MHD Dynamo Simulation Using the GeoFEM Platform - Comparison with a Spectral Method -

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ABSTRACT
We investigate differences of results of MHD dynamo simulations using the Finite-Element method (FEM) with that by the spherical harmonics expansion. The simulations are performed with a total of 7.8 x 10^4 elements in the simulation domain. The results show that the magnetic energy is generated approximately 4 times of the kinetic energy in both cases. The outline of present results has consistence with that by the spectral method. However, small-scale patterns can be observed in the spectral method case. These discrepancies are caused by limitation of the spatial resolution in both of the present simulations.
Abstract

We investigate differences of results of MHD dynamo simulations using the Finite-Element method (FEM) with that by the spherical harmonics expansion. The simulations are performed with a total of $7.8 \times 10^4$ elements in the simulation domain. The results show that the magnetic energy is generated approximately 4 times of the kinetic energy in both cases. The outline of present results has consistence with that by the spectral method. However, small-scale patterns can be observed in the spectral method case. These discrepancies are caused by limitation of the spatial resolution in both of the present simulations.

Key words: Geodynamo, magnetohydrodynamics, rotating shell, Finite-Element method

1. Introduction

Investigations of the processes generating the Earth’s and other planets’ magnetic fields entered a new stage in 1995, when several magnetohydrodynamic (MHD) simulations of a rotating spherical shell were used to represent some basic characteristics of the geomagnetic field (Glatzmaier and Roberts, 1995a [1], 1995b [2]; Kageyama et al., 1995 [3]). Following these studies, several groups analyzed the strong, dipole-like magnetic fields by which the geomagnetic field is characterized (Kuang and Bloxham, 1997 [4], 1999[5]; Christensen et al., 1999 [6], Sakuraba and Kono, 1999 [7]). Most of these simulations used a spherical harmonics expansion in the azimuthal and elevation directions to obtain high spectral accuracy. With this approach, the magnetic fields in the spherical shell are easily connected to the potential field outside the shell via boundary conditions. Kageyama et al. (1995) [3] used the finite difference method (FDM), but applied a different magnetic boundary condition at the boundaries of the shell from that estimated to exist at the Earth’s core-mantle boundary (CMB). However, some disadvantages of the spectral method come to be pointed out; specifically, the approach is not suitable for massively parallel computations because a significant number of global operations are required for the computation of nonlinear terms. In the present study, we choose the FEM because it is possible to represent any unstructured mesh and domain decomposition pattern in the FEM. A numerical simulation using the parallel FEM platform of thermal convection in a rotating spherical shell was reported (Matsui and Okuda, 2002a [10]). On the other hand, Chan et al. (2001a [9], 2001b [11]) solved a nonlinear kinematic dynamo problem within the FEM platform. We reported the first result of MHD dynamo simulation using the FEM (Matsui and Okuda, 2002b [12]). To verify the simulation results, we compared the simulation results with those obtained results using the spectral method. The results shows that the calculated magnetic energy in the fluid shell is approximately 90% that determined with the spectral method, and further that similar characteristics of the convection and magnetic field patterns are obtained. Furthermore, behaviors of the magnetic field and convection patterns are also similar to each other.

2. Simulation model and methods

Simulation model

Consider a rotating spherical shell modeled on the Earth’s outer core. The ratio of the inner boundary to the outer boundary of the spherical shell is set to be $r_i/r_o = 0.4$, where, $r_i$ and $r_o$ are the radii of the inner and outer boundaries of the fluid shell, respectively. The shell is filled with an electrically conducting fluid and rotates with a uniform angular velocity $\Omega = \Omega \hat{z}$. The fluid has a constant kinetic viscosity $\nu$, thermal diffusivity $\kappa$, thermal expansion coefficient $\alpha$, and magnetic diffusivity $\eta$. We assume that the inner core is electrically insulated and co-rotates with the mantle to simplify the model. The nondimensional motion of a fluid in a rotating frame is fully described by the mass conservation equation, the momentum equation...
(Navier-Stokes equation), incorporating the Boussinesq approximation and the Coriolis and Lorentz terms, the thermal diffusion equation, Ohm’s law and the Maxwell’s equations with the magnetohydrodynamics (MHD) approximation; that is,

