UNIFORM DISSIPATIVENESS, ROBUST SYNCHRONIZATION AND LOCATION OF THE ATTRACTOR OF PARAMETRIZED NONAUTONOMOUS DISCRETE SYSTEMS

HILDEBRANDO M. RODRIGUES
Departamento de Matemática Aplicada e Estatística,
Instituto de Ciências Matemáticas e de Computação,
Universidade de São Paulo, Caixa Postal 668,
13560-970, São Carlos, SP, Brazil
hmr@icmc.usp.br

JIANHONG WU
Laboratory for Industrial and Applied Mathematics,
Department of Mathematics and Statistics,
York University, Toronto, Canada M3J1P3
wujh@mathstat.yorku.ca

LUÍS R. A. GABRIEL
Departamento de Matemática Aplicada e Estatística,
Instituto de Ciências Matemáticas e de Computação,
Universidade de São Paulo, Caixa Postal 668,
13560-970, São Carlos, SP, Brazil

Received March 13, 2010

In this series of papers, we study issues related to the synchronization of two coupled chaotic discrete systems arising from secured communication. The first part deals with uniform dissipativity with respect to parameter variation via the Liapunov direct method. We obtain uniform estimates of the global attractor for a general discrete nonautonomous system, that yields a uniform invariance principle in the autonomous case. The Liapunov function is allowed to have positive derivative along solutions of the system inside a bounded set, and this reduces substantially the difficulty of constructing a Liapunov function for a given system. In particular, we develop an approach that incorporates the classical Lagrange multiplier into the Liapunov function method to naturally extend those Liapunov functions from continuous dynamical system to their discretizations, so that the corresponding uniform dissipativeness results are valid when the step size of the discretization is small. Applications to the discretized Lorenz system and the discretization of a time-periodic chaotic system are given to illustrate the general results. We also show how to obtain uniform estimation of attractors for parametrized linear stable systems with nonlinear perturbation.

Keywords: Discrete system; uniform dissipativeness; attractor; synchronization; constructing a Liapunov function.
1. Introduction

In this paper we study the uniform dissipativeness, with respect to parameter variation, in order to obtain uniform estimates of global attractors for general discrete systems. The central idea is based on the classical Liapunov direct method. The novelty is the use of a Liapunov function, whose derivative along solutions of the considered systems is allowed to be positive in a bounded set, to obtain a uniform estimation of the attractors for those discrete systems arising from the discretization of some well-known chaotic continuous dynamical systems.

As will be shown, to adopt the Liapunov function of a continuous system to its discrete analogue we will need to impose some restrictions on the step size of the discretization, and one goal of this work is to use two important examples to show how to derive these restrictions by incorporating the Language multiplier method into the Liapunov direct approach.

There are many application problems where estimating the global attractors is an important step towards the full understanding of the complex dynamical behaviors of given systems, and normally many parameters are involved in these systems and the parameter values vary due to measurement errors and imperfection of hardware implementation. Estimation of the attractors for a given dynamical system exhibiting chaotic behaviors is particularly important for secure communication, but obtaining such an estimation is a very challenging task since chaotic behaviors are normally associated with the coexistence of both expansion and contraction in disjoint subspaces of the high dimensional phase space. Historically, this task has been achieved by using a Liapunov function, but due to the aforementioned coexistence of expansion and contraction in different subspaces the construction of a Liapunov function with negative derivative along solutions of the considered system is difficult. Our general results will require the negative derivative of the Liapunov function along solutions of the considered system only in a restricted region. This reduces substantially the difficulty of constructing a Liapunov function from the physical background of the system, as will be illustrated by various applications presented in the paper.

In 2004, Luís R. A. Gabriel, under the advice of one of the authors (H. M. Rodrigues), presented his Master Dissertation “Comportamento Assintótico de Sistemas não Lineares Discretos”.

That work includes some results related to those of this paper, but for autonomous systems and can be found in the address www.teses.usp.br/teses/dispersivos/55/55135/tde-12012005-230105/. Similar theoretical results were published by [Alberto et al., 2007], for autonomous systems. The general results in our paper can be regarded as the generalization of the two above papers and analogue of the uniform invariance principle for continuous dynamical systems developed previously in [Rodrigues et al., 2000, 2001] and [Gumeiro & Rodrigues, 2001]. We also obtain some previous related results in [Rodrigues, 1996] for autonomous systems, in [Affraimovich & Rodrigues, 1998] for nonautonomous systems and in [Carvalho et al., 1998] for infinite dimensional parabolic systems.

Here, however, will be shown, to use the same Liapunov function of a continuous system for its discrete analogue we will need to impose some restrictions on the step size of the discretization, and our general results, coupled with the Language multiplier method, allow us to derive these restrictions explicitly.

Our work is mainly motivated by the synchronization problem of secured communication system, such as the discretization of the following master-slave Lorenz system consisting of a Master System

\[
\begin{align*}
\dot{x}(t) &= -ax(t) + ay(t), \\
\dot{y}(t) &= -y(t) - \frac{r}{4}(x(t) + \alpha(t)) - (x(t) + \alpha(t))z, \\
\dot{z}(t) &= -bz(t) + (x(t) + \alpha(t))y(t) - \frac{5}{4}br,
\end{align*}
\]

and a Slave System

\[
\begin{align*}
\dot{u}(t) &= -au(t) + av(t), \\
\dot{v}(t) &= -b(v(t) - \frac{r}{4}(x(t) + \alpha(t)) - (x(t) + \alpha(t))w(t), \\
\dot{w}(t) &= -bw(t) + (x(t) + \alpha(t))v(t) - \frac{5}{4}br,
\end{align*}
\]

where \(\alpha\) is the signal to be transmitted and the parameter values \(a, b, r\) are chosen so that the master system alone should exhibit chaotic behaviors in order to maintain the security of the communication. As indicated above, due to the arbitrary choice of the signal \(\alpha\), we will have to deal with nonautonomous systems. Roughly speaking, we say that two similar coupled systems synchronize if the
distance between their solutions is sufficiently close to zero for sufficiently large values of $t$.

A general approach in the continuous case was developed in [Gameiro & Rodrigues, 2001] to establish the synchronization property for the above master-slave system, that requires a uniform estimation of the attractors for both the master and the slave systems. Our ultimate goal is to develop a parallel theory for the discretized analogue and indeed a very general theory for robust synchronization of coupled discrete systems: this has obvious implications in terms of both hardware implementation and numerical simulations. This part of our work focuses on the first part of this theory — the uniform estimation of the global attractor.

