Generalization of Shannon-Khinchin Axioms
to Nonextensive Systems and the Uniqueness Theorem

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Abstract

The Shannon-Khinchin axioms are generalized to nonextensive systems and the uniqueness theorem for the nonextensive entropy is proved rigorously. In the present axioms, Shannon additivity is used as additivity in contrast to pseudoadditivity in Abe’s axioms. The results reveal that Tsallis entropy is the simplest among all nonextensive entropies which can be obtained from the generalized Shannon-Khinchin axioms.

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Tsallis entropy, which was first introduced by Tsallis in 1988 [1], is defined by

$$S_q^T (p_1, \ldots, p_n) \equiv \frac{1 - \sum_{i=1}^{n} p_i^q}{q - 1},$$

(1)

for a given probability distribution \( \{p_i\}_{i=1}^{n} \), where \( n \) is a number of accessible configurations, \( q \) is a real parameter assumed to be positive, and Boltzmann’s constant \( k \) has been set equal to unity.

Tsallis entropy is a one-parameter generalization of Shannon entropy in the sense that

$$\lim_{q \to 1} S_q^T = S_1 \equiv -\sum_{i=1}^{n} p_i \ln p_i.$$  

(2)

The characteristic property of Tsallis entropy is called *pseudoadditivity*, i.e.

$$S_q^T (A, B) = S_q^T (A) + S_q^T (B) + (1 - q) S_q^T (A) S_q^T (B)$$

(3)

holds true for two mutually independent finite event systems \( A \) and \( B \)

$$A = \begin{pmatrix} A_1 & \ldots & A_n \\ p_1^A & \ldots & p_n^A \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & \ldots & B_m \\ p_1^B & \ldots & p_m^B \end{pmatrix},$$

(4)

such that \( p_{ij}^{AB} = p_i^A p_j^B \) for any \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).  

(5)

The property (3) represents nonextensive additivity of Tsallis entropy, i.e. Tsallis entropy is a *nonextensive entropy*. When \( q = 1 \) the pseudoadditivity (3) reverts to the standard additivity, so that Shannon entropy is considered to be *extensive entropy* in contrast to Tsallis entropy.

Introduction of Tsallis entropy opened a new research area in statistical physics, providing an advantageous generalization of traditional Boltzmann-Gibbs statistical mechanics [2]. The generalization enables us to find a consistent treatment of dynamics in many nonextensive physical systems such as long-range interactions, long-time memories, and multifractal structures, which cannot be coherently explained within the conventional Boltzmann-Gibbs statistics [3]. Thus Tsallis entropy inspires many physicists to establish a generalized Boltzmann-Gibbs statistical mechanics leading to numerous applications [4 3].
Along with the rapid progress of research in the generalized Boltzmann-Gibbs statistical mechanics, some axioms of Tsallis entropy (1) were presented as generalization of Shannon-Khinchin axioms [4, 5]. We proceed with a discussion of the Shannon-Khinchin axioms and their possible generalized counterparts.

Let $\Delta_n$ be defined by the $n$-dimensional simplex:

$$
\Delta_n \equiv \left\{ (p_1, \ldots, p_n) \right\} \quad \text{where} \quad p_i \geq 0, \quad \sum_{i=1}^{n} p_i = 1.
$$

The Shannon-Khinchin axioms [3] are given by the following four conditions:

[SK1] continuity: for any $n \in \mathbb{N}$ the function $S_1(p)$ is continuous with respect to $p \in \Delta_n$,

[SK2] maximality: for given $n \in \mathbb{N}$ and for $(p_1, \ldots, p_n) \in \Delta_n$, the function $S_1(p_1, \ldots, p_n)$ takes its largest value for $p_i = \frac{1}{n}$ ($i = 1, \ldots, n$),

[SK3] additivity (Shannon additivity): if

$$
p_{ij} \geq 0, \quad p_i = \sum_{j=1}^{m_i} p_{ij} \quad \text{for any} \quad i = 1, \ldots, n \text{ and } j = 1, \ldots, m_i, \text{ and } \sum_{i=1}^{n} p_i = 1,
$$

then the following equality holds:

$$
S_1 (p_{11}, \ldots, p_{nm_n}) = S_1 (p_1, \ldots, p_n) + \sum_{i=1}^{n} p_i S_1 \left( \frac{p_{i1}}{p_i}, \ldots, \frac{p_{im_i}}{p_i} \right),
$$

[SK4] expandability: $S_1(p_1, \ldots, p_n, 0) = S_1(p_1, \ldots, p_n)$.

