Egalitarianism Under Earmark Constraints *

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Abstract

We consider a model in which a homogeneous commodity (the resource) is shared by several agents with single-peaked preferences and capacity constraints, and the resource is coming from different suppliers under arbitrary bilateral feasibility constraints: each supplier can only deliver to a certain subset of agents. Examples include balancing the workload of machines, sharing earmarked funds between different projects, and assigning students to schools under geographic constraints.

Unlike in the one supplier model (Sprumont [18]), that we generalize, in a Pareto optimal allocation agents who get more than their peak typically coexist with agents who get less. A variant of the Gallai-Edmonds decomposition identifies these two subsets of agents, that we call respectively the over-demanded and the under-demanded side of the market. Like in the one supplier model, there is a Lorenz dominant Pareto optimal allocation. We call it the egalitarian solution, and characterize it, in the case of identical capacity constraints, by the combination of strategyproofness (truthful revelation of peaks), efficiency, and a variant of equal treatment of equals. The analysis relies on submodular optimization techniques as in Dutta and Ray [8].

Keywords: Bipartite graph, egalitarianism, Lorenz dominance, single-peaked preferences. strategy-proofness

JEL codes: C72, D63, D61, C78, D71.

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1 Introduction

Egalitarianism, the central principle of fair division, may conflict with incentives, feasibility or efficiency constraints. Maximizing the leximin ordering over profiles of relevant characteristics (a.k.a. the Rawlsian approach) is the most common implementation of egalitarianism under constraints. It is however a controversial method. Indeed, it recommends to take arbitrarily large amounts of resources from the “rich” if this allows to raise by even a tiny amount the lot of the “poor”. The only case where egalitarianism eschews this critique is when we can find a Lorenz dominant distribution of welfare, or resources: at the Lorenz dominant outcome, we simultaneously maximize the share of the $k$ poorest individuals, for any number $k$ of agents.¹ Unlike the leximin ordering that always reaches a unique maximum in any closed convex set, a Lorenz dominant outcome may not exist. We know in fact very few fair division models admitting Lorenz dominating solutions over a reasonably rich domain of problems. The two main exceptions follow.

Dutta and Ray ([8]) observed that the core of a supermodular (convex) cooperative game is one general instance where a Lorenz dominant solution exists; this solution has been known after their work as the egalitarian selection in the core. The second model, due to Sprumont, is the fair division of a single commodity under single-peaked preferences and no free disposal ([3], [18]). The uniform solution selects for each agent either his peak, or a common share in such a way that the resource is fully distributed. Although the original motivation of the uniform solution was its incentive properties ([3]), its most compelling fairness property, and its shortest definition, is to be Lorenz dominant among all Pareto optimal allocations of the resource ([7]).

We study a considerable generalization of the Sprumont model, where a homogeneous commodity (the resource) is still shared by several agents with single-peaked preferences, but the resource is coming from any number of different suppliers, under arbitrary bilateral feasibility constraints: each supplier can only deliver to a certain subset of agents. Examples where such constraints are critical include:

- Balancing the workloads of several machines, when each machine can only process certain jobs, but the processing speed is uniform. For instance assigning customers to groups of agents when language constraints limits the set of customers each group of agents can handle.

- Sharing earmarked resources, for instance dividing funds between different research projects, when the foundations, agencies, or private donors attach overlapping “strings” on the use of their gift.² For instance one donor funds projects relevant to global warming, another donor looks for projects with a Latin American component, a third one for those involving minorities, etc..

- Assigning students to schools when each school can only handle a certain subset of students—e.g., those coming from certain neighborhoods—, and these subsets overlap.

Assuming that each recipient of the resources wants to maximize her share if the resource is a “good” (money), or minimize it if it is a “bad” (workload) is a reasonable first approximation. But in most concrete examples the situation is more nuanced. Under the widespread bureaucratic constraint that funds must be spent in a given calendar year, and the belief that returning funds

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¹References on the Lorenz optimum: [16], [11], [10].

²In United States politics, an earmark is a congressional provision that directs approved funds to be spent on specific projects.
has a negative impact on future funding, a project manager does not want her budget to be too large, lest it becomes difficult to find justifiable ways to spend it. If the workload of a worker is too small, his machine or his job may soon be deemed redundant, so his most preferred workload is not zero. Similarly a school principal has in mind an ideal amount of students she would like to handle, not too large so that classes will not be crowded, not too small lest some of his faculty or staff becomes idle. And so on.

Single-peaked preferences provide a rich model of the agents’ goals, from always increasing (more commodity is always better) to always decreasing (less is always better), and much in between. But the target share of an agent depends upon many subjective, privately known factors, therefore any division rule, fair or otherwise, to allocate the resources must worry about the agents’ incentives to truthfully reveal their true target. We take strategyproofness (truthful report is a dominant strategy) as our incentive compatibility design constraint. We identify a canonical division rule that simultaneously aligns incentives with efficiency (is strategyproof and selects a Pareto optimal allocation), and is egalitarian-fair, in the sense that it selects the Lorenz dominant Pareto optimal allocation.

Our Egalitarian solution generalizes the familiar Uniform solution in Sprumont’s one supplier model. While both share some common features, there are important differences as well. We start with the latter.

Besides bilateral constraints on transfers, our model accommodates capacity constraints, i.e. arbitrary exogenous lower and upper bounds on each agent’s allocation. This is important in all examples we discussed, where shares cannot be arbitrarily large, or small.

In a Pareto optimal allocation some agents always get at most their peak (the over-demanded side of the market), while some others get at least their peak (the under-demanded side of the market); and there is a third set of agents who always get their peak allocation (Proposition 1). This three-pronged partition of the agents and resources is key to our analysis. It only depends upon the profile of peaks and resources, and the bilateral constraints (see a simple example in the next section). The mathematical result driving the partition is a variant of the Gallai-Edmonds (henceforth, GE) decomposition for bipartite graphs,(see [13] for a formal treatment, or [5] and [14] for applications in matching).

The two equity tests No Envy and the less demanding Equal Treatment of Equals, must be formulated with more care than in the one supplier model: given the bilateral constraints, a transfer of resources between two given agents may require to alter the share of a third one. We postulate that Ann’s envy of Bob’s share is legitimate only if it is feasible to improve her share at the expense only of Bob, i.e., while preserving the shares of every agent other than Bob. Similarly Equal Treatment of Equals is violated only if we can bring Ann’s and Bob’s shares closer together without altering any other share. Our solution meets both properties.

Two features common to our and the one supplier model are discussed above: it is Lorenz dominant among Pareto optimal allocations (Proposition 2), and strategyproof (Proposition 4). Finally, when capacity constraints are identical for all participants, the egalitarian solution is characterized –like the uniform solution ([18], [6])– by the combination of strategy-proofness, Pareto optimality and equal treatment of equals (Theorem 1).

