1, it is easy to know that every edge cut of $D(k,n-1)$ can transform to an edge cut of $S(k,n)$. So the line-connectivity of $S(k,n)$ is at most $k - 1$. Since the line-connectivity of a graph is always larger than or equal to the connectivity of the same graph, the line-connectivity of $S(k,n)$ is at least $k - 1$.

**Example 3.2.** In $S(3,3)$, the two possible node-disjoint paths from $000$ to $111$ are:

$$000 \rightarrow 001 \rightarrow 010 \rightarrow 011 \rightarrow 110 \rightarrow 111,$$
$$000 \rightarrow 002 \rightarrow 020 \rightarrow 021 \rightarrow 210 \rightarrow 211 \rightarrow 112 \rightarrow 111.$$
4 Modified Shuffle-Exchange Digraphs MS(k, n)

A loop is a link from a node to itself which doesn’t serve any useful purpose for communication. Furthermore, since the degree of each node is fixed at \( k \) in \( S(k, n) \), the existence of a loop at a node immediately reduces the connectivity to \( k - 1 \). In the following, we replace all self-loops of \( S(k, n) \) by a directed cycle, denoting the new graph as \( MS(k, n) \). Since the order of the self-loop nodes in the directed cycle does not have any impact on the properties of the resulting digraph, we assume the cycle is

\[
0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow (k - 1) \rightarrow 0,
\]

without loss of generality. See Figure 4.1 for an example.

From [4], we know that if we replace all self-loops of \( D(k, n) \) by directed cycles, denoted as \( MD(k, n) \), the connectivity is \( k \). In the following, we prove that similar situation holds for \( MS(k, n) \).

**Theorem 4.1.** The connectivity of \( MS(k, n) \) is \( k \).

**Proof.** As we know, the connectivity of \( MD(k, n - 1) \) is \( k \). That is, between any two nodes \( u \) and \( v \), there exist \( k \) disjoint paths from \( u \) to \( v \). We can construct disjoint paths of \( MS(k, n) \) based on \( MD(k, n - 1) \). Let \( u = u_1u_2\cdots u_n \) and \( v = v_1v_2\cdots v_n \). Since we can transform \( S(k, n) \) to \( D(k, n - 1) \), we consider two nodes \( u' = u_1u_2\cdots u_{n-1} \) and \( v' = v_1v_2\cdots v_{n-1} \) of \( MD(k, n - 1) \).
It is easy to see that the diameter of $\mathcal{P}$ and $v\mathcal{k}$

Case I. $P_t'$ does not contain edges of the form $xx\cdots x \rightarrow y\cdots y$. Assume $P_t'$ is

$$u_1u_2\cdots u_{n-1} \rightarrow u_2\cdots u_{n-1}x_1 \rightarrow u_3\cdots u_{n-1}x_1x_2 \rightarrow \cdots \rightarrow v_1v_2\cdots v_{n-1}.$$

Construct $P_t$ as follows:

$$u_1\cdots u_{n-1}u_n \rightarrow u_1\cdots u_{n-1}x_1 \rightarrow u_2\cdots u_{n-1}x_1u_1 \rightarrow u_2\cdots u_{n-1}x_1x_2,$$

$$\rightarrow u_3\cdots u_{n-1}x_1x_2u_2 \rightarrow \cdots \rightarrow v_1\cdots v_{n-1}x_t \rightarrow v_1\cdots v_{n-1}v_n.$$

Although $P_t$ may not be a path, it is a walk and contains a path.

Case II. $P_t'$ contains edges of the form $xx\cdots x \rightarrow yy\cdots y$. Assume $P_t'$ is

$$u_1u_2\cdots u_{n-1} \rightarrow u_2\cdots u_{n-1}x_1 \rightarrow u_3\cdots u_{n-1}x_1x_2 \rightarrow \cdots \rightarrow wx\cdots x,$$

$$\rightarrow xx\cdots x \rightarrow yy\cdots y \rightarrow y\cdots yz \rightarrow v_1v_2\cdots v_{n-1}.$$

Construct $P_t$ as follows:

$$u_1\cdots u_{n-1}u_n \rightarrow u_1\cdots u_{n-1}x_1 \rightarrow u_2\cdots u_{n-1}x_1u_1 \rightarrow u_2\cdots u_{n-1}x_1x_2,$$

$$\rightarrow u_3\cdots u_{n-1}x_1x_2u_2 \rightarrow \cdots \rightarrow wx\cdots x \rightarrow x\cdots x \rightarrow x\cdots x,$$

$$\rightarrow yy\cdots y \rightarrow y\cdots yz \rightarrow \cdots \rightarrow v_1v_2\cdots v_{n-1}x_t \rightarrow v_1v_2\cdots v_{n-1}v_n.$$

Although $P_t$ may not be a path, it is a walk and it contains a path. It is easy to see that $P_1, P_2, \ldots P_k$ are disjoint. So the connectivity of $MS(k, n)$ is $k$.