In order to discriminate between the pseudoadditivity (3) and the above additivity property (8), we call the additivity (8) Shannon additivity throughout the paper.

Until now, two sets of axioms for Tsallis entropy were presented as generalizations of the above Shannon-Khinchin axioms [4, 5]. In the former axioms presented in [4], the four conditions are given as axioms for Tsallis entropy $S_q^T(p)$: [Sa1] continuity - same as [SK1] for $S_q^T$, [Sa2] increasing monotonicity - when $p_i = \frac{1}{n}$ ($i = 1, \ldots, n$), $S_q^T$ is a monotonic increasing function of $n$ for any $q \in \mathbb{R}^+$, [Sa3] pseudoadditivity - the equality (3) holds, and [Sa4] generalized Shannon additivity - under the same constraints as (7), the following equality holds

$$
S_q^T (p_{11}, \ldots, p_{nm_n}) = S_q^T (p_1, \ldots, p_n) + \sum_{i=1}^{n} p_i^q S_q^T \left( \frac{p_{i1}}{p_i}, \ldots, \frac{p_{im_i}}{p_i} \right).
$$
This condition (9) is a generalization of the original one that appeared in [4], which is a case of (9) when \( n = 2 \). This generalization is a straightforward one [2, 8].

In [4], the uniqueness theorem stating that the only function satisfying all requirements [Sa1]~[Sa4] is Tsallis entropy is proved, but actually these four axioms are verbose. In fact, it is sufficient to have two axioms, pseudoadditivity [Sa3] and generalized Shannon additivity [Sa4], to determine Tsallis entropy uniquely, as has been shown in our paper [9]. Thus the four axioms [Sa1]~[Sa4] in [4] are clearly redundant.

On the other hand, in the latter axioms presented in [5], the three conditions given as axioms for Tsallis entropy \( S^T_q \) are [Ab1] continuity and maximality - same conditions as [SK1] and [SK2] for \( S^T_q \), [Ab2] pseudoadditivity using the conditional entropy, and [Ab3] expandability - same condition as [SK4] for \( S^T_q \). The uniqueness theorem for Tsallis entropy is also proved.

In this paper, the Shannon-Khinchin axioms are generalized to nonextensive systems and the uniqueness theorem for nonextensive entropies including Tsallis entropy is proved rigorously. In our axioms, Shannon additivity is used as additivity in contrast to pseudoadditivity using the conditional entropy in Abe’s axioms [5].

It was found recently that the original Tsallis entropy (1) needs to be renormalized to satisfy the form invariance of the maximum entropy principle or the pseudoadditivity [10, 11]. The normalized Tsallis entropy is defined by

\[
\hat{S}^T_q(p_1, \ldots, p_n) \equiv \frac{1 - \sum_{i=1}^{n} p_i^q}{(q - 1) \sum_{j=1}^{n} p_j^q}.
\]  

(10)

In order to avoid confusion, we call \( S^T_q \) given by (1) and \( \hat{S}^T_q \) given by (10) \textit{original Tsallis entropy} and \textit{normalized Tsallis entropy} respectively.

In the next section, the axioms for nonextensive entropy including the original Tsallis entropy are presented, and the uniqueness theorem is proved rigorously.