In the companion paper [4] we develop a related model with suppliers at one end of the bilateral constraints and demanders at the other end. Each supplier (resp. demander) has single peaked preferences over the amount of commodity he wants to supply (resp. receive); the homogenous commodity can only be transferred across the bilateral edges. There is a close formal analogy with
the current paper, in the sense that the set of critical Pareto optimal allocations is described by the same GE decomposition as in the current paper, the egalitarian solution is again Lorenz dominant among such allocations, and defines a strategyproof revelation mechanism. The difference is that, in order to guarantee voluntary participation in the mechanism, no supplier (or demander) ever gets to supply (or receive) more than her ideal level, thus the algorithmic definition of the egalitarian solution (in section 5 below) is significantly different. Its properties are also different: importantly, the destruction of one edge through which transfers are possible is (weakly) detrimental to the agents at both ends of the edge. By contrast, in our model, dropping an edge can be good or bad news for the agent at one end of this edge, as well as to other agents (see the concluding section). Finally, we use in [4] the techniques of flows on graphs, in particular the max-flow min-cut theorem, instead of the Gallai Edmonds decomposition. The flow approach simplifies some proofs there, but in the current model it does not appear to work well.

2 A motivating example

We illustrate our egalitarian solution in a blood donation example. A blood bank must divide the (objective) blood needs of a group of patients between a set of donors; patients and donors are partitioned by blood type and transfusions must respect the familiar compatibility constraints: (i) type O are universal donors (ii) type AB are universal receivers (iii) type A can also give to A , (iii) type B can also give to B (iv) type AB can only give to AB. Figure 1 shows on the right side the quantity of blood needed by each group of receivers of the same type. On the left side a node is a group of donors of a given type. The total demand of 40 units must be served, and we assume first that the bank wishes to share the burden equally among the blood types. Taking 10 units from each group of donors is clearly not feasible. The most egalitarian division of the burden compatible with Figure 1 is $E_0$: $O : 13, A : 13, B : 8, AB : 6$. Type AB donors cannot contribute more than 6 units toward the patients’ demands. Given AB’s share, type B donors give as much as type AB and B receivers can still accept. Finally, types A and O donors supply equal amounts to cover the rest of the demand.
A more refined version of the model takes into account the fact that the blood bank wishes to spend the blood it receives from various groups of donors in proportion to the representation of these groups in the population. We assume that those proportions are $O: 25\%, A: 37.5\%, B: 25\%, AB: 12.5\%$, so the ideal distribution of the 40 units of demand is $p = (10, 15, 10, 5)$ respectively for types O,A,B,AB. We illustrate this situation in Figure 2 where we interpret the numbers on the left as the *ideal shares* (peak allocation) of each donor type. We look for a fair compromise faithful to the unfeasible profile of ideal shares $p$.

Consider the egalitarian allocation $E_0 = (13, 13, 8, 6)$ identified above. It ignores ideal shares, and when the latter is taken into account, Pareto improvements emerge: $E_0$ uses more type AB blood than the target of 5, and less of type B than the target of 10. Similarly type A is giving more than its target 10, and type O is giving less than 15. Two feasible Pareto improving moves rearrange the shares of types B and AB as $B: 9$, $AB: 5$, and those of types O and A as $O: 11$, $A: 15$. The resulting allocation $E = (11, 15, 9, 5)$ is the one our solution proposes. Another Pareto optimal allocation may be $x = (11, 15, 10, 4)$, but the difference between AB’s and B’s shares is larger in $x$ than in $E$, and $E$ Lorenz dominates $x$.

In the example of Figure 2 the canonical partition of the agents (Lemma 2) is as follows: \(\{AB, B\}\) is the underdemanded subset of agents because these donors cannot meet more than 14 units of demand but they would like to give 15 units; \(\{O\}\) is the overdemanded subset because it must supply at least 11 units but would like to contribute only 10; and A always gets exactly its peak 15 in any Pareto optimal allocation.

### 3 Feasible allocations

We have a set $M$ of agents with generic elements $i, j, k, \ldots$, and $m = |M|$; a set $Q$ of resources with generic element $r, s, \ldots$, and $q = |Q|$. Resource $r$ is of size $\omega_r$, with $\omega_r > 0$. Agent $i$ has capacity constraints $c_i^-, c_i^+$, where $0 \leq c_i^- < c_i^+ \leq \infty$: this means that her allocation $x_i$ (share of resources) is feasible only if $c_i^- \leq x_i \leq c_i^+$. We let $[c^-, c^+] = ([c_i^-, c_i^+])_{i \in M}$ be the profile of capacity constraints.
All resources must be allocated between the agents, but each resource can only be assigned to some of the agents. The bipartite graph $G$, a subset of $M \times Q$, represents the compatibility constraints between resources and agents: $ir \in G$ means that it is possible to transfer resource $r$ to agent $i$. We assume throughout that the graph $G$ is connected, else we can treat each connected component of $G$ as a separate problem.

We use the following notation: for any subsets $S \subseteq M, T \subseteq Q$ the restriction of $G$ is $G(S, T) = G \cap \{S \times T\}$ (not necessarily connected); the set of resources compatible with agents in $S$ is $f(S) = \{r \in Q | G(S, \{r\}) \neq \emptyset\}$, the set of agents compatible with resources in $T$ is $g(T) = \{i \in M | G(\{i\}, T) \neq \emptyset\}$.

A transfer of resources from $Q$ to $M$ is described by a $G$-flow $\varphi$, i.e., a vector $\varphi \in \mathbb{R}_+^Q$ such that $\varphi_{ir} > 0 \Rightarrow ir \in G$. We call a $G$-flow $\varphi$ feasible if it allocates all the resources and we write $x(\varphi)$ for the allocation it realizes:

$$
\text{for all } r \in Q : \sum_{i \in g(r)} \varphi_{ir} = \omega_r; \text{ for all } i \in M : x_i(\varphi) = \sum_{r \in f(i)} \varphi_{ir} \tag{1}
$$

We write $\mathcal{F}(G; \omega)$ for the set of feasible $G$-flows, and $\mathcal{A}(G, \omega) = x(\mathcal{F}(G; \omega))$ for the set of allocations achieved by some feasible $G$-flow. Both sets are obviously non empty, but we need additional assumptions to ensure that some feasible allocations respect the capacity constraints $x \in [c^-, c^+]$.

We write $x_S = \sum_{i \in S} x_i$, $\omega_T = \sum_{r \in T} \omega_r$ etc..

**Lemma 1:** Feasible allocations.

1) The three following statements are equivalent:
   
   i) $x \in \mathcal{A}(G, \omega)$;
   
   ii) for all $S \subseteq M, x_S \leq \omega_f(S)$ and $x_M = \omega_Q$;
   
   iii) for all $T \subseteq Q, \omega_T \leq x_g(T)$ and $\omega_Q = x_M$

2) The set $\mathcal{A}(G, \omega, c)$ of feasible allocations respecting the capacity constraints, is non empty if and only if

$$
\text{for all } S \subseteq M, c^- \leq \omega_f(S); \text{ and for all } T \subseteq Q, \omega_T \leq c^+ \tag{2}
$$

3) Assume (2) holds. Then the three following statements are equivalent:

   i) $x \in \mathcal{A}(G, \omega, c)$

   ii) $x \leq c^+$ and $x$ is in the (lower) core$^3$ of the supermodular TU game $(M, w)$, where

$$
w(S) = \max_{T : g(T) \subseteq S} \{\omega_T + c^-_{S \setminus g(T)}\} \tag{3}
$$

   ii) $x \geq c^-$ and $x$ is in the (upper) core$^4$ of the submodular TU game $(M, v)$, where

$$
v(S) = \min_{S \subseteq S'} \{w_f(S') + c^+_S\} \tag{4}
$$

**Proof:** Statement 1) is a standard application of the Marriage Lemma, see [1]. The only if part of statement 2) is obvious. The proof of if is postponed until after that of statement 3).