The remaining part of this paper is organized as follows. In Sec. 2, we state and prove our main general results of uniform dispativeness for nonautonomous systems and uniform invariance for autonomous systems. These general results are then illustrated by applications to the estimation of the global attractor for the discretized analogue of the well-known chaotic Lorenz system (Sec. 3) and a periodic-time chaotic system (Sec. 4) when the parameters are in the region where chaotic behaviors are observed. Illustrated by these two examples is also the approach to extend the Liapunov function from continuous systems to their discrete analogues, that involves a delicate treatment of the step size of the discretization. We will also obtain, in Sec. 4, a uniform estimation for the attractor of a nonlinear perturbation of a linear stable parametric system that requires some careful analysis of the spectral radius for a family of linear bounded operators in Banach spaces.

2. Main Results

Let $\Lambda$ be a subset of a Banach space $E$ and let $V : \mathbb{R}^p \times \Lambda \to \mathbb{R}_+$ and $T : \mathbb{R}^p \times \mathbb{Z} \times \Lambda \to \mathbb{R}^n$ be continuous functions. Consider the discrete system

$$x(n+1) = T(x(n), n, \lambda).$$  (1)

A solution of (1) with $n \in J$ for a given subset $J$ of $\mathbb{Z}$ is a sequence $x(n)$ that satisfies (1) for all $n \in J$. Unless specified otherwise, by a solution of (1), we will mean a solution defined for all $n \geq 0$ (namely, $J = \mathbb{Z}_+$) and $x(0)$ will be called the initial value.

For each $(x, n, \lambda) \in \mathbb{R}^p \times \mathbb{Z} \times \Lambda$ we define

$$V(x(n), n, \lambda) = V(T(x(n), n, \lambda), \lambda) - V(x, \lambda).$$

Therefore, if $x(n)$ is a solution of (1) for $n \in J$, then

$$V(x(n), n, \lambda) = V(x(n+1), \lambda) - V(x(n), \lambda)$$

provided that $n \in J$ and $n + 1 \in J$.

**Theorem 1 (Uniform Dispativeness).** Let $a, b : \mathbb{R}^p \to \mathbb{R}_+$ be continuous functions such that

$$a(x) \leq V(x, \lambda) \leq b(x), \quad \forall (x, \lambda) \in \mathbb{R}^p \times \Lambda \quad (2)$$

and $a(x) \to \infty$ as $|x| \to \infty$. For every $p > 0$, define $A_p := \{x \in \mathbb{R}^p : a(x) \leq \rho\}$ and $B_p := \{x \in \mathbb{R}^p : b(x) \leq \rho\}$. We assume that there exists $H > 0$ so that for every $p \in [0, H]$ the set $A_p$ is connected.

Assume further that there exists a continuous function $c : AH \to \mathbb{R}$ such that

$$-V(x, n, \lambda) \geq c(x), \quad \forall (x, n, \lambda) \in AH \times \mathbb{Z} \times \Lambda, \quad (3)$$

and that the set $\{x \in AH : c(x) < 0\} \neq \emptyset$, and the set $\mathcal{C} := \{x \in AH : c(x) \leq 0\}$ is bounded. Finally, assume that we can find positive constants $R, \mu$ and $H$ so that

$$\max_{x \in \mathcal{C}} b(x) < R < \infty - \mu < \min_{x \in \mathcal{C}} c(x)$$

and $R + \mu < H$.  

(4)

Then for each $\lambda \in \Lambda$ the following holds:

(i) If $x_0 \in B_H$ and if $x(n)$ is the solution of (1) with initial $x_0$, then $x(n) \in A_{R+\mu}$ for every $n \in \mathbb{Z}_+$.

(ii) If $x_0 \in A_H$ then there exists $n_0 = n_0(x_0, \lambda) \geq 0$ such that $x(n_0) \in B_R$ and so $x(n) \in A_{R+\mu}$ for every $n \geq n_0$.

(iii) If $x_0 \in A_H$ and $x(n)$ is the solution of (1) with initial value $x_0$, then there exists an increasing subsequence $n_j$ such that $x(n_j) \in B_R$.

(iv) If $x(n)$ satisfies (1) for every $n \in \mathbb{Z}$ and if $x(n) \in AH$ for every $n \in \mathbb{Z}$, then $x(n) \in A_{R+\mu}$ for every $n \in \mathbb{Z}$.

**Proof.** Whenever the argument does not need to specify the dependence on $\lambda$, we shall write $T(\cdot, n, \lambda)$ and $V(\cdot) := V(\cdot, \lambda)$. We first note from the above assumptions that if $0 \leq \rho \leq H$ then $B_\rho \subset V_\rho \subset A_\rho$, where $V_\rho := \{x \in AH : V(x) \leq \rho\}$. Also, $\mathcal{C} \subset B_R$ and $\mathcal{C} \neq B_R$.

We now claim that if $R + \mu \leq \rho \leq H$ then $V_\rho$ is positively invariant with respect to (1). To show this, we note that if $x_0 \in B_R$ then $-V(x(0), n) \geq c(x_0) \geq -\mu$ and so

$$V(x(1)) - V(x(0)) = V(x(0), 0) \leq \mu.$$

Therefore, if $x(n)$ is a solution of (1) for $n \in J$, then

$$V(x(n), n, \lambda) = V(x(n+1), \lambda) - V(x(n), \lambda)$$

provided that $n \in J$ and $n + 1 \in J$.
Therefore, \( V(x(1)) \leq V(x(0) + \mu \leq b(x_0) + \mu \leq R + \mu \leq \rho \). This shows that if \( x_0 \in B_R \) then \( x(1) \in V_{\rho} \). On the other hand, if \( x(0) \in V_{\rho} \setminus B_R \), then \( c(x(0)) > 0 \) and so
\[
-V(T(x(0), 0)) + V(x(0)) = -V(T(x(0), 0)) \geq c(x(0)) > 0,
\]
from which it follows that \( V(x(1)) = V(T(x(0), 0)) < V(x(0)) \leq \rho \). This, via the standard induction argument, implies that \( V_{\rho} \) is positively invariant.

In particular, if we take \( \rho = R + \mu \) and if \( x(0) \in B_R \) then \( x(n) \in V_{\rho} \) for all \( n \in \mathbb{Z} \), and this completes the proof of (i).