\[ \text{div } \mathbf{u} = 0, \]  
\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + P_r \nabla^2 \mathbf{u} - \nabla T_a \Omega \times \mathbf{u} + P_r R_a (T - T_0) \mathbf{r} + P_r (\nabla \times \mathbf{B}) \times \mathbf{B}, \]  
\[ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \nabla^2 T, \]  
\[ \frac{\partial A}{\partial t} = -\nabla \varphi + \frac{P_r}{P_m} \nabla^2 A + \mathbf{u} \times \mathbf{B}, \]  
\[ \text{div } A = 0, \]  
\[ B = \nabla \times A, \]  

and

\[ T_0 = \frac{r_1 r_o - r_i r}{r (r_o - r_i)}. \]  

where \( \mathbf{u}, P, r, T, T_0, A, \) and \( B \) are the velocity, modified pressure, position vector, temperature, reference temperature, vector potential of the magnetic field, and magnetic field, respectively. Basic equations for the vector potential in an electrical insulator must also be taken into account;

\[ \nabla^2 A = 0, \]  
\[ \nabla \cdot A = 0. \]

To obtain the above normalized equations, a shell width of \( L = r_o - r_i \) and a thermal diffusion time of \( L^2/\kappa \) were selected as the length and time scales, respectively. The magnetic field is also normalized by \( B_0^2 = \rho \mu_0 \kappa^2/\mu^2 \), where \( \rho \) and \( \mu_0 \) are the density of the fluid and the magnetic permeability, respectively. There are four dimensionless numbers in the equations above; namely, the Prandtl number \( P_r \), the Rayleigh number \( R_a \), the taylor number \( T_a \), and the magnetic Prandtl number \( P_m \). The Taylor and Rayleigh numbers in the Earth’s outer core are both estimated to be of the order of \( 10^{30} \) with molecular viscosities (Gubbins [14], 1987). Even if turbulent viscosities are considered, these dimensionless numbers remain greater than \( 10^{10} \). However, such high estimated values cannot be represented directly because of limits to our computational power. The dimensionless numbers are set to be the following values;

\[ P_r = \frac{\nu}{\kappa} = 1.0, \]  
\[ R_a = \frac{g \alpha \Delta T L^3}{\kappa \nu} = 1.2 \times 10^4, \]  
\[ T_a = \left( \frac{2 \Omega L^2}{\nu} \right)^2 = 9.0 \times 10^4, \]  

and

\[ P_m = \frac{\nu}{\eta} = 10.0, \]

where \( \Delta T \) and \( g \) are difference in temperature between the inner and outer boundaries of the shell and gravity, respectively.

The boundary conditions exert a significant influence on the motion of the fluid and the overall dynamo process. Rigid velocity boundary conditions and constant temperature boundary condition at both boundaries of the fluid shell were employed:

\[ \mathbf{v} = 0 \quad \text{at } r = r_o, r_i, \]  
\[ T = 1 \quad \text{at } r = r_i, \]  

and

\[ T = 0 \quad \text{at } r = r_o. \]

The externals \((0 \leq r < r_i \text{ and } r > r_o)\) of the fluid shell are assumed to be electrical insulators in the present model, requiring a conductive fluid contact at \( r = r_o \) and \( r_i \). However, within the insulating regions we have
to solve for the vector potential using the Laplace equation, as given in eqs. (8) and (9). The boundary conditions can be described that the vector potential and its differential have continuity. These boundary conditions reveal that no special treatment is required to determine the vector potential at the boundaries within the FEM framework. The vector potential for $0 \leq r < r_i$ can be determined by these treatments because the inner sphere is surrounded by the inner boundary of the fluid shell. To determine the vector potential outside the fluid shell, we impose boundary conditions on the vector potential at infinite radius. Mathematically, the boundary condition is given as:

$$A \rightarrow O(r^{-2}) \text{ for } r \rightarrow \infty. \quad (17)$$

In the present study, however, the following simplified boundary conditions on the vector potential are considered instead of those represented by eq. (17),

$$A = 0 \text{ at } r = r_m, \quad (18)$$

where $r_m$ is the radius of the outermost boundary of the simulation domain.

The simulation code is based on the GeoFEM thermal-hydraulic subsystem (Okuda et al. [13], 2002), which is designed for numerical simulation of thermally driven convection by a parallel FEM. The simulation domain is divided into tri-linear hexahedral elements, and the physical parameters are defined at each node and interpolated by a tri-linear function within each hexahedron. We use the fractional step scheme for time integration. The Crank-Nicolson scheme is used for solving the diffusion terms, and the 2nd-order Adams-Bashforth scheme is used for the other terms. The conjugate gradient (CG) solver in GeoFEM is used to determine the diffusion terms and solve the Poisson equations for $P$ and $\varphi$, because an iterative solver with preconditioning is one of the most powerful methods of parallel computation (Nakajima and Okuda, 1999 [15]).