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$^3$That is, $x_M = w(M)$, and $x_S \geq w(S)$ for all $S \subseteq M$.

$^4$That is, $x_M = v(M)$, and $x_S \leq v(S)$ for all $S \subseteq M$. 

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Statement 3). We check first \( w(M) = \omega_Q \). Inequality \( w(M) \geq \omega_Q \) holds by taking \( T = Q \) in (3). Fix now any \( T \) and show \( \omega_T + c_{M\setminus g(T)}^- \leq \omega_Q \iff c_{M\setminus g(T)}^- \leq \omega_{Q\setminus T} \). This follows from (2) because \( f(M\setminus g(T)) \subseteq Q\setminus T \). A similar proof that \( v(M) = \omega_Q \) is omitted, as is the straightforward proof that the game \((M, w)\) is supermodular, while \((M, v)\) is submodular.\(^5\)

To prove \( 3) \iff 3) \), pick \( x \in \mathcal{A}(G, \omega, c) \) and \( g(T) \subseteq S \). Statement 1) implies

\[
x_S = x_{g(T)} + x_{S\setminus g(T)} \geq \omega_T + c_{S\setminus g(T)}^-
\]

hence \( x_S \geq w(S) \). Conversely \( x_S \geq w(S) \) implies \( x_i \geq c_i^- \) for \( S = \{i\} \), and for \( S = g(T) \) gives \( x_{g(T)} \geq \omega_T \), so by statement \( 1) \iff 3) \), \( x \in \mathcal{A}(G, \omega) \). The similar proof of \( 3) \iff 3) \) is omitted.

Statement 2). The "only if" part is clear by statement 1). For "if", property (2) implies \( \omega_T \leq c_{g(T)}^- \) for all \( T \subseteq Q \). Combining this with \( c^- \leq c^+ \) gives \( \omega_T + c_{S\setminus g(T)}^- \leq c_S^+ \), thus \( w(S) \leq c_S^- \) for all \( S \subseteq M \). Then we use the fact that a supermodular game has a large core ([17]): there is a core allocation \( x \) such that \( x \leq c^+ \). An alternative proof uses \( c_S^- \leq v(S) \) for all \( S \), and the submodular version of Sharkey's result ([17]).

4 Preferences and Pareto optimality

Agent \( i \) has single-peaked preferences over her share of resources. A single-peaked preference \( R_i \) is transitive and complete over \([c_i^-, c_i^+]\). The symmetric and asymmetric parts of \( R_i \) are denoted by \( I_i \) and \( P_i \), respectively. Preference \( R_i \) has a "peak" \( p[R_i] \in [c_i^-, c_i^+] \) such that for each \( x_i, x'_i \in [c_i^-, c_i^+] \),

\[
x_i' < x_i \leq p[R_i] \implies x_i P_i x'_i,
\]

\[
p[R_i] \leq x_i < x'_i \implies x_i P_i x'_i.
\]

Let \( \mathcal{R}(c_i) \) be the set of single-peaked preferences over \([c_i^-, c_i^+]\); for simplicity, we omit the reference to \( c_i \) and write \( \mathcal{R} \). A preference profile is then \( R = (R_i)_{i \in M} \in \mathcal{R}^M \). For each \( R \in \mathcal{R}^M \), we let \( p[R] = (p[R_i])_{i \in M} \in [c^-, c^+] \) be the associated profile of peaks.

Several of our definitions and results use only a single profile. Whenever this causes no confusion, we simply write \( p_i \) in place of \( p[R_i] \). Notice that if there are no capacity constraints, i.e. \([c_i^-, c_i^+] = [0, +\infty)\) for each \( i \in M \), then \( \mathcal{R} = \mathcal{R} \) for each \( i, j \in M \).

An allocation \( x \in \mathcal{A}(G, \omega, c) \) is Pareto optimal at profile \( R \) if for any other \( x' \) in \( \mathcal{A}(G, \omega, c) \), we have

\[
\{x_i'R_i x_i \text{ for all } i \in M\} \Rightarrow \{x'_i I_i x_i \text{ for all } i \in M\}
\]

We write \( \mathcal{PO}(G, \omega, c, R) \), for the set of Pareto optimal allocations at \( R \).

In order to characterize Pareto optimality, we will use a variant of the Gallai-Edmonds decomposition for bipartite graphs ([13]). This result depends upon \( G \), the profile of peaks \( p \) and the profile of resources \( \omega \), but not on the capacity constraints or the other aspects of preferences. It relies on the three following definitions for the triple \((G, \omega, p)\):

- \((G, \omega, p)\) is balanced if \( p \in \mathcal{A}(G, \omega) \);
- \((G, \omega, p)\) exhibits under-demand if for all \( S \subseteq M \), \( p_S < \omega f(S) \);

\(^5\)In United States politics an earmark is a congressional provision that directs approved funds to be spent on specific projects.
• \((G, \omega, p)\) exhibits **over-demand** if for all \(T \subseteq Q, \omega_T < p_g(T)\);

In view of Lemma 1, in a balanced problem we can give exactly his peak allocation to every agent; in a problem with under-demand we can give each agent at least his peak, and must give to at least one strictly more; and in a problem with over-demand we can give each agent at most his peak, and must give to at least one strictly less.

Any allocation problem \((G, \omega, p)\) can be decomposed in three subproblems, one of each type, and at most two types may be absent. When we speak of the subproblem restricted to \(S\peak\), and must give to at least one strictly less.

For any problem \((G, \omega, p)\) where \(G\) is connected, and \(p \geq 0, \omega \gg 0\), there exists unique partitions \(M_+, M_0, M_- \) of \(M\), and \(Q_+, Q_0, Q_- \) of \(Q\) such that

i) \(G(M_-, Q_0) = G(M_-, Q_-) = G(M_0, Q_-) = \emptyset\)

ii) \(G(M_+, Q_-, \omega, p)\) exhibits under-demand;

iii) \(G(M_0, Q_0, \omega, p)\) is balanced;

iv) \(G(M_-, Q_+), \omega, p)\) exhibits over-demand.

We stress that up to two of the pairs \((M_+, Q_-), (M_0, Q_0)\), or \((M_-, Q_+)\) may be empty. For instance, if there are no bilateral constraints \((G = M \times Q)\), our model is a simple generalization of Sprumont’s where we add capacity constraints for each agent. The GE decomposition reduces to a single component: if \(\omega_Q < p_M\) we have overdemand, \(M = M_-, Q = Q_+\); if \(\omega_Q > p_M\) underdemand and \(M = M_+, Q = Q_-\); if \(\omega_Q = p_M\) the problem is balanced and \(M = M_0, Q = Q_0\). See the discussion of this special case after Proposition 2.