We now prove (ii). We claim that if \( x(0) \in A_H' \setminus B_R \) then there exists \( m \geq 1, m = m(x(0), \lambda) \) such that \( x(m) \in B_R \). Let \( \rho = \rho(x(0), \lambda) = \max \{V(x(0)), R + \mu\} \). Then the set \( V_{\rho} \) is positively invariant. Let \( W := V_{\rho} \setminus B_R \). It is easy to see that if \( x \in W \) then \( c(x) > 0 \). Let \( \beta := \min \{c(x) > 0\} \). Since \( x(0) \in W \) we have \( c(x(0)) > 0 \). Then
\[
-V(T(x(0), 0)) + V(x(0)) = -V(T(x(0), 0)) \geq c(x(0)) \geq \beta
\]
and
\[
V(x(1)) = V(T(x(0), 0)) \leq V(x(0)) - \beta.
\]
If \( x(1) \in B_R \), we are done, otherwise if \( x(1) \in V_{\rho} \setminus B_R \), then we can repeat the above procedure to obtain
\[
V(x(2)) \leq V(x(1)) - \beta \leq V(x(m - 1)) - \beta.
\]
If \( x(2) \in B_R \), we are done, otherwise we continue the above procedure. Inductively, if after \( n - 1 \) times we still have \( x(n - 1) \notin B_R \), then
\[
V(x(n)) \leq V(x(0)) - n \beta.
\]
This shows that there exists \( m \) such that \( x(m) \in B_R \), otherwise we would have a contradiction because \( V \geq 0 \). This proves the claim.

Conclusions (ii) and (iii) are obvious consequences of the above claim.

It remains to prove (iv). Under the hypothesis of (iv), and suppose that there exists \( m \in \mathbb{Z} \) such that \( x(m) \in A_H' \setminus V_{\rho} \). Let \( \nu := \min \{c(x) : x \in A_H \setminus V_{\rho} \} \). It is clear that \( \nu > 0 \). From (ii) it follows that \( x(m - 1) \in A_H \setminus V_{\rho} \). Then
\[
-V(x(m)) - V(x(m - 1)) \geq c(x(m - 1)) \geq \nu.
\]
This implies that
\[
V(x(m)) + \nu \leq V(x(m - 1)).
\]
Since \( x(n - 2) \in A_H' \setminus V_{\rho} \), if we repeat the above procedure we obtain,
\[
V(x(m)) + 2\nu \leq V(x(m - 1)) + \nu \leq V(x(m - 2)).
\]
Inductively, after \( n \) steps we will obtain
\[
V(x(m)) + n\nu \leq V(x(m - n)) \leq b(x(m - n)).
\]
This would give a contradiction because \( b \) is a bounded function on \( A_H \).

We now consider a special case where the system is autonomous. The next proposition can be found in [LaSalle, 1976].

**Proposition 1 (Invariance Principle).** Let \( V : G \rightarrow \mathbb{R} \) be continuous, where \( G \) is an open set of \( \mathbb{R}^p \). Let \( T : \mathbb{R}^p \rightarrow \mathbb{R}^p \) be a continuous function. Suppose that
\[
V(x) = V(T(x)) - V(x) \leq 0 \text{ for every } x \in G.
\]
If \( x_0 \in G \) and \( T^n(x_0) \in G \) for every \( n \in \mathbb{N} \) and is bounded, then the \( \omega \)-limit set \( \omega(x_0) \neq \emptyset \) and there exists a real number \( c \) such that \( T^n(x_0) \rightarrow M \cap V^{-1}(c) \) where \( M \) is the largest positively invariant set contained in the set \( E := \{x \in \mathbb{R}^p : V(x) = 0\} \).

**Theorem 2 (Uniform Invariance Principle).** Let \( T : \mathbb{R}^p \times \Lambda \rightarrow \mathbb{R}^p, V : \mathbb{R}^p \times \Lambda \rightarrow \mathbb{R}_+ \). Suppose that there are continuous functions \( a, c : \mathbb{R}^p \rightarrow \mathbb{R} \) such that \( 0 \leq a(x) \leq V(x, \lambda) \), \( a(x) \rightarrow \infty \) as \( |x| \rightarrow \infty \), and \( -V(x, \lambda) \geq c(x) \geq 0 \) for every \( x \in \mathbb{R}^p \) and every \( \lambda \in \Lambda \). Let \( x_0 \in \mathbb{R}^p \) and \( \lambda_0 \in \Lambda \). Then \( E_\nu := \{x \in \mathbb{R}^p : c(x) = 0\} \neq \emptyset \) and \( T^n(x_0, \lambda_0) \) (ends, as \( n \rightarrow \infty \), to the largest invariant set of \( T(\cdot, \lambda_0) \) contained in \( E_\nu \).

**Proof.** We denote \( T(\cdot) = T(\cdot, \lambda_0) \) and \( V(\cdot) = V(\cdot, \lambda_0) \). Since \( c(x) \geq 0, \forall x \in \mathbb{R}^p \), \( V(x) \geq c(x) \geq 0 \) \( \Rightarrow -V(x) \geq 0, \forall x \in \mathbb{R}^p \). Moreover,
\[
V(x) \leq 0
\]
\[
\Rightarrow V(T(x_0)) - V(x_0) \leq 0
\]
\[
\Rightarrow V(T(x_0)) \leq V(x_0)
\]
\[
\Rightarrow V(T^n(x_0)) \leq \cdots \leq V(T^2(x_0)) \leq V(T(x_0)) \leq V(x_0)
\]
\[
\Rightarrow V(T^n(x_0)) \leq V(x_0) = \rho(x_0, \lambda_0) = \rho
\]
\[
\Rightarrow T^n(x_0) \in V_\rho = V(\lambda_0)
\]
\[
:= \{x \in \mathbb{R}^m : V(x) \leq \rho\},
\]
Then $T'(x_0)$ is bounded because $x \in V \Rightarrow V(x, \lambda) \leq \rho \Rightarrow a(x) \leq V(x, \lambda) \leq \rho \Rightarrow a(x) \leq \rho$. Since $a(x) \to \infty$ as $|x| \to \infty$, then $A_\rho := \{x \in \mathbb{R}^p : a(x) \leq \rho\}$ is bounded. We conclude that $V$ is a Lyapunov function in $\mathbb{R}^p$. Since every solution $x(n) = x(n, \lambda_0) = T^n(x_0)$ is bounded in $\mathbb{R}^p, \forall n \in \mathbb{N}$, then by Proposition 1, $x(n)$ tends to the largest invariant set contained in $\mathcal{E} = \mathcal{E}_\rho = \{x \in \mathbb{R}^p : V(x) = 0\}$ as $n \to \infty$.

Furthermore,

$x \in \mathcal{E} \Rightarrow V(x) = 0 \Rightarrow \dot{V}(x) = -\dot{V}(x) \geq c(x) \geq 0.$

$\Rightarrow c(x) = 0 \Rightarrow x \in \mathcal{L} = \mathcal{E} \subset \mathcal{L}_\rho$.