Grid pattern

The finite element mesh covers the exterior of the shell as well as the shell itself. The grid pattern was generated by projecting a cubic pattern onto a spherical surface in the present study as shown in Fig. 1. In the present study, the outer boundary of the simulation domain is set to be $r_m = 14.7$, which corresponds to 5.1 times the Earth’s radius. This radius is required to fully describe the potential field adjacent to the outer boundary of the fluid shell. Outside the fluid shell, the mesh size coarsens with increasing radius because variations of the vector potential are small.

Symmetry with respect to the equatorial plane is considered in the present study to reduce computation times. The finite element mesh for the northern hemisphere as given in Fig. 1 in total 77760 elements make up the simulation domain, of which 46656 elements are within the hemispherical fluid shell. This mesh has $5^\circ$ resolution in the azimuthal direction at the equatorial plane and 49 unequally spaced nodes in the radial direction for the fluid shell. The mesh has 13 unequally spaced nodes in the radial direction for the exterior of the shell. Domain decomposition is required for the present simulation. The finite element mesh is divided with respect to the azimuthal and elevation directions (see Fig. 1).

Simulation methods by the spectral harmonics expansion

To verify the results of the simulation by GeoFEM, the same simulation was carried out with the spectral method. In this case, the simulation scheme is based on those of Frazer (1974) [16] and Honkura et al.
It is well established that arbitrary solenoidal vector fields can be separated into poloidal and toroidal components (Bullard and Gellman (1954) [18], Chandrasekhar (1961) [19]). Scalar functions of the poloidal and toroidal components of the velocity and magnetic fields, and the temperature perturbation are expanded in spherical harmonics. The coefficients are determined, and find a solution in the radial direction, the second-order FDM is applied. To calculate the temporal evolution, the Crank-Nicolson scheme is adopted for the diffusion terms and the second-order Adams-Bashforth scheme is used to obtain the remaining terms. The advection terms, the Coriolis term, the Lorentz term, and the induction term are represented by the coefficients of the spherical harmonics. Here, the truncation level of the spherical harmonics is set to be 18 degrees with 64 equally spaced grid points in the radial direction.

3. Results of the simulation

We compared the following results obtained with the FEM technique with those of the spectral method: i) Kinetic and magnetic energies averaged over the spherical shell, ii) characteristics of the convection and magnetic field patterns, and iii) behavior of the magnetic field and convection patterns.

Kinetic energy and magnetic energies

The kinetic energy $E_k$ and magnetic energy $E_m$ averaged over the fluid shell are plotted as functions of the thermal diffusion time in Fig. 2. As seen Fig. 2, the magnetic energy in the FEM case grows much earlier than that in the spectral method because the initial magnetic energy in the FEM is set approximately $10^4$ times as large as that in the spectral method. The magnetic and kinetic energies are averaged from $t = 10.0$ in the GeoFEM case, and from $t = 18.8$ in the spectral method case for ease of comparison. The results are given in Table 1.

![Figure 2: Time evolution of the kinetic energy and magnetic energy averaged throughout the fluid shell. The FEM results are given by the solid lines, and the spectral method results are given by the dotted lines. The kinetic and magnetic energies are denoted by thin lines and thick lines, respectively.](image)

<table>
<thead>
<tr>
<th></th>
<th>$E_m$</th>
<th>$E_k$</th>
<th>$E_m/E_k$</th>
<th>Drift angular frequency $m = 3$</th>
<th>Drift angular frequency $m = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GeoFEM</td>
<td>$86.0 \pm 11.4$</td>
<td>$21.5 \pm 3.7$</td>
<td>4.00</td>
<td>$-2.1 \pm 1.3$</td>
<td>$-2.4 \pm 1.3$</td>
</tr>
<tr>
<td>Spectral</td>
<td>$96.6 \pm 16.0$</td>
<td>$21.4 \pm 4.1$</td>
<td>4.51</td>
<td>$-2.3 \pm 1.1$</td>
<td>$-2.5 \pm 1.3$</td>
</tr>
</tbody>
</table>

Comparing the magnetic energies in the two cases, we observe that the intensity of the magnetic energy in the GeoFEM case is approximately 89% of that obtained with the spectral method, while the intensities of the kinetic energy in each case are approximately the same. This difference is addressed with respect to the magnetic field pattern generated in the following section.