**Proof of Lemma 2:** The Gallai-Edmonds decomposition of a bipartite graph gives precisely the statements when \(p_i = \omega_r = 1\) for all \(i\) and all \(r\). When each \(p_i, \omega_r\) is a positive integer, we make \(p_i\) copies of agent \(i\), and \(\omega_r\) copies of resource \(r\), and connect all copies of \(i\) to all copies of \(r\) iff \(ir \in G\). Again the statements follow by the GE decomposition of this new bipartite graph. By a common rescaling of \(p\) and \(\omega\), we cover the case where \(p_i, \omega_r\) are rational and positive, and by a straightforward limit argument that of real numbers as well, including possibly zero for some peaks.

For future reference we note that the elements of the partition can be defined as the solutions of simple maximization problems.\(^6\) Define \(D = \arg \max_{S \subseteq M} \{p_S - \omega_f(S)\}\) if there is at least one \(S\) such that \(p_S > \omega_f(S)\), \(D = \emptyset\) otherwise. As \(S \rightarrow p_S - \omega_f(S)\) is supermodular, \(D\) is stable by intersection and union, and \(M_+\) is its smallest element, while \(M_- \cup M_0\) is its largest element. Define similarly \(B = \arg \max_{T \subseteq Q} \{\omega_T - p_g(T)\}\) if there is at least one \(T\) such that \(\omega_T > p_g(T)\), \(B = \emptyset\) else. Then \(B\) is stable by intersection and union, \(Q_-\) is its smallest element, and \(Q_- \cup Q_0\) its largest element. We omit the straightforward proof.

There are algorithms polynomial in the number of nodes \(|M| + |Q|\) to compute the GE decomposition ([13]). In the blood donors example (Figure 2), the GE decomposition is \(M_+ = \{O\}, M_- = \{B, AB\}, M_0 = \{A\}, Q_- = \{O\}, Q_+ = \{B, AB\}, Q_0 = \{A\}\).

For another example consider a variant of Figure 2 in which 17 units of type A blood, instead of 15, must be supplied. This is shown in In Figure 3 where \((M_+, Q_-)\) is the upper part of the

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\(^6\)See the proof of Proposition 4.
Figure 3: The donor-receiver example III

Proposition 1: Fix $G, \omega, c$, and assume inequalities (2).

i) For any profile $R \in \mathcal{R}^M$ with peaks $p \in [c^-, c^+]$, the set $\mathcal{PO}(G, \omega, c, R)$ is non empty, convex and compact, and the property of Pareto optimality is peak-only;

ii) A flow implementing a Pareto optimal allocation is null on $G(M_+, Q_-)$, $G(M_+, Q_0)$, and $G(M_0, Q_+)$;

iii) The allocation $x$ is Pareto optimal if and only if

- on $M_+$: $x \in \mathcal{A}(G(M_+, Q_-), \omega)$, and $p \leq x \leq c^+$ (5)
- on $M_0$: $x = p$ (6)
- on $M_-$: $x \in \mathcal{A}(G(M_-, Q_+), \omega)$, and $c^- \leq x \leq p$ (7)

In words, agents in $M_+$ consume precisely all the resources in $Q_-$, each one gets at least his peak, and at least one, strictly more (Lemma 2); those in $M_-$ share the resources in $Q_+$, consume no more than their peak, and at least one gets strictly less; those in $M_0$ consume the resources in $Q_0$ and each gets precisely his peak.

An important consequence of Proposition 1 is that Pareto optimality is a peak-only property: it only depends upon the profile of peaks $p$. Thus we will write simply $\mathcal{PO}(G, \omega, c, p)$ for the Pareto optimal set.

By adapting statement 3) in Lemma 1, we can also describe $\mathcal{PO}(G, \omega, c, p)$ as the cartesian product of three sets: on $M_+$, the subset of the (upper) core of $(M_+, v^+)$ such that $x \geq p$, where $v^+$ is given by (4) for $G(M_+, Q_-)$; on $M_-$ the subset of the (lower) core of $(M_-, w^-)$ such that $x \leq p$, where $w^-$ is given by (3) for $G(M_-, Q_+)$; and $p$ on $M_0$. 

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Proof of Proposition 1:

**Step 1** We first prove the “if” part of Statement iii). By statement 2 in Lemma 1, and the fact that \((G(M_+, Q_-), \omega, p)\) exhibits under-demand (Lemma 2), the set \(\mathcal{A}(G(M_+, Q_-), \omega) \cap [p, c^+]\) is non empty; the set \(\mathcal{A}(G(M_-, Q_+), \omega) \cap [-c, p]\) is similarly non empty because \((G(M_-, Q_+), \omega, p)\) exhibits over-demand. Finally, \(\mathcal{A}(G(M_0, Q_0), \omega, p, p)\) is non empty because \((G(M_0, Q_0), \omega, p)\) is balanced. Suppose now that an allocation \(x\) satisfying (5),(6),(7) is Pareto dominated by some \(y \in \mathcal{A}(G, \omega, c)\). Clearly \(y = x\) on \(M_0\). Because \(G(M_- \cup M_0, Q_-) = \emptyset\) we have \(y_{M_+} \geq \omega_{Q_-} - x_{M_+}\); on the other hand if \(y_i > x_i\) for some \(i \in M_+\), this agent with peak \(p_i \leq x_i\) strictly prefers \(x_i\) to \(y_i\) which our assumption precludes. We conclude \(y = x\) on \(M_+\). The argument establishing \(y = x\) on \(M_-\) is entirely similar.

**Step 2** We next both prove the “only if” part of Statement iii), and Statement ii). Note that statement i) is then clear, for (5),(6),(7) together define a peak-only, convex and compact set of allocations.

We fix throughout Step 2 an economy \((G, \omega, c, R)\), a Pareto optimal allocation \(x\), and a flow \(\varphi\) implementing \(x\). We color agent \(i\) in green if \(x_i < p_i\), in red if \(x_i > p_i\), and in black if \(x_i = p_i\). We also construct a directed graph \(G^\varphi\) as follows: all edges in \(G\) are oriented from \(M\) to \(Q\); if \(\varphi_{ij} > 0\), and only then, we add a “backward” edge from resource \(j\) to agent \(i\).

We claim there is no path in \(G^\varphi\) from a green agent to a red one. If there was such a path from \(i\) to \(i'\), we could increase a little the flow along that path (with the convention that increasing the flow on a backward edge amounts to decrease by the same amount the flow \(\varphi_{ij}\) in \(G\)), and obtain a new allocation where \(i\) consumes a little more, \(i'\) a little less, and everyone else as before; this would contradict Pareto optimality.

Define now \(X\) as the set of all green nodes in \(M\) together with the nodes in \(M \cup Q\) that one can reach in \(G^\varphi\) from a green node; \(Y\) as the set of nodes in \(M \cup Q\) that are either a red agent, or a node from which one can reach a red node in \(G^\varphi\); and \(Z\) as the remaining subset of \(M \cup Q\). Thus \(X, Y, Z\) partition \(M \cup Q\), and every agent in \(X \cup M\) (resp. \(Y \cap M\), resp. \(Z \cap M\)) is green or black (resp. red or black, resp. black). Moreover there is no path in \(G^\varphi\) from \(X\) to \(Z\) or \(Y\), or from \(Z\) to \(Y\).

**Step 2.1** We focus on \(M_-\) and \(Q_+\): we show first \(x \leq p\) on \(M_-\), i.e., there is no red agent in \(Y \cap M_-\).