By Proposition 1, $\theta \neq \omega(x(0)) \subset \mathcal{E}$, and this completes the proof. \blackslug

3. Applications to Discretized Lorenz System

As an application of the general results in the previous sections, we consider the map

$T(x, y, z, n, \lambda) := (T_1(x, y, z, n, \lambda), T_2(x, y, z, n, \lambda),$ $T_3(x, y, z, n, \lambda)),$

where $x, y, z \in \mathbb{R}, n \in \mathbb{Z}$, and $\lambda := (a, r, b, (\alpha_n), h)$, where $a \in [a_m, a_M], r \in [r_m, r_M], b \in [b_{\min}, b_{\max}], h$ are positive real numbers and $(\alpha_n) \in \mathcal{E}_\rho$. The map $T$ is obtained by discretizing the Lorenz equation, and hence

$T_1(x, y, z, n, \lambda) = x + h[-ax + ay],$

$T_2(x, y, z, n, \lambda) = y + h[-y - r(x + \alpha)],$

$T_3(x, y, z, n, \lambda) = z + h[-bz + (x + \alpha_n)y - \frac{5}{4}br].$

(5)

Our goal in this section is to use the above uniform dissipativity theorem to obtain an uniform estimate of the attractor of $T$. Let us consider the function $V(x, y, z, \lambda) := r_\alpha x^2 + 4a_\alpha y^2 + 4a_\alpha z^2$. Then for every $(x, y, z, \lambda)$ we have the following inequalities:

$a(x, y, z) := r_\alpha x^2 + 4a_\alpha y^2 + 4a_\alpha z^2 \leq V(x, y, z, \lambda),$ and

$V(x, y, z, \lambda) \leq r_\alpha x^2 + 4a_\alpha y^2 + 4a_\alpha z^2 := b(x, y, z).$

Let us compute the derivative:

$-\dot{V}(x, y, z, \lambda, n) = -V(T_1, T_2, T_3) + V(x, y, z)$

$= rx^2 + 4ay^2 + 4az^2$

$- \left\{ r[x + h(-ax + ay)]^2 + 4a[y + h\left(-y - \frac{r}{4}(x + \alpha) - (x + \alpha)z\right)]^2 + 4a\left[z + h\left(-bz + (x + \alpha_n)y - \frac{5}{4}br\right)\right]^2 \right\}$

$= rx^2 + 4ay^2 + 4az^2$

$- \left\{ r[\alpha x + h(2x)](-ax + ay) + h^2(-ax + ay)^2 \right\}$

$+ 4a\left[g^2 + h(2r)\left(-y - \frac{r}{4}(x + \alpha) - (x + \alpha)z\right)\right]^2$

$+ h^2\left(-y - \frac{r}{4}(x + \alpha) - (x + \alpha)z\right)^2$

$+ 4a\left[z + h(2z)\left(-bz + (x + \alpha_n)y - \frac{5}{4}br\right)\right]^2$

$+ h^2\left(-bz + (x + \alpha_n)y - \frac{5}{4}br\right)^2 \right\}$

$= h(2ar^2x^2 - 2arxy + 8ay^2 + 2ary(x + \alpha_n)$

$+ 8abz^2 + 10abr)$ - $h^2 g(x, y, z, \lambda, n).$

where

$g(x, y, z, \lambda, n) := r(-ax + ay)^2$

$+ 4a\left[y - \frac{r}{4}(x + \alpha_n) - (x + \alpha)z\right]^2$

$+ 4a\left[z + h(x + \alpha_n)y - \frac{5}{4}br\right]^2.$

Therefore,

$-\dot{V}(x, y, z, \lambda, n)$

$= 2ar^2x^2 + 8ay^2 + 2ary\alpha_n + 8abz^2 + 10abrz$

$+ h g(x, y, z, n, \lambda)$
where we assumed that $|a_n| \leq \gamma$, $\forall n \in \mathbb{Z}_+$.

Let $d(x, y, z) := 2\alpha_m r_m x^2 + 8\alpha_m y^2 - 2\alpha_M r_M |y| - 8\alpha_m b_m z^2 - 10\alpha_M b_M |z| - h|g(x, y, z, n, \lambda)|$, where $h := h_0$.

The minimum of $d(x, y, z)$ is given by

$$d(x, y, z) = 2\alpha_m r_m x^2 + 8\alpha_m \left[ y^2 - 2|y|\frac{\alpha_M r_M}{S_m} \right] + 8\alpha_m b_m \left[ z^2 - 2|z|\frac{\alpha_M b_M}{S_m b_m} \right] + 5\alpha_M b_M r_M \left[ \frac{\alpha_M}{S_m b_m} \right]^2 - \frac{(\alpha_M r_M)^2}{S_m} - \frac{(\alpha_M b_M r_M)^2}{S_m b_m}.$$
Substituting these values to \( q(y, z) = 0 \), we get

\[
8a_m \left( \frac{a_M r_M \gamma}{8a_m - 4a_M} \right)^2 + 8a_m b_m \left( \frac{5a_M b_M r_M \gamma}{5a_M b_M - 4a_M} \right)^2 - \left( \frac{a_M r_M \gamma}{8a_m - 4a_M} \right)^2 - \left( \frac{5a_M b_M r_M \gamma}{5a_M b_M - 4a_M} \right)^2 = -699.799 \frac{156.641}{1176.490} \frac{756.499}{756.499} \frac{594.819468623618}{594.819468623618}.
\]

we have

\[
-\hat{V}(x, y, z, \lambda, n) \geq d_1(x, y, z) \geq -595 - \delta = -596,
\]

and so

\[
-\hat{V}(x, y, z, \lambda, n) \geq bd_1(x, y, z) = c(x, y, z) > -596 h > -596 h_0
\]

for all \( h \in (0, h_0) \).

If we restrict \( h_0 \) so that \( h_0 < 1/596 \), then we have

\[
c(x, y, z) > -1 := -\mu.
\]

Consequently, we can apply Theorem 1 to conclude that the attractor is contained in the ellipsoid:

\[
A_{R^{+}\mu} = A_{96.999+1} = A_{97.000} = \{(x, y, z) \in A_H : r_m x^2 + 4a_m y^2 + 4a_m z^2 \leq 97000\}
\]

and

\[
\{(x, y, z) \in A_H : \left\{ \frac{49}{3} x^2 + \frac{196}{3} y^2 + \frac{196}{3} z^2 \leq 300000 \right\}
\]

is the basin of attraction.

Fig. 1. Ellipsoid that contains discrete Lorenz attractor.
We refer to Fig. 1 for a schematic illustration of various functions and their graphs involved.