Convection and magnetic field patterns

The results of the average energy calculations (Fig. 2) suggest that the convection and magnetic field patterns are unstable, complicating a comparison of the GeoFEM and the spectral method calculation. For this reason,
we looked for snap shots having similar patterns to each other for comparison. Convection and magnetic field patterns in a cross section at $z = 0.35$ are shown in Fig. 3. The snap shots are for $t = 20.0$ in GeoFEM case and for $t = 21.72$ for Spectral case. The magnetic field patterns exhibit common characteristics. As seen the convection and magnetic field patterns by GeoFEM case in Fig. 3, an intense $z$-component of the magnetic field is generated in the anti-cyclonic negative $z$-component vortices. Moreover, these anti-cyclones have a pronounced structure, while the positive $z$-component cyclones lose their shapes. These characteristics have been noted in the previous MHD simulation of a rotating spherical shell (Kageyama et al., 1995 [3]). On the other hand, small-scale patterns can be seen in the spectral method results. To investigate this difference, the $B_z$ along a circle at the mid-depth and the equator is expanded into the coefficients of the trigonometric functions by $B_z(\phi) = \sum_m B_{zm} \cos(m\phi - \phi_0m)$, where $\phi$ is the longitude. The results are plotted in Fig. 4.

Although these spectra show characteristics of the magnetic field in a very limited area, the spectra also

Figure 3: Intensity of $z$-component of the magnetic field (middle and right panels) and vorticity (left panel) in the cross section at $z = 0.35$. Results by GeoFEM for $t = 20.0$ are given in the middle and right panels, and these by spectral method are given in left panels. The northward component is given by positive values in the all contours.

Figure 4: Spectra of the $z$-component of the magnetic field with respect to the longitudinal direction along a circle on the equatorial plane and the mid-depth of the shell. The result by GeoFEM is averaged after generated the magnetic field.

suggest that the small scale components of the magnetic fields in the spectral case have larger amplitude than that in the case of GeoFEM, and these components may be intensified by the small truncation level in the spectral method case. This difference is manifest as the large magnetic energy illustrated in the previous subsection (see Fig. 2).

Behavior of the magnetic field

After Busse’s linear analysis (Busse, 1970 [20]), convection patterns propagate in the zonal direction. It is well known that the magnetic fields also propagate with the convection patterns by the previous MHD dynamo simulations. To investigate behavior of the magnetic fields in the zonal direction, the $z$-component of the
magnetic field along the circle at mid-depth of the shell and at the equatorial plane is plotted in Fig. . The field patterns basically propagate to the westward throughout the simulation in both cases. Furthermore, the propagation of the propagation of the intense $z$-component of the magnetic field is sometimes stopped and finally disappeared. In this moment, the anti-cyclones with the intense $z$-component of the magnetic field are broken by the Lorentz force. These phenomena are also seen in both cases.

To verify these zonal propagation of the magnetic field more quantitatively, the drift angular frequency of the field patterns is estimated from the $m = 3$ and $m = 4$ components of the $B_{zm}$. The results is given in Table 1. The difference of the drift frequency is less than 10 %. We consider this difference is quite small value considering the unstable behaviors of the magnetic field patterns in these cases.

4. Discussions

As seen in the previous section, the simulation results obtained with GeoFEM and the spectral method exhibit common characteristics in terms of the convection and magnetic field patterns and behaviors. However, there are some outstanding discrepancies to be addressed. In particular, the magnetic energy and the magnetic field patterns clearly differ between the two cases. In order to evaluate these causes, we must bear in mind that the spatial resolution is different for the two methods. As seen in Fig. 4, the magnetic fields which belong to around the truncation level are emphasized because the spatial resolution is not fully in the case of the spherical harmonics expansion. The spectra of magnetic energy averaged from $t = 18.8$ in the spectral case is shown in Fig. 6. The intensity of the truncation level of magnetic energy remains approximately 2% of that of the dipole component. If we considered higher truncation level in the spectral method than that in the present study, the magnetic energy generated may be smaller than the present results because small scale patterns will diffuse rapidly.

![Figure 5: Z-component of the magnetic field along a circle at the mid-depth of the fluid shell and the equatorial plane. The positive (the northward) values are indicated by black.](image)

![Figure 6: Spectra of the magnetic energy averaged over the fluid shell from $t = 18.0$ in the spectral method as functions of the harmonic degree $l$.](image)

To fully evaluate discrepancies, a more quantitative approach is required. At present, there is only
one benchmark test for MHD simulation in a rotating spherical shell, namely the dynamo benchmark test described by Christensen et al. (2001) [21]. We performed this test with the mesh which has the same number of elements as the present study. The results by GeoFEM and suggested solution by Christensen et al. are shown in Table 2. Comparing with the Christensen’s solutions, our results still have approximately 10% of differences. Some differences may appear because their solutions were derived only from spectral method. However, the present discrepancy is mainly caused by low spatial resolution of the present simulation. More verification with higher resolution mesh is required.