We have \(x_{(Y \cup Z) \cap M_-} \geq p_{(Y \cup Z) \cap M_-}\), because all such agents are red or black. We also have \(x_{(Y \cup Z) \cap M_-} \leq \omega_{(Y \cup Z) \cap Q_+}\), because the only positive flow out of \((Y \cup Z) \cap M_\) goes to \((Y \cup Z) \cap Q_+\); it cannot go to \(X\) without creating a path in \(G^\varphi\) from \(X\) to \(Y \cup Z\), and \(G(M_-, Q_+ \cup Q_0) = \emptyset\). Finally

\[
\omega_{(Y \cup Z) \cap Q_+} \leq p_{(Y \cup Z) \cap Q_+} \cap M_- \leq p_{(Y \cup Z) \cap M_-}
\]

where the first inequality follows from statements iii) and iv) in Lemma 2, and the second from the fact that there is no edge between an agent in \(X\) and a resource in \(Y \cup Z\). Taken together, these inequalities must all be equalities, so \(x = p\) on \((Y \cup Z) \cap M_-\) and there is no red agent anywhere in \(M_-\), as desired.

Moreover \(x_{(Y \cup Z) \cap M_-} = \omega_{(Y \cup Z) \cap Q_+}\), and there is no positive flow between an agent in \(Y \cup Z\) and a resource in \(X\): hence the resources in \((Y \cup Z) \cap Q_+\) are entirely consumed in \(M_-\), and we conclude that there is no flow between \(M_+ \cup M_0\) and \((Y \cup Z) \cap Q_+\).

**Step 2.2** We focus now on \(M_+ \cup M_0\) and \(Q_- \cup Q_0\), and show first \(x \geq p\) on \(M_+ \cup M_0\), i.e., there is no green agent in \(X \cap (M_+ \cup M_0)\). We have \(x_{(X \cup Z) \cap (M_+ \cup M_0)} \leq p_{(X \cup Z) \cap (M_+ \cup M_0)}\) (these agents
are green or black); \( x_{(X \cup Z) \cap (M_+ \cup M_0)} \geq \omega_{(X \cup Z) \cap (Q_- \cup Q_0)} \) because the only edges in \( G \) to \( Q_- \cup Q_0 \) are from \( M_+ \cup M_0 \), and a resource in \( X \cup Z \) cannot receive a positive flow from one in \( Y \); finally

\[
P((X \cup Z) \cap (M_+ \cup M_0) \leq \omega_{((X \cup Z) \cap (M_+ \cup M_0)) \cap (Q_- \cup Q_0)} \leq \omega_{(X \cup Z) \cap (Q_- \cup Q_0)}
\]

where the first inequality is by Lemma 2, the second one because the only edges in \( G \) from an agent in \( X \cup Z \) are to a resource in \( X \cup Z \). Therefore \( x = p \) on \((X \cup Z) \cap (M_+ \cup M_0)\). Moreover \( x_{(X \cup Z) \cap (M_+ \cup M_0)} = \omega_{(X \cup Z) \cap (Q_- \cup Q_0)} \), and the fact that a resource in \((X \cup Z) \cap (Q_- \cup Q_0)\) cannot receive a positive flow from \( Y \) or from \( Q_- \), imply that there is no positive flow between \((X \cup Z) \cap (M_+ \cup M_0)\) and \( Q_- \).

**Step 2.3** There is no positive flow between \( Y \cap (M_+ \cup M_0) \) and \( X \cap Q_- \) (or from any agent in \( Y \) to a resource in \( X \)). Combining this with the two steps above, we see that there is no positive flow between \( M_+ \cup M_0 \) and \( Q_- \). We have also shown \( x \leq p \) on \( M_- \) and \( x \geq p \) on \( M_+ \cup M_0 \). A symmetrical argument, omitted for brevity, focuses first on \( M_+ \) and \( Q_- \), next on \( M_- \cup M_0 \) and \( Q_- \cup Q_0 \), to establish that there is no positive flow between \( M_+ \) and \( Q_- \cup Q_0 \), and that \( x \leq p \) on \( M_0 \).

We illustrate Proposition 1 in two load balancing examples.

**Example 1:** *Load balancing I*
In Figure 4, four agents \( A, B, C, D \), handle jobs of type 1, 2, 3, 4. As usual the figure shows the preferred load of each agent, the amount of each type of job that need to be processed, and the compatibilities between agents and job types. The two dashed line boxes indicate the GE decomposition: \( A, B, C \), are over-demanded, while \( D \) is underdemanded (\((M_0, Q_0)\) is absent). Most of the inequalities in the system (5),(6),(7) are redundant, and the Pareto optimal set is given by

\[
x_A + x_B + x_C = 29, \quad x_D = 9 \quad x_A \geq 11, \quad x_B \geq 10, \quad x_C \geq 5
\]
Example 2: Load balancing II
In Figure 5, we use the same profile of peaks and resources as in Example 1 but the feasibility constraints have changed. The GE decomposition has now agent \(A\) overdemanded, while \(B, C, D\) are underdemanded. Note that the graph \(G(M_-, Q_+)\) is disconnected. The system (5),(6),(7) reduces to

\[
x_A = 21, \ x_B + x_C = 8, \ x_C \leq 5, \ x_D = 9
\]

5 The egalitarian solution
In this section we give an algorithmic definition of the egalitarian solution, then characterize it as the Lorenz dominant element of the Pareto set.

Given an economy \((G, \omega, c, p)\), such that (2) holds true, we define separately the solution on \(M_+\) and on \(M_-\). It is a selection from, respectively \(A(G(M_+, Q_-), \omega) \cap [p, c^+]\) (see (5)), and \(A(G(M_-, Q_+), \omega) \cap [c^-, p]\) (see (7)). By Pareto optimality ((6)) there is nothing to choose on \(M_0\).

**Computing the egalitarian allocation in \(M_+\):** The definition in \(M_+\) uses an ascending algorithm based on the following system \(\Theta(\lambda)\) of inequalities where \(\lambda\) is a non negative parameter:

\[
\gamma_S(\lambda) \leq \omega_f(S) \cap Q_- \quad \text{for all } S \subseteq M_+
\]

where for all \(i \in M_+, \ \gamma_i(\lambda) = \text{med}\{\lambda, p_i, c_i^+\}\) (the median), so that \(p \leq \gamma(\lambda) \leq c^+\) for all \(\lambda\).

For \(\lambda = 0, \ \gamma(0) = p\) and \(\Theta(0)\) holds true, even strictly, because there is underdemand in \((G(M_+, Q_-), \omega, p)\) (Lemma 2). For \(\lambda = \infty, \ \gamma(\infty) = c^+\) and \(\Theta(\infty)\) requires \(c^+_M \leq \omega_{Q_-}\), while (2) applied to \(T = Q_-\) implies the opposite inequality. Therefore there is a smallest number \(\lambda^l\), strictly positive, such that one of the inequalities in \(\Theta(\lambda^l)\) is tight. As \(S \rightarrow \omega_f(S) \cap Q_- - \gamma_S(\lambda^l)\) is submodular, the equality \(\gamma_S(\lambda^l) = \omega_f(S) \cap Q_-\) is stable by union and intersection of the sets \(S\). We
call $S^1$ the largest such subset. By statement 1) ii) in Lemma 1 applied to $G(S^1, f(S^1) \cap Q_-)$, the (restricted) allocation $\gamma_i(\lambda^1)$ for the agents in $S^1$ is feasible by using all the resources in $f(S^1) \cap Q_-$ and no more.