4. Applications to a Time-Periodic System

The following continuous chaotic system was studied in [Afraimovich et al., 1986], where they were interested in analyzing coupled damped-excited Duffing equations

\[ x + \alpha x + \omega x + (q \cos t + x^3) x = 0. \]

We are here considering a corresponding discrete system:

\[ T_1(x, y, \lambda) = x + hy, \]

\[ T_2(x, y, \lambda) = y + h[-\omega y - (q \cos t + x^3)x], \]

where \( \lambda := (\alpha, \omega, q, t). \)

To find an estimate of the attractor, we use the following Liapunov function:

\[ V(x, y, \lambda) := \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}y^2 + 5xy. \]

It is easy to see that \( a \leq V \leq b, \) where

\[ a(x, y) = \frac{1}{2}\omega_m x^2 + \frac{1}{4}x^4 + \frac{1}{2}y^2 + 5xy, \]

\[ b(x, y) = \frac{1}{2}\omega_M x^2 + \frac{1}{4}x^4 + \frac{1}{2}y^2 + 5xy. \]

Then

\[
V = \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}y^2 + 5xy - \left\{ \frac{1}{2}(x + hy)^2 + \frac{1}{4}(x + hy)^4 \right\}
\]

\[ + \frac{1}{2}(y + h(-\omega y - (q \cos t + x^3) x)^2 + 5(x + hy)(y + h(-\omega y - (q \cos t + x^3)x))) \]

\[ = \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}y^2 - \left\{ \frac{1}{2}(x^2 + 2hx\alpha y + \frac{1}{4}(x^4 + 4hx^3y) \right\}
\]

\[ + \frac{1}{2}(y^2 + 2hy(-\omega y - (q \cos t + x^3)x) + 5xy + 5hx(-\omega y - (q \cos t + x^3)x) + O(h^2) \]

\[ = -h(\omega y x + x^4 y + [p(-\omega y - (q \cos t + x^3) x)] + 5x(-\omega y - (q \cos t + x^3) x) + O(h^2) \]

\[ = h(\omega y x + x^4 y - \omega xy - qyx \cos t - yx^3 - 5axy - 5x^2 y - 5q\omega x^2 \cos t - 5x^4) + O(h^2) \]

\[ \geq h(\alpha_m y^2 - (q_M + 5\alpha_M) x\lvert y \rvert + 5\omega_m x^2 - 5q_M x^2 + 5x^4) + O(h^2) \]

\[ = h \left\{ \alpha_m y^2 - \frac{(q_M + 5\alpha_M)^2}{\sqrt{\alpha_m}} x\lvert y \rvert + 5\omega_m x^2 - 5q_M x^2 + 5x^4 \right\} + O(h^2) \]

\[ \geq h \left\{ \alpha_m y^2 - \frac{(q_M + 5\alpha_M)^2}{\sqrt{2\alpha_m}} y^2 - \frac{\alpha_m}{2} y^2 + 5\omega_m x^2 - 5q_M x^2 + 5x^4 \right\} + O(h^2) \]

\[ \geq h \left\{ \alpha_m y^2 - \frac{(q_M + 5\alpha_M)^2}{\sqrt{2\alpha_m}} + 5q_M - 5\omega_m \right\} x^2 + 5x^4 \right\} + O(h^2). \]

Let \( d(x, y) := (\alpha_m/2)y^2 - [(q_M + 5\alpha_M)^2/2\alpha_m] + 5\omega_m - 5\omega_m x^2 + 5x^4 \) and let \( D := \{(x, y) : d(x, y) \geq 0 \}. \) If we let \( J := (q_M + 5\omega_m)^2/2\omega_m + 5q_M - 5\omega_m, \) we have \( d(x, y) = (\alpha_m/2)y^2 - Jx^2 + 5x^4. \)

Now we consider the parameters varying with 2\% of uncertainty from the basic values \( \alpha = 1, \omega = 1, q = 50, \) that is:

\[ \alpha_m = 1 - \frac{2}{100} = \frac{49}{50}, \quad \alpha_M = 1 + \frac{2}{100} = \frac{51}{50}, \]

\[ \omega_m = 1 - \frac{2}{100} = \frac{49}{50}, \quad \omega_M = 1 + \frac{2}{100} = \frac{51}{50}, \]

\[ q_m = 50 - 1 = 49, \quad q_M = 50 + 1 = 51. \]
Similarly, or

\begin{equation}
\begin{aligned}
    x & = 25a_m(20p - 1) - 2(p \alpha m - 1)^2(20p - 1)J \\
    & \quad + 10(p \alpha m - 1)[25 + (p \alpha m - 1)(w_M + 2pJ)] \\
    & = 0.
\end{aligned}
\end{equation}

Using Mathematica we solve numerically the above equation and we find the following Lagrange multipliers: $p = 1.1437812609853224$, $p = 0.0937535609612867$, $p = 0.0935563158936598$. The maximum will be obtained using the first Lagrange multiplier. Using (10) and (9) we obtain the maximum point:

\[
(x_M, y_M) = (14.267595144549922, 500.0301939691485).
\]

This gives $b(x_M, y_M) = 226.623$. As in the previous example, we take $R_1 = 226.624$ and $R = 226.625$. We can also compute the minimum of the function $d$, and use Theorem 1 to conclude that for sufficiently small $b$ the attractor is contained in the set $A = \{(x, y) : a(x, y) \leq 226.625\}$.

\[
\begin{aligned}
    A & = \left\{(x, y) : \frac{1}{2}w_M x^2 + \frac{1}{2}w_M y^2 + 5xy \right. \\
    & \leq 226.625 \right\}.
\end{aligned}
\]

Figure 3 illustrates the estimate and the above argument.

### 4.1. Nonlinear perturbations of parametrized linear stable systems

In this section, we consider nonlinear perturbations of a linear stable system with parameters, and we wish to obtain a uniform bound for the attractor. We start with some notations. Let $X$ be a Banach space and let $\mathcal{L}(X)$ be the space of bounded operators from $X$ to $X$, with its usual norm, and let $A \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. If $A - \lambda I$ is invertible with $R(I(A)) := (A - \lambda)^{-1} \in \mathcal{L}(X)$, the resolvent operator of $A$, then we say that $\lambda \in \sigma(A)$, the resolvent set of $A$. We indicate its complement in $\mathbb{C}$ by $\sigma(A)$, the spectrum of $A$. It is well known that $\sigma(A)$ is compact and $\lim_{n \to \infty} |A^n|^{1/n} = \inf_{n=1,2\ldots} |A^n|^{1/n} = \max\{|\lambda|, \lambda \in \sigma(A)\} := r(A)$, the spectral radius of $A$. See [Kato, 1976, p. 27].
We will also need the following technical lemma (see [Kato, 1976, p. 280]):

Lemma 1. A map \( L(X) \ni A \mapsto \sigma(A) \) is upper semicontinuous, that is, given \( A \in L(X) \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \sigma(B) \subset V_\varepsilon(\sigma(A)) \) if \( |B - A| < \delta \) and \( B \in L(X) \).