A huge problem for the present simulation is the computation time: It took almost 1 month to obtain the present results on 4 nodes of an SR8000. We obtained only ~ 5 % of the peak performance, and solution of the Poisson equations and diffusion terms required more than 80% of the total time. We are developing the simulation code to run on the Earth Simulator (ES). ES has 640 nodes containing SMP type 8 vector processors with peak performance of 8 GFLOPS each. We anticipate that we can perform the simulation with a mesh comprising over $10^8$ elements at $<10$ (sec/step) if we use 600 nodes. On the ES, we will improve the MHD simulations with much larger Taylor number (smaller Ekman number) models than that of the present study.

In addition, the present simulation code can treat a finite conductive inner core easily because the finite element mesh is already set for the inner core. On the other hand, it is difficult to consider super-rotation of the inner core in the FEM. The viscous and magnetic torque is integrated over the inner boundary of the fluid shell in the FEM, while the rotation can be solved using boundary conditions for degree $l = 1$ components of the toroidal velocity in the spectral method. Application of these model remains as a next step of the present study.

5. Conclusions

We have developed a MHD simulation code for a fluid in a rotating hemispherical shell modeled on the Earth’s outer core using the parallel finite element method to aid in the understanding of the geodynamo process and fluid dynamics in the Earth’s outer core. A total of 77760 elements were used 46656 of which were set in the fluid shell. Because the Rayleigh number and Taylor number of the outer core are too large to be treated at present, they were set in a range suitable for the simulation; the Prandtl number was set to 1.0, the Taylor number to $9.0 \times 10^4$, Rayleigh number to $1.2 \times 10^4$, and the magnetic Prandtl number to 10.0. A Non-slip boundary condition was imposed at the boundaries of the fluid shell, and the temperature was set to 1.0 and 0.0 on the inner and outer boundaries of the shell, respectively. The vector potential was set to be 0.0 at the outer boundary of the simulation box instead of using a boundary condition at infinite radius. Simulations were performed to 2.5 times the magnetic diffusion time.

Simulations with the two results revealed broadly similar characteristics of the magnetic field and convection; that is, the magnetic energy in the fluid shell was formed to be approximately 4 times the kinetic energy in both cases, and intense z-component magnetic fields were generated in anti-cyclones. These characteristics of the magnetic field and convection patterns have also been seen in previous studies of MHD dynamo simulations. The magnetic fields propagates to the westward in both cases, and the propagation velocity decreases when the convection column has intense z-component of the magnetic field. Discrepancies between the two simulations were also observed. In particular, intensity of the magnetic energy calculated by GeoFEM was approximately 89% of that by the spectral method, which smaller scale patterns of the magnetic field were also observed. These small-scale components is intensified by the low truncation level of the spherical harmonics. However, considering the results of the dynamo benchmark test using the present simulation code, the results suggests that the present spatial resolution in the GeoFEM is also lower to obtain the sufficient solution.

Table 2: Solutions of the dynamo benchmark test.

<table>
<thead>
<tr>
<th></th>
<th>Christensen(2002)</th>
<th>GeoFEM (Ratio to the solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kinetic energy</td>
<td>30.733 ± 0.020</td>
<td>34.466 (112.1%)</td>
</tr>
<tr>
<td>Magnetic energy</td>
<td>626.41 ± 0.40</td>
<td>663.93 (106.0%)</td>
</tr>
<tr>
<td>Angular drift</td>
<td>-3.1017 ± 0.0040</td>
<td>3.1784 (102.5%)</td>
</tr>
<tr>
<td>Local Temperature</td>
<td>0.37338 ± 0.00040</td>
<td>0.36167 (96.9%)</td>
</tr>
<tr>
<td>Local zonal velocity</td>
<td>-7.6250 ± 0.0060</td>
<td>-6.7932 (89.1%)</td>
</tr>
<tr>
<td>Local magnetic field</td>
<td>-4.9280 ± 0.0060</td>
<td>-5.082 (103.1%)</td>
</tr>
</tbody>
</table>
Although some discrepancies are seen in the results of the present simulation, our simulation code appears to reasonably represent basic processes of the geodynamo. We intend to perform the MHD simulations incorporating rapid rotation and intense buoyancy in a future study.

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