In the restricted problem $(G(M_+ \setminus S^1, Q_- \setminus f(S^1)), \omega)$ the bilateral graph is described by $f^1(S) = (f(S) \setminus f(S^1)) \cap Q_-$. We claim $\gamma_S(\lambda^1) < \omega f^1(S)$ for all non empty $S \subseteq M_+ \setminus S^1$. Indeed $\Theta(\lambda^1)$ is true and $S^1$ is the largest set such that the corresponding inequality is tight, therefore $\gamma_{S \cup S^1}(\lambda^1) < \omega_{(S \cup S^1) \cap Q_-} \Leftrightarrow \gamma_S(\lambda^1) + \gamma_{S^1}(\lambda^1) < \omega f^1(S) + \omega f^1(S^1) \cap Q_- \cdot$

Applying (2) to $T = f^1(M_+) = f^1(M_+ \setminus S^1)$ and noticing that $g(T) = M_+ \setminus S^1$, gives $\omega f^1(M_+ \setminus S^1) \leq c^+_M \setminus S^1 = \gamma_{M_+ \setminus S^1}(\infty)$. Therefore, repeating the argument above, there is a smallest number $\lambda^2$ strictly above $\lambda^1$, at which one of the inequalities $\gamma_S(\lambda) < \omega f^1(S), S \subseteq M_+ \setminus S^1$, becomes an equality. We call $S^2$ the largest such subset of $M_+ \setminus S^1$. The allocation $\gamma_i(\lambda^2)$ for the agents in $S^2$ is achievable by using precisely all the resources in $f(S^2) \setminus f(S^1)$ (Lemma 1).

Continuing in this fashion, we obtain a partition $S^1, S^2, \ldots, S^K$, of $M_+$, and a strictly increasing sequence $\lambda^1 < \lambda^2 \cdots < \lambda^K$, such that for all $k, 1 \leq k \leq K$, the allocation $\gamma_i(\lambda^k)$ to the agents in $S^k$ is feasible by assigning the resources in $f(S^k) \setminus f(S^1 \cup \cdots \cup S^{k-1})$ to these agents. This allocation is in $[c^-, p]$ by construction. It is our egalitarian solution for the agents of $M_+$.

**Computing the egalitarian allocation in $M_-$:** Turning to the agents in $M_-$, we use a descending algorithm based on the system $\Xi(\mu)$ with non negative parameter $\mu$:

$$\omega_T \leq \delta_{g(T) \cap M_-}(\mu) \text{ for all } T \subseteq Q_+$$

where for all $i \in M_-$, $\delta_i(\mu) = \text{med}\{\mu, c^-, p_i\}$, so that $c^- \leq \delta(\mu) \leq p$ for all $\mu$.

We have $\delta(\infty) = p$, so $\Xi(\infty)$ is true, even strictly, because there is overdemand in $(G(M_-, Q_+), \omega, p)$ (Lemma 2). At $\mu = 0$, $\delta(0) = c^-$ and $\Xi(0)$ requires $\omega_{Q_+} \leq c^-_{M_-}$, but (2) implies the opposite inequality. Therefore there is a largest number $\mu^1$ such that one of the inequalities in $\Xi(\mu^1)$ is tight. We let $T^1$ be the largest subset of $Q_+$ for which we have an equality (its existence guaranteed by the submodularity of $T \rightarrow \delta_{g(T) \cap M_-}(\mu^1) - \omega_T$). The allocation $\delta_i(\mu^1)$ to the agents in $g(T^1) \cap M_-$ is feasible by using exactly the resources in $T^1$ (statement 1) iii) in Lemma 1 applied to $G(g(T^1) \cap M_-, T^1)$). We repeat this construction in the restricted problem $(G(M_-, g(T^1), Q_+, T^1), \omega)$, etc.

We end up with a partition $T^1, \cdots, T^L$ of $Q_+$, and a strictly decreasing sequence $\mu^1 > \cdots > \mu^L$, such that for all $l, 1 \leq l \leq L$, the allocation $\delta_i(\mu^l)$ to the agents in $g(T^l) \setminus g(T^1 \cup \cdots \cup T^{l-1})$ is feasible by assigning exactly the resources in $T^l$ to these agents. This allocation is in $[c^-, p]$ by construction. It is our egalitarian solution for the agents in $M_-$.

By Proposition 1, the entire egalitarian allocation (over $M$) is Pareto optimal. This concludes the definition of the egalitarian solution.

To introduce the main normative property of our solution, we recall first some well known facts. For any finite set $N$ and any $z \in \mathbb{R}^N$, denote by $z^*$ the order statistics of $z$, obtained by rearranging the coordinates of $z$ in increasing order: $z^1 \leq z^2 \leq \cdots \leq z^n$. Say that $z$ Lorenz dominates $w$, written $z \text{ LD } w$, if for all $k, 1 \leq k \leq n$

$$\sum_{t=1}^k z^{*t} \geq \sum_{t=1}^k w^{*t}$$

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Finally \( z \) is Lorenz dominant in the set \( A \) if \( z \) LD \( z' \) for all \( z' \in A \). Lorenz dominance is a partial ordering: not every set, even convex and compact, admits a Lorenz dominant element. On the other hand, in a convex set \( A \) there can be at most one Lorenz dominant element.

**Proposition 2:** Fix \((G, \omega, c^-, c^+)\), and assume inequalities (2). For any profile \( R \in \mathcal{R}^M \) with associated peaks \( p \), the egalitarian solution \( E(p) \) is the Lorenz dominant Pareto optimal allocation:

\[
E(p) \text{ LD } x \text{ for all } x \in \mathcal{PO}(G, \omega, c, p)
\]

In view of the decomposition of \( \mathcal{PO}(G, \omega, c, p) \) (Proposition 1), this result is equivalent to the combination of two statements: the restriction of \( E(p) \) to \( M_+ \) (resp. \( M_- \)) is Lorenz dominant within \( \mathcal{A}(G(M_+, Q_-), \omega) \cap [p, c^+] \) (resp. \( \mathcal{A}(G(M_-, Q_+), \omega) \cap [-c^-, p] \)).

**Proof of Proposition 2:** We set \( x = E(p) \). We need two independent arguments showing respectively that the restriction of \( x \) to \( M_+ \) is Lorenz optimal in \( \mathcal{A}(G(M_+, Q_-), \omega) \cap [p, c^+] \), as is its restriction to \( M_- \) within \( \mathcal{A}(G(M_-, Q_+), \omega) \cap [-c^-, p] \). There is nothing to prove for the \( M_0 \) components. We start with the former statement.