Proposition 2. The map \( L(X) \ni A \mapsto r(A) \) is upper semicontinuous, that is, given \( A \in L(X) \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( r(B) < r(A) + \varepsilon \) if \( |B - A| < \delta \) and \( B \in L(X) \).

Proof. Let \( A \in L(X) \) and \( \varepsilon > 0 \). Since the map \( L(X) \ni A \mapsto \sigma(A) \) is upper semicontinuous, there exists \( \delta > 0 \) such that \( \sigma(B) \subset V_{2\varepsilon}(\sigma(A)) \) if \( |B - A| < \delta \) and \( B \in L(X) \). There exists \( \lambda_1 \in \sigma(B) \) such that \( r(B) = |\lambda_1| \). Since \( \lambda_1 \in V_{\varepsilon}(\sigma(A)) \), there exists \( \lambda_0 \in \sigma(A) \) such that \( |\lambda_1 - \lambda_0| < \varepsilon \). Therefore, \( r(B) < r(A) + \varepsilon \), because

\[
|\lambda_1| - |\lambda_0| \leq |\lambda_1 - \lambda_0| < \varepsilon \Rightarrow |\lambda_1| < |\lambda_0| + \varepsilon \leq r(A) + \varepsilon.
\]

Proposition 3. Let \( X \) be a Banach space, \( \Gamma \) a compact subset of another Banach space and \( f : \Gamma \to \mathbb{R} \) a lower semicontinuous function, that is, given \( \gamma \in \Gamma \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( f(\gamma') < f(\gamma) - \varepsilon \) if \( |\gamma' - \gamma| < \delta \) and \( \gamma' \in \Gamma \).

Proof. We claim that \( f \) is bounded from above in \( \Gamma \). In fact, let us suppose that there exists a sequence \( \gamma_n \in \Gamma \) such that \( f(\gamma_n) \to \infty \). We can assume that \( f(\gamma_n) > 0 \), \( \forall n \in \mathbb{N} \). Since \( X \) is compact, we can also suppose that \( \gamma_n \to \gamma_0 \in \Gamma \). Let \( \varepsilon > 0 \). As \( f \) is upper semicontinuous, there exists \( \delta > 0 \) such that \( f(\gamma) < f(\gamma_0) + \varepsilon \) if \( \gamma \in B_\delta(\gamma_0) \). Since \( \gamma_n \to \gamma_0 \), there exists \( n_0 \in \mathbb{N} \) such that for \( n \geq n_0 \), we have \( \gamma_n \in B_\delta(\gamma_0) \), which implies that \( f(\gamma_n) < f(\gamma_0) + \varepsilon \) and this is a contradiction. Then \( f \) is bounded from above.

Let \( S := \sup_{\gamma \in \Gamma} f(\gamma) \in \mathbb{R} \). Then, \( S \geq f(\gamma), \forall \gamma \in \Gamma \). There exists \( \gamma_0 \in \Gamma \) such that \( f(\gamma_0) \to S \). We can suppose that \( \gamma_0 \to \gamma_0 \in \Gamma \). Let us show that \( f(\gamma) = S \). In fact, let \( \varepsilon > 0 \) be a real number. Since \( f \) is upper semicontinuous, there exists \( \delta > 0 \) such that \( f(\gamma) < f(\gamma_0) + \varepsilon \), if \( \gamma \in B_\delta(\gamma_0) \). Since \( \gamma_0 \to \gamma_0 \), let \( n_0 \in \mathbb{N} \) be such that for \( n \geq n_0 \), we have \( \gamma_n \in B_\delta(\gamma_0) \), which implies that \( f(\gamma_n) < f(\gamma_0) + \varepsilon \). Then by letting \( n \to \infty \), we have \( S \leq f(\gamma) + \varepsilon \) for every \( \varepsilon > 0 \). Therefore, \( S \leq f(\gamma) \) and so, \( S = f(\gamma) \).

In a similar way, we obtain

Proposition 4. Let \( X \) be a Banach space, \( \Gamma \) a compact subset of another Banach space and \( f : \Gamma \to \mathbb{R} \) a lower semicontinuous function, that is, given \( \gamma \in \Gamma \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( f(\gamma') < f(\gamma) + \varepsilon \) if \( |\gamma' - \gamma| < \delta \) and \( \gamma' \in \Gamma \).

Theorem 3. Let \( X \) be a Banach space, \( \Gamma \) a compact subset of another Banach space. We assume that the map \( \Gamma \ni \gamma \mapsto A_\gamma \in L(X) \). We also suppose that \( |\lambda| < 1, \forall \lambda \in \sigma(A_\gamma) \). Then there exists \( \theta \in (0,1) \) such that \( r(A_\gamma) \leq \theta < 1, \forall \gamma \in \Gamma \).

Proof. From Proposition 1, the map \( L(X) \ni A \mapsto r(A) \) is upper semicontinuous. By Proposition 3, there exists \( \gamma \in \Gamma \) such that \( r(A_\gamma) \) is also upper semicontinuous. By Proposition 3, there exists \( \gamma \in \Gamma \) such that \( r(A_\gamma) \geq r(A_\gamma), \forall \gamma \in \Gamma \). Taking \( \theta := r(A_\gamma) = \sup_{\gamma \in \Gamma} |\sigma(A_\gamma)| \), we have \( r(A_\gamma) \leq \theta < 1, \forall \gamma \in \Gamma \).

We note that for any compact \( K \subset C, K \) compact, there exists a Hilbert space \( H \) and an operator \( A \in L(H) \) such that \( \sigma(A) = K \). This result can be found in [Taylor, 1958, p. 263]. We also note that for a given Banach space \( (X, |\cdot|) \), and given \( A \in L(X) \) and \( \varepsilon > 0 \), there exists \( \delta = \frac{|\varepsilon|}{\|A\|} \), a norm equivalent to \( |\cdot| \), such that \( r(\lambda) \leq |\lambda| \leq r(\lambda) \). This result can be found in [Irwin, 1980] or in [Rodrigues & Solórzano-Morales, 2004] for a more general case. As a consequence,
we have

**Corollary 1.** Let $(X, |·|)$ be a Banach space and $A \in \mathcal{L}(X)$. Suppose $|\lambda| < 1, \forall \lambda \in \sigma(A)$. Then $A$ is a contraction with respect to some norm equivalent to $|·|$ on $X$.

**Proposition 5.** Let $X$ be a Banach space, $\Gamma$ a compact subset of another Banach space. Suppose that the map $\gamma \mapsto A_\gamma \in \mathcal{L}(X)$ is continuous. Then there exist $M > 0$ and $\theta \in (0, 1)$ such that $|A_\gamma| \leq M\theta^k$ for every $\gamma \in \Gamma$ and $k \in \mathbb{N}$.