II. GENERALIZED AXIOMS AND UNIQUENESS THEOREM FOR NONEXTENSIVE ENTROPY

Generalized Shannon-Khinchin axioms and the uniqueness theorem for nonextensive entropy are given below.
Let $\Delta_n$ be a $n$-dimensional simplex defined by (10). The following axioms [N1]–[N4] determine the function $S_q : \Delta_n \to \mathbb{R}^+$ such that

$$S_q (p_1, \ldots, p_n) = \frac{1 - \sum_{i=1}^{n} p_i^q}{\phi (q)},$$

where $\phi (q)$ satisfies properties (i) $\sim$ (iv):

(i) $\phi (q) \begin{cases} > 0, & \text{if } q > 1 \\ < 0, & \text{if } 0 \leq q < 1 \end{cases}$.

(ii) $\phi (q)$ is differentiable with respect to $q \in \mathbb{R}^+$,

(iii) $\lim_{q \to 1} \frac{d\phi (q)}{dq} = 1$,

and

(iv) $\lim_{q \to 1} \phi (q) = \phi (1) = 0, \quad \phi (q) \neq 0 \ (q \neq 1)$.

[N1] $S_q$ is continuous in $\Delta_n$ and $q \in \mathbb{R}^+$,

[N2] For any $q \in \mathbb{R}^+$ and any $n \in \mathbb{N}$,

$$S_q (p) \leq S_q \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \quad \text{for any } p \in \Delta_n,$$

[N3] Under the constraints (10), the following equality holds:

$$S_q (p_1, \ldots, p_{nmn}) = S_q (p_1, \ldots, p_n) + \sum_{i=1}^{n} p_i^q S_q \left( \frac{p_{i1}}{p_i}, \ldots, \frac{p_{imi}}{p_i} \right),$$

[N4]

$$\lim_{q \to 1} S_q = S_1 = - \sum_{i=1}^{n} p_i \ln p_i.$$

Note that the Shannon-Khinchin axioms and ours differ in the formulation of [N3] and [N4]. [N3] is a slight generalization of the Shannon additivity (10) in the sense that instead of the exponent “1” of the coefficient $p_i$ in front of $S_q \left( \frac{p_{i1}}{p_i}, \ldots, \frac{p_{imi}}{p_i} \right)$, any positive real number $q$ in $p_i^q$ can be taken. [N4] shows that Tsallis entropy is a one-parameter generalization of the Shannon entropy. [N4] can be replaced by the following expandability:

[N4*] For any $q \in \mathbb{R}^+$ and any $(p_1, \ldots, p_n) \in \Delta_n$,

$$S_q (p_1, \ldots, p_n, 0) = S_q (p_1, \ldots, p_n).$$
Furthermore, in case of \( q = 1 \), the axioms \([N1]\sim[N3]\) and \([N4^*]\) coincide with the Shannon-Khinchin axioms \([3, 4]\). Therefore when \( q = 1 \) we can obtain Shannon entropy from \([N1]\sim[N3]\) and \([N4^*]\). In the present axioms, for simplicity, \([N4]\) is used instead of \([N4^*]\).

The proof is given as follows.

Consider the case of \( q \neq 1 \). In the axiom \([N3]\), we apply the following special case: for any \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \), if we take

\[
m = m_1 = \ldots = m_n \quad \text{and} \quad p_{ij} = \frac{1}{mn},
\]

then the identity (17) can be written as

\[
S_q \left( \frac{1}{mn}, \ldots, \frac{1}{mn} \right) = S_q \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) + \sum_{i=1}^{n} \frac{1}{n^q} S_q \left( \frac{1}{m}, \ldots, \frac{1}{m} \right).
\]

Let \( f_q(n) \) be defined by

\[
f_q(n) \equiv S_q \left( \frac{1}{n}, \ldots, \frac{1}{n} \right),
\]

so the equation (21) becomes

\[
f_q(mn) = f_q(n) + n^{1-q} f_q(m).
\]

Exchanging the variables \( m \) and \( n \) in (23), we have

\[
f_q(n) + n^{1-q} f_q(m) = f_q(m) + m^{1-q} f_q(n).
\]