**Step 1** We write \( x^+ \) for the restriction to \( M_+ \) of \( E(p) \). Recall that \( M_+ \) is partitioned by \( S^1, \cdots, S^K \), such that for all \( k \), \( \gamma_{S^k}(\lambda^k) = \omega_{f(S^k) \setminus f(S^k \cup \cdots \cup S^{k-1})} \), and in \( S^k \) we have \( x^+_i = \text{med}\{\lambda^k, p_i, c_i^+\} \). Moreover \( \lambda^k \) is strictly increasing in \( k \). We further partition \( S^k \) as follows

\[
A^k = \{i \in S^k | x^+_i > p_i\}; \quad B^k = \{i \in S^k | x^+_i = p_i\}
\]

Note that agent \( i \) is in \( A^k \) iff \( \lambda^k > p_i \) and \( c_i^+ > p_i \); \( i \) is in \( B^k \) iff \( \lambda^k \leq p_i \). We check first that \( A^k \) is non empty for all \( k \). By Lemma 2 ii)

\[
p_{S^1} < \omega_{f(S^1)} = \sum_{S^1} \text{med}\{\lambda^1, p_i, c_i^+\}
\]

so \( A^1 \) is non empty. Next

\[
p_{S^2} \leq \gamma_{S^2}^+(\lambda^1) < \omega_{f(S^2) \setminus f(S^1)} = \sum_{S^2} \text{med}\{\lambda^2, p_i, c_i^+\}
\]

where the strict inequality is explained in the construction of \( E(p) \). So \( A^2 \) is non empty. And so on.

Now we label the agents in \( M_+ \) as \( \{1, 2, \cdots, m_+\} \) in such a way that \( x^+_i \) is weakly increasing in \( i \), and moreover

- the first \(|A_1|\) terms cover \( A^1 \)
- the next terms cover a possibly empty subset \( \tilde{B}^1 \) of \( B^1 \)
- the next \(|A^2|\) terms cover \( A^2 \)
- the next terms cover a possibly empty subset \( \tilde{B}^2 \) of \( B^1 \cup B^2 \)
and so on. This is possible because in $A^k$ agent $i$ gets $\min\{\lambda^k, c_i^+\}$, so in $A^1 \cup \cdots \cup A^{k-1}$ no one receives more than $\lambda^{k-1}$; on the other hand, in $B^k, k' \geq k$, every agent receives no less than $\lambda^k$; and the sequence $\lambda^k$ increases strictly.

We fix $y \in A(G(M_+, Q_-), \omega) \cap [p, c^+]$ and check that it is Lorenz dominated by $x^+$. We use the notation $y^*(i) = \sum_{j=1}^i y^j$. We have $y_S \geq y^*(|S|)$ for all $S$, and if $S \subseteq M_+$ is such that $y_S = y^*(|S|)$ we say that $S$ is a $y$-tail. Note that our labeling of $M_+$ implies that any subset of the form $\{1, \cdots, i\}$ is an $x^+$-tail.

By feasibility (Lemma 1) $y_S \leq \omega_f(S) = x_S^+$ and on the other hand $y \geq x^+$ in $B^1$. Therefore

$$y_S \leq x_S^+ \text{ for all } S \text{ such that } A^1 \subseteq S \subseteq A^1 \cup \tilde{B}_1$$

When the above $S$ takes the form $\{1, \cdots, i\}$, it is an $x^+$-tail, hence we have $x^+([S]) = x_S^+ \geq y_S \geq y^*(|S|)$. Next we note that $2x^+_i$ increases weakly in $i$, so that for $i \leq |A_1|$ we have

$$\frac{y^*(i)}{i} \leq \frac{y^*(|A_1|)}{|A_1|} \leq \frac{y(A_1)}{|A_1|} \leq \frac{x^+([A_1])}{|A_1|} = \frac{x^+(i)}{i}$$

where the equality is because $x^+$ is egalitarian in $A^1$. We have proved the desired inequality $y^*(i) \leq x^+([S])$ up to $i = |A_1 \cup \tilde{B}_1|$. Next consider $S^2$: feasibility implies $y_{S^1 \cup S^2} \leq \omega_f(S^1 \cup S^2) = x_{S^1 \cup S^2}^+$ and on the other hand $y \geq x^+$ in $B^1 \cup B^2$. Therefore

$$y_S \leq x_S^+ \text{ for any } S \text{ such that } S \subseteq S^1 \cup S^2 \text{ and } S^1 \cup S^2 \setminus S \subseteq B^1 \cup B^2.$$  

In particular

$$y_S \leq x_S^+ \text{ for all } S \text{ such that } A^1 \cup \tilde{B}_1 \cup A^2 \subseteq S \subseteq A^1 \cup \tilde{B}_1 \cup A^2 \cup \tilde{B}_2$$

Again such a set $S$ is an $x^+$-tail if of the form $\{1, \cdots, i\}$, so the inequality $y^*(i) \leq x^+([S])$ follows as above for $i$ such that $|A^1 \cup \tilde{B}_1 \cup A^2| \leq i \leq |A^1 \cup \tilde{B}_1 \cup A^2 \cup \tilde{B}_2|$. For $i = |A^1 \cup \tilde{B}_1| + a \leq |A^1 \cup \tilde{B}_1 \cup A^2|$, we pick $S$ such that $A^1 \cup \tilde{B}_1 \subseteq S \subseteq A^1 \cup \tilde{B}_1 \cup A^2$, with $|S| = i$. Because $x^+$ is egalitarian in $A^2$, we have

$$x^+([S]) = x_{A^1 \cup \tilde{B}_1} + \frac{a}{|A^2|} x_{A^1 \cup \tilde{B}_1 \cup A^2} = (1 - \frac{a}{|A^2|}) x_{A^1 \cup \tilde{B}_1} + \frac{a}{|A^2|} x_{A^1 \cup \tilde{B}_1 \cup A^2}$$

We claim

$$y^*(i) \leq y_{A^1 \cup \tilde{B}_1} + \frac{a}{|A^2|} y_{A^1 \cup \tilde{B}_1 \cup A^2} = (1 - \frac{a}{|A^2|}) y_{A^1 \cup \tilde{B}_1} + \frac{a}{|A^2|} y_{A^1 \cup \tilde{B}_1 \cup A^2}$$

which will imply $y^*(i) \leq x^+([S])$ because $y_S \leq x_S^+$ is true both for $A^1 \cup \tilde{B}_1$ and $A^1 \cup \tilde{B}_1 \cup A^2$. Observe that if $X, Y, Z$ are three disjoint subsets, we have

$$y^*(|X| + |Y|) \leq y_X + \frac{|Y|}{|Y| + |Z|} y_{Y \cup Z}$$

Indeed $\frac{|Y|}{|Y| + |Z|} y_{Y \cup Z}$ is no less than the sum of the $|Y|$ lowest terms in the $Y \cup Z$-coordinates of $y$, and $y_X$ is no less than the sum of the $|X|$ lowest terms in its $X$-coordinates. Applying the claim to $X = A^1 \cup \tilde{B}_1, Y = S \setminus (A^1 \cup \tilde{B}_1)$ and $Z = (A^1 \cup \tilde{B}_1 \cup A^2) \setminus S$ gives (12).