**Proof.** Let $\gamma \in \Gamma, k \in \mathbb{N}, A = A_\gamma \in \mathcal{L}(X), M := \sup_{\lambda \in C_0} (|A_\lambda - \lambda|^{-1}) = \max_{\lambda \in C_0} (|A_\lambda - \lambda|^{-1}), C_0$, the circle in the complex plane with center at the origin and radius $\theta$. Then by [Taylor, 1958, p. 287], we have

\[
A_\gamma = \frac{1}{2\pi i} \int_{C_0} \lambda^k R\lambda d\lambda
\]

\[
\Rightarrow A_\gamma = \frac{1}{2\pi i} \int_{C_0} \lambda^k (A - \lambda)^{-1} d\lambda
\]

\[
|A_\gamma| \leq \frac{1}{2\pi i} \sup_{\lambda \in C_0} |\lambda^k (A_\lambda - \lambda)^{-1}| 2\pi \theta^k \leq M\theta^k
\]

Let $H$ be a Hilbert space and $A$ a bounded linear operator in $H$. By [Halmos, 1951, p. 39], there exists a unique operator $A^*$, the adjoint operator of $A$, such that $(Ax, y) = (x, A^*y)$ for every $x, y \in H$, and $|A^*| = |A|$.

**Proposition 6.** Let $X$ be a Banach space, $\Gamma$ a compact subset of another Banach space and $f_k : \Gamma \to X$, $k = 1, 2, \ldots$ a sequence of continuous functions with $|f_k(\gamma)| \leq \alpha_k, \forall \gamma \in \Gamma$ and $\sum \alpha_k < \infty$. Then the map $\gamma \mapsto \sum_{k=0}^\infty f_k(\gamma) =: F(\gamma)$ is uniformly continuous on $\Gamma$.

**Proof.** Given $\varepsilon > 0$, $\exists k_0(\varepsilon)$ such that $n \geq k_0$ and

\[
\sum_{k=0}^{k_0} f_k(x) - F(\gamma) < \varepsilon
\]

\[
\Rightarrow \sum_{k=k_0+1}^\infty f_k(\gamma) \leq \sum_{k=k_0+1}^\infty \alpha_k < \varepsilon, \forall \gamma \in \Gamma.
\]

So, we have

\[
|F(\gamma_1) - F(\gamma_2)| = \left| \sum_{k=0}^{k_0} f_k(\gamma_1) + \sum_{k=k_0+1}^\infty f_k(\gamma_1) - \sum_{k=0}^{k_0} f_k(\gamma_2) - \sum_{k=k_0+1}^\infty f_k(\gamma_2) \right|
\]

\[
= \left| \sum_{k=k_0}^\infty (f_k(\gamma_1) - f_k(\gamma_2)) \right| + 2\varepsilon.
\]

We consider now $f_k, k = 0, \ldots, k_0$. Then, $\exists \delta_k = \delta_k(\varepsilon) > 0$ for $k = 0, \ldots, k_0$

\[
|f_k(\gamma_1) - f_k(\gamma_2)| < \frac{\varepsilon}{k_0 + 1}, \text{ provided } |\gamma_1 - \gamma_2| < \delta_k.
\]

Take $\delta = \min\{\delta_k, k = 0, \ldots, k_0\}$. Then

\[
|\gamma_1 - \gamma_2| < \delta \Rightarrow |F(\gamma_1) - F(\gamma_2)| < \frac{\varepsilon}{k_0 + 1} + 2\varepsilon = 3\varepsilon.
\]

From now on we will assume that $H = \mathbb{R}^n$ or $H = C^0$, and consider the $p \times p$ matrices $A_\gamma, C$ and $B_\gamma := \sum_{k=0}^\infty (A_\gamma)^k C(A_\gamma)^k, \forall \gamma \in \Gamma$.

**Theorem 4.** If $|\lambda| < 1, \forall \lambda \in \sigma(A_\gamma)$, then the series $\sum_{k=0}^\infty (A_\gamma)^k C(A_\gamma)^k$ is uniformly and absolutely convergent and the map $\gamma \mapsto B_\gamma = \sum_{k=0}^\infty (A_\gamma)^k C(A_\gamma)^k$ is continuous.

**Proof.** In fact, by Proposition 4, there exists $M > 0$ such that $|A_\gamma| \leq M\theta^k$ for every $\gamma \in \Gamma, \theta \in (0, 1)$ and $k \in \mathbb{N}$. Then

\[
|B_\gamma| \leq \sum_{k=0}^\infty |(A_\gamma)^k| |C||A_\gamma|^k = |C| \sum_{k=0}^\infty |A_\gamma|^{2k}
\]

\[
\leq |C|M^2 \sum_{k=0}^\infty \theta^{2k} < \infty
\]

Then the series $\sum_{k=0}^\infty (A_\gamma)^k C(A_\gamma)^k$ is absolutely and uniformly convergent and the map $\gamma \mapsto B_\gamma$ is continuous by Proposition 5. □
Recall that for a given \( V : \mathbb{R}^p \to \mathbb{R} \), we say that \( V \) is positive definite if \( V(0) = 0 \) and \( V(x) > 0 \) for \( x \neq 0 \) in a neighborhood of the origin. Let \( D \) be a \( p \times p \) matrix. We say that \( D \) is positive definite if the quadratic form \((Dx)x\) is positive definite.

The following results can be verified easily.

**Proposition 7.** If \( B_c = \sum_{k=0}^{\infty} (A_k')^t C A_k \), then the following statements hold:

(i) \( A'_k B_k A_k - B_k = -C \);

(ii) If \( C \) is positive definite so is \( B_c \);

(iii) If \( C \) is self-adjoint so is \( B_c \).

We can now consider the linear parametrized system:

\[ x_{n+1} = A_n x_n \]

**Theorem 5.** Let \( A, A_n, B \) be defined as above. Let \( C \) be positive definite and suppose \(|\lambda| < 1, \forall \lambda \in \sigma(A_n) \). Let \( V_c(x) = (B_c x|x) = x^t B_c x, \forall x \). Then for every \( x \) and every \( \gamma \in \Gamma \), we have:

(i) \( V_c \) is positive definite in the whole space and there exists a constant \( c_0 > 0 \) such that \( -V_c(x) = x^t C x \geq c_0|x|^2, \forall x \in \mathbb{R}^p \).

(ii) There exist constants \( c_1 \) and \( c_2 > 0 \) such that \( c_1|x|^2 \leq V_c(x) \leq c_2|x|^2, \forall x \in \mathbb{R}^p \).