This can be rearranged to give

\[
\frac{f_q(n)}{1 - n^{1-q}} = \frac{f_q(m)}{1 - m^{1-q}}.
\]

This identity holds for any \( m, n \in \mathbb{N} \), and it depends on the variable \( q \) only. Thus, there exists a function \( \phi(q) \) such that

\[
f_q(n) = \frac{1 - n^{1-q}}{\phi(q)},
\]

where \( \phi(q) \) is a function of \( q \in \mathbb{R}^+ \), satisfying \([12] \sim [13]\). The properties \([12] \sim [13]\) are required for \( \phi(q) \) to satisfy the other axioms \([N2]\) and \([N4]\).

If all \( p_i \) in \( (p_1, \ldots, p_n) \in \Delta_n \) are rational numbers, then for every \( p_i \) there exist nonnegative integers \( m_i \in \mathbb{N} \) satisfying

\[
p_i = \frac{m_i}{\sum_{i=1}^{n} m_i} \quad (i = 1, \ldots, n).
\]
Here if we take $p_{ij}$ as
\[
p_{ij} = \frac{1}{\sum_{i=1}^{n} m_i}
\]
for any $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$, then from (27) we have $p_i = \sum_{j=1}^{m_i} p_{ij}$. Applying these formulas (27) and (28) to (17), the identity (17) can be reduced to
\[
S_q \left( \frac{1}{\sum_{i=1}^{n} m_i}, \ldots, \frac{1}{\sum_{i=1}^{n} m_i} \right) = S_q (p_1, \ldots, p_n) + \sum_{i=1}^{n} p_i^q S_q \left( \frac{1}{m_i}, \ldots, \frac{1}{m_i} \right),
\]
that is, using $f_q (n)$ defined in (22),
\[
f_q \left( \sum_{i=1}^{n} m_i \right) = S_q (p_1, \ldots, p_n) + \sum_{i=1}^{n} p_i^q f_q (m_i).
\]
Substitution of $f_q (n)$ obtained in (26) into (30) yields
\[
S_q (p_1, \ldots, p_n) = f_q \left( \sum_{i=1}^{n} m_i \right) - \sum_{i=1}^{n} p_i^q f_q (m_i)
\]
\[
= \frac{1 - \sum_{i=1}^{n} p_i^q}{\phi(q)} + \sum_{i=1}^{n} p_i^q m_i^{1-q} - \left( \sum_{i=1}^{n} m_i \right)^{1-q}.
\]
The identity (27) implies
\[
\sum_{i=1}^{n} p_i^q m_i^{1-q} = \left( \sum_{i=1}^{n} m_i \right)^{1-q},
\]
so that (31) becomes
\[
S_q (p_1, \ldots, p_n) = \frac{1 - \sum_{i=1}^{n} p_i^q}{\phi(q)}.
\]
The continuity [N1] of $S_q$ and the fact that any real number can be approximated by rational numbers with any precision enable us to validate the above formula for any $(p_1, \ldots, p_n) \in \Delta_n$. 

Here $1 - \sum_{i=1}^{n} p_i^q$ in the right hand side of (33) is clearly a concave function with respect to $(p_1, \ldots, p_n) \in \Delta_n$ when $q > 1$, and a convex function when $0 \leq q < 1$. Moreover, the obtained entropy (33) is clearly a symmetric function with respect to its arguments $p_1, \ldots, p_n$. Thus the axiom [N2] requires the property (12) for $S_q (p_1, \ldots, p_n)$ given by (33), because the property (12) makes $S_q (p_1, \ldots, p_n)$ given by (33) a concave function on $\Delta_n$ for any fixed $q \in \mathbb{R}^+$. 