**Step 2** Let $x^-$ be the restriction of $E(p)$ to $M_-$. Then $Q_+$ is partitioned as $T^1, \cdots, T^L$, such that the resources of $T^i$ are entirely assigned to agents in $S^i = \{g(T)^i \setminus \text{g}(T^i \cup \cdots \cup T^{i-1})\}$, and $\omega_{T^i} = \delta_{S^i}(\mu^i)$ for all $k$, where $\mu^i$ is strictly decreasing; moreover $x^-_i = \text{med}\{\mu^i, c^-_i, p_i\}$ for $i \in S^i$. As in Step 1 we partition $S^i$ as follows

$$A^i = \{i \in S^i | x^-_i < p_i\}; B^i = \{i \in S^i | x^-_i = p_i\}$$
so that an agent \(i \in S_l\) is in \(A'\) iff \(\mu^i \leq p_i\) and \(c^-_i < p_i\), while \(i\) is in \(B'\) iff \(\mu^i \geq p_i\).

The set \(A'\) is non empty because \(\sum_{S_1} med\{\mu^1, c^-_1, p_1\} = \omega_{T1} < p_{S1}\); \(A'\) is non empty because

\[
p_{S2} \geq \delta_{S2}(\mu^1) > \omega_{T2} = \sum_{S^2} med\{\mu^2, c^-_2, p_1\}
\]

and so on. We label the agents in \(M_-\) so that an agent \(i\) \(A\) and so on. This is possible because a coordinate in \(A\)

\[
\text{similar to the one in step 1, upon reversing the direction of inequalities. That is, the feasibility of } y(10) \text{ follows (up to a change of sign). Similarly the inequality }
\]

\[
\text{iff constraints are feasible}
\]

\[
\text{model is a simple variant with capacity constraints of Sprumont’s original model. The capacity}
\]

\[
\text{solution for the concave TU game } (M, v)
\]

\[
\text{want all or nothing: This is another special case of interest. Suppose there are no capacity}
\]

\[
\text{Example 1 (revisited): Recall that by Pareto optimality, it remains to divide 29 units between}
\]

\[
A, B, C. \text{ We solve a variant of the one resource problem constrained by } x_A \geq 11 \text{ (which could be}
\]
viewed as a capacity constraint): so $x_A = 11, x_B = 10, x_C = 8$. It illustrates the familiar lack of “progressivity” of the egalitarian solution, that “taxes” agent $C$ but not agent $B$.

**Example 2 (revisited):** Similarly, in Example 2 we only need to divide 8 units between $B, C$. As full equality $x_B = x_C = 4$ is feasible and stays below both peaks, it is the egalitarian solution.

### 6 Other properties of the egalitarian rule

From now on, we always assume inequalities (2) (lest the economy is empty).

**Definition:** Given the problem $(G, \omega, c)$, a rule selects for every preference profile $R \in R^M$ a feasible allocation $\psi(R) \in \mathcal{A}(G, \omega, c)$.

We start with the familiar equity test of No Envy, that must be adapted to our model because of the feasibility constraints. In a classic fair division problem, individual shares can always be exchanged between two agents, say agents 1 and 2, without affecting other agents’ shares. This is not true in our model. First, the bilateral constraints may prevent us from exchanging $x_1$ and $x_2$. But more importantly, even if this exchange is possible, it may require to alter the allocation of agents other than 1 and 2. We postulate that agent 1’s envy of agent 2’s allocation is legitimate only if it is feasible to improve upon agent 1’s allocation without altering the allocation of anyone other than agent 2.

**No Envy:** A rule $\psi$ satisfies No Envy if for each $R \in R^M$ and $i, j \in M$ such that $\psi_j(R) \in [c_i^-, c_i^+]$ and $\psi_j(R) P_i \psi_i(R)$, there exists no $x \in \mathcal{A}(G, \omega, c)$ such that

$$\psi_k(R) = x_k$$

for each $k \neq i, j$ and $x_i P_i \psi_i(R)$

In Example 1, consider the Pareto optimal allocation where $(x_A, x_B, x_C) = (11, 11, 7)$, where the burden of overdemand is shared by all 3 agents in $M_+$. If $7P_B 11$, $B$ envies $C$, and this is legitimate because $(x_B, x_C) = (10, 8)$ is feasible and improves upon $B$ while affecting only $C$. At the egalitarian allocation $(x_A, x_B, x_C) = (11, 10, 8)$, $A$ envies $B$ but no feasible allocation allows a lower load for $A$.

The basic horizontal equity property Equal Treatment of Equals must be similarly adapted to take feasibility constraints into account.

**Equal Treatment of Equals:** A rule $\psi$ satisfies ETE if for each $R \in R^M$ and $\{i, j\} \subset M$ such that $R_i = R_j$ and $c_i = c_j$, if $\psi_j(R) \neq \psi_i(R)$ there exists no $x \in \mathcal{A}(G, \omega, c)$ such that

$$\psi_k(R) = x_k$$

for each $k \neq i, j$ and $|x_i - x_j| < |\psi_j(R) - \psi_i(R)|$

**Proposition 3:**

i) Pareto optimality and No Envy imply ETE.

ii) The egalitarian rule satisfies No Envy.

**Proof:** We prove both statements in turn.
Statement i) Suppose the rule \( \psi \) violates ETE and check it violates either No Envy or Pareto optimality. Fix a profile \( R \) and two agents 1, 2 as in the premises of (14). Write \( y = \psi(R) \) and assume without loss of generality \( y_1 < y_2 \). Distinguish two cases. If 1 and 2’s common peak \( p \) is in \([y_1, y_2]\), then for \( \varepsilon \) small enough the allocation \( z = (1 - \varepsilon)y + \varepsilon x \), feasible by convexity of \( A(G, \omega, c) \), is Pareto superior to \( y \). If \( p \leq y_1 < y_2 \), then 2 envies 1, and the allocation \( x \) satisfies (13), contradiction.

Statement ii) Let \( \psi \) be the egalitarian rule, and \( R \) be a profile at which 1 envies 2 via allocation \( x \). From \( x_1 + x_2 = \psi_2(R) + \psi_1(R) \) and the fact that \( \psi(R) \) Lorenz dominates \( x \) we must have \( |x_1 - x_2| > |\psi_2(R) - \psi_1(R)| > 0 \). If \( x_1 - x_2 \) and \( \psi_2(R) - \psi_1(R) \) have the same sign, then we have \( x_1 < \psi_1(R) < \psi_2(R) < x_2 \) (or a symmetric condition by exchanging 1 and 2). Now \( \psi_2(R)P_1\psi_1(R) \) implies \( p_1 > \psi_1(R) \), hence \( \psi_1(R)P_1x_1 \), contradiction. If \( x_1 - x_2 \) and \( \psi_2(R) - \psi_1(R) \) have opposite signs, convexity of \( A(G, \omega, c) \) implies that the allocation \( x', x'_1 = x'_2 = \frac{1}{2}(x_1 + x_2), x'_k = \psi_k(R) \) otherwise, is feasible. But \( x' \) Lorenz dominates \( \psi(R) \), a contradiction.

We turn to strategyproofness. As in the classic model, we decompose it into a monotonicity and an invariance condition. For clarity, we go back to the notation \( p[R_i] \) for the peak of \( R_i \).

**Monotonicity:** A rule \( \psi \) is monotonic if for all \( R \in R^M, i \in M, \) and \( R'_i \in R^i \)

\[
p[R'_i] \leq p[R_i] \Rightarrow \psi_i(R'_i, R_{-i}) \leq \psi_i(R)
\]

Note that Monotonicity imply own-peak-only, namely \