**Proof.** We first prove (i). By Theorem 4, \( B_1 \) is a continuous function of \( \gamma \). Since \( C \) is positive definite then so is \( B_c \), thanks to Proposition 7(ii). This implies that \( V \) is positive definite. To simplify the notations we set \( A := A_n, B := B_c \) and \( V := V_c \). Since from Proposition 7(i), we have \( A' B - B = -C \), we have

\[
\dot{V}(x) = V(Ax) - V(x) = (Ax)'B(Ax) - x'Bx
= x'(-C + B) x - x'Bx = -x'Cx.
\]

Since \( C \) is positive definite it follows that \(-V\) is definite positive in the whole space. Therefore, \( V \) is a Liapunov function for the dynamical system defined by \( x_{n+1} = A_n(x_n) \). Let us prove now that there exists constant \( c_0 > 0 \) such that \(-V_c(x) = x^t C x \geq c_0|x|^2, \forall x \in \mathbb{R}^p, \forall \gamma \in \Gamma \). In fact by the continuity of the function \( x \mapsto x^t C x \) and the compactness of \( S := \{ x \in \mathbb{R}^p : |x| = 1 \} \), it follows that there exists a constant \( c_0 > 0 \) such that \( c_0 := \min_{(\gamma, \gamma) \in S^t} x^t C x \). Let \( x \neq 0 \) and \( y = x/|x| \in S \). Then,

\[
c_0 \leq y'C y \Rightarrow c_0 \leq \frac{x^t C x}{|x|^2}
\Rightarrow c_0 \leq \frac{1}{|x|^2} x^t C x \Rightarrow x^t C x \geq c_0|x|^2.
\]

We now prove (ii). By the continuity of \( V_c \) and the compactness of \( S \), it follows that there exist constants \( c_1, c_2 > 0 \) such that \( c_1 := \min_{(\gamma, \gamma) \in S^t} x^t B_c x \) and \( c_2 := \max_{(\gamma, \gamma) \in S^t} x^t B_c x \). Let \( x \neq 0 \) and \( y = (x/|x|) \in S \). Then,

\[
c_1 \leq V_c(y) \leq c_2 \Rightarrow c_1 \leq y'B_c y \leq c_2
\Rightarrow c_1 \leq \frac{x'^t B_c x}{|x|^2} \leq c_2
\Rightarrow c_1 \leq \frac{1}{|x|^2} x'^t B_c x \leq c_2
\]

and so, \( c_1|x|^2 \leq V_c(x) \leq c_2|x|^2, \forall x \in \mathbb{R}^p \).

We can now state the main result of this section for nonlinear perturbations of a linear parametrized system:

**Theorem 6.** Let \( A_n, B_n \) be real matrices as in (12), \( V_c(x) = (B_c x|x) = x^t B_c x, x \in \mathbb{R}^p, C \) positive definite and self-adjoint. We suppose that \(|\lambda| < 1, \forall \lambda \in \sigma(A_n) \). Let \( f_\gamma : \mathbb{R}^p \to \mathbb{R}^p \) be continuous with \(|f_\gamma(x)| \leq L, \forall x \in \mathbb{R}^p, t_\Gamma := A_n + f_\gamma \)

(i) Then there exists a function \( c \in \mathbb{R}^p \to \mathbb{R} \) such that \(-V_c(x) \geq c(x), \forall x \in \mathbb{R}^p \) and there exists a constant \( M > 0 \) such that \( c(x) > 0, \forall x \geq M \), and \( c(x) \to \infty \) as \( |x| \to \infty \).

(ii) There exist constants \( c_1, c_2 > 0 \) such that \( c_1|x|^2 \leq V_c(x) \leq c_2|x|^2, \forall x \in \mathbb{R}^p, \gamma \in \Gamma \)

**Proof.** We first prove (i). By Theorem 4, \( B_1 \) is a continuous function of \( \gamma \). To simplify the notations we let \( A := A_n, B := B_n, f := f_\gamma, T := T_\Gamma, V := V_c \). By Proposition 7, we have \( A' B - B = -C, B \) is positive definite and self-adjoint.

Next we are going to show that function \( c \in \mathbb{R}^p \to \mathbb{R} \) such that \(-V_c(x) \geq c(x) \). In fact,

\[
\dot{V}(x) = V(Tx) - V(x)
= V(Ax + f(x)) - V(x)
= (Ax + f(x))'B(Ax + f(x)) - x'Bx
= (x'A + f'(x))B(Ax + f(x)) - x'Bx
\]
= x^*A^*B^*A x + x^*A^*B f x + f^*(x)B A x \\
+ f^*(x)B f x - x^*B x \\
= x^*A^*B^*A x + x^*A^*B f x + (x^*A^*B f x)^* \\
+ f^*(x)B f x - x^*B x \\
= x^*A^*B^*A x + 2x^*A^*B f x \\
+ f^*(x)B f x - x^*B x \\
= x^*(-C + B) x + (2x^*A^*B f x) \\
+ f^*(x)B f x (x) - x^*B x \\
= -x^*C x + (2x^*A^*B f x + f^*(x)B f x).

Then, $V(x) = -x^*C x + (2x^*A^*B f x + f^*(x)B f x)$. Since $f$ is bounded there exist constants $P > 0$ and $Q > 0$ such that $2x^*A^*B f x + f^*(x)B f x \leq P|x| + Q$, because

$$2x^*A^*B f x + f^*(x)B f x$$

$$\leq |2x^*A^*B f x + f^*(x)B f x|$$

$$\leq |2x^*A^*B f x| + |f^*(x)B f x|$$

$$\leq |2x^*A^*B f x| + |B|$$

$$\leq P|x| + Q.$$ 

For $c_0 > 0$ we have $c(x) = c_0|x|^2 - P|x| - Q$, with positive constants $c_0$, $P$ and $Q$. Moreover, $c(x) \to \infty$ as $|x| \to \infty$. Then there exists $\eta > 0$ such that $c(x) > 0$ for $|x| > \eta$.

We conclude with a simple remark that since $-V(x) \geq c(x)$, we can use Theorems 1 and 2 to obtain uniform dissipativeness and uniform invariance for the nonlinear system.

5. Conclusions

Estimations of the global attractors for dynamical systems with parameters is important for a number of application problems including the synchronization of coupled chaotic systems arising from secured communications, the main issue that motivates this series of papers. Obtaining such an estimate is highly nontrivial since the desired chaotic behavior (of the master system) are usually associated with the coexistence of both expansion and contraction in different subspaces. Not to mention that any such reasonable estimation should involve small step size of the discretization if the considered discrete system is the discretization of a continuous system (of differential equations). In this paper, we establish some general results based on the Liapunov direct method, for the uniform estimation of the attractor of nonautonomous discrete systems with parameters, and we also show how the classical Lagrange multiplier method can be effectively used to construct the required Liapunov function for two important examples from secured communications.

This is the basis for some general synchronization results of a discrete Master-slave system, to be presented in a future study.

References