7
Substitution of (33) into [N4] implies
\[ \lim_{q \to 1} \frac{1 - \sum_{i=1}^{n} p_i^q}{\phi(q)} = - \sum_{i=1}^{n} p_i \ln p_i. \] (34)

Clearly \[ \lim_{q \to 1} \left(1 - \sum_{i=1}^{n} p_i^q\right) = 0, \]
so that the axiom [N4] requires the property (13) for \( \phi(q) \).
Moreover, the numerator \( 1 - \sum_{i=1}^{n} p_i^q \) is obviously differentiable with respect to \( q \) and its first derivative is \( -\sum_{i=1}^{n} p_i^q \ln p_i \). Thus the axiom [N4] requires the property (13) and (14) for \( \phi(q) \). Accordingly, we can apply l’Hôpital’s rule to the left hand side of (34):
\[ \lim_{q \to 1} \frac{-\sum_{i=1}^{n} p_i^q \ln p_i}{d\phi(q)/dq} = - \sum_{i=1}^{n} p_i \ln p_i. \] (35)

Thus [N4] is satisfied as required.

Therefore, given a certain function \( \phi \), we can uniquely determine nonextensive entropy (11). Thus the proof is complete.

Note that there are many functions \( \phi(q) \) satisfying all properties (12) \( \sim \) (13). For instance,
\[ \phi(q) = \frac{(q - 1)(q^2 + 1)}{2} \] (36)
is one such function, and
\[ \phi(q) = q - 1 \] (37)
is the simplest one. Therefore, the original Tsallis entropy (1) is the simplest of all nonextensive entropies which can be obtained from the generalized Shannon-Khinchin axioms [N1] \( \sim \) [N4].

In particular, when \( \phi(q) = 1 - 2^{1-q} \), \( S_q \) given by (11) is exactly the same as Havrda-
Charvat entropy \( S_q^{HC} \) (13) or Daróci entropy \( S_q^{D} \) (14):
\[ S_q^{HC}(p_1, \ldots, p_n) = S_q^{D}(p_1, \ldots, p_n) \equiv \frac{1 - \sum_{i=1}^{n} p_i^q}{1 - 2^{1-q}}. \] (38)

Thus Havrda-Charvat entropy and Daróci entropy are also particular examples of the nonextensive entropies. Note that \( \phi(q) = 1 - 2^{1-q} \) does not satisfy the condition (14) because of the difference between 2 and \( e \) as base of logarithm used in each formulation.
It is easy to verify that $S_q$ obtained in our uniqueness theorem satisfies the additivity property
\[ S_q(A, B) = S_q(A) + S_q(B) - \phi(q) S_q(A) S_q(B). \] (39)

When $\phi(q) = q - 1$, this equality (39) takes the form of (3). Thus the form (39) is a generalization of the pseudoadditivity given by (3). In this way the pseudoadditivity (3) or (39) of the nonextensive entropy can be derived naturally from the slight generalization of the Shannon-Khinchin axioms.

III. CONCLUSION

There exist two main methods for axiomatic definition of Shannon entropy. The first is to introduce inherent properties as axioms for Shannon entropy, as done by Shannon in 1948 and Khinchin in 1953 [6, 7]. The other way is to treat satisfactory properties as axioms for information content of Shannon entropy, which was used in many references such as [12]. This method is based on the fact that Shannon entropy is given by an expectation of information content.

In this paper we present generalized Shannon-Khinchin axioms for nonextensive entropy and prove the uniqueness theorem rigorously in accordance with the Shannon-Khinchin approach. These results reveal that the Tsallis entropy is the simplest of all nonextensive entropies which can be obtained from the generalized Shannon-Khinchin axioms. Moreover, the remaining problem such as redundancy in the previously presented axioms in [4] is successfully solved.

On the other hand, the latter way to define nonextensive entropy by means of information content has been already presented in our paper [11]. Thus the two manners to define nonextensive entropy as similar to Shannon entropy are clearly observed in nonextensive systems framework. Comparing these two ways to define nonextensive entropy, we can point out certain advantages of using information content over the Shannon-Khinchin approach. For instance, the latter allows systematic introduction of other entropies [11]. In our forthcoming paper, another application of information content will be presented for the formalism of Tsallis information theory.
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