Since $MS(k, n)$ contains more useful edges than $S(k, n)$, the diameter of $MS(k, n)$ may be smaller than that of $S(k, n)$. When $k \geq n + 1$, it is easy to show that the diameter of $MS(k, n)$ is $2n - 1$. (The path from node $00\cdots 0$ to node $23\cdots n1$ needs to take $n$ exchange operations and so needs $n - 1$ shuffles). But when $3 \leq k \leq n$, it is not easy to see that the diameter of $MS(k, n)$ is still $2n - 1$.

**Theorem 4.2.** The diameter of $MS(k, n)$ is $2n - 1$ for $k \geq 3$.

**Proof.** It is easy to see that the diameter of $MS(k, n)$ is at most $2n - 1$. Now we will show that the shortest path in $MS(k, n)$ from $1\cdots 121$ to $020\cdots 0$ has length $2n - 1$. If the path wants to take the modified edge, it must follow the following steps:

$$1\cdots 121 \rightarrow 11\cdots 11 \rightarrow 00\cdots 00 \rightarrow 020\cdots 0.$$
The shortest path from $1 \cdots 121$ to $11 \cdots 1$ has $n - 1$ shuffles, one exchange, for a total of $n$ edges. The shortest path from $11 \cdots 1$ to $020 \cdots 0$ has at least one modified edge and $n - 2$ shuffles, for a total of $2n - 1$ edges. Otherwise, if the path will either take $n$ exchanges and at least $n - 1$ shuffles, for a total of $2n - 1$ edges, or take $n - 1$ exchanges (change every 1 to 0, leave 2 there) and at least $2n - 3$ shuffles, for a total of $3n - 4$ edges. So the diameter of $MS(k, n)$ is $2n - 1$.

When $k = 2$, we can find that the diameter of $MS(2, 3)$ is 3. Determining the diameter of $MS(2, n)$ is an open problem.

5 Conclusion

Although the shuffle-exchange network $S(k, n)$ has been used as topology for VLSI systems, parallel architectures, and communication networks, it does not exist with number of nodes between $k^n$ and $k^{(n+1)}$. We have proposed a class of digraphs $GS(k, N)$ which generalizes $S(k, n)$ to any number $N$ of nodes. We have in this paper and in previous investigation ([16]) shown that:

- the diameter of $GS(k, N)$ is at most $2\lceil \log_k N \rceil + 1$, when $k|N$, the diameter is $2\lceil \log_k N \rceil - 1$;
- the number of self loops in $GS(k, N)$ is $g+1$, where $g = \gcd(k-1, p-b-1) = g$, $p = \lceil N/k \rceil$, $b = kp - N$, $p > b + 1$, and $k \geq 2$;
- when $k|N$, the connectivity of $GS(k, N)$ is $k - 1$;
- the $k$-ary balanced tree of height $2t - 1$ can be embeded in $GS(k, N)$, where $t = \lceil \log_k N \rceil$;
- if $k|N$, $GS(k, N)$ has the self-routing property; and
- $GS(k, k(k+1))$, for any $k > 2$, contains a Hamiltonian circuit.

In this paper, we also showed that:

- the shuffle-exchange network $S(k, n)$ has both connectivity and link connectivity equal to $k - 1$;
- the wide-diameter $d_w(S(k, n))$, fault-diameter $D_w(S(k, n))$, and rabin number $r_w(S(k, n))$ are $2n + 1$ for $1 < w \leq k - 1$; and
- the modified shuffle-exchange network $MS(k, n)$ has connectivity $k$.

The generalized shuffle-exchange network $GS(k, N)$ we proposed and studied has maintained most of the good properties, yet it is much more general than the shuffle-exchange network $S(k, n)$. When $N = k^n$, $GS(k, N)$ is the same as $S(k, n)$. The current study suggests several other problems worthy of further investigation.
• study combinatorial and fault tolerant properties of the undirected version of \( GS(k, N) \), \( UGS(k, N) \);

• study the wide-diameter of \( d_w(MS(k, n)) \) of the modified shuffle-exchange network with \( w = k \); and

• study the Hamiltonian property of \( S(k, n) \) and \( GS(k, N) \) when \( k \mid N \).

Let the network \( G \) have degree \( d \). \( G \) is said to be strongly fault-tolerant if the resulting network \( G_f \) (with at most \( d - 2 \) faulty nodes removed) has a container \( C_w(u, v) \) of width \( w \) between any two nodes \( u, v \) in \( G_f \), where \( w = \) minimum degree of \( u \) and \( v \) in \( G_f \). In [17], Oh and Chen showed that the star-network is strongly fault tolerant. Are \( S(k, n) \), \( MS(k, n) \), \( GS(k, N) \) or \( UGS(k, N) \) strongly fault tolerant?

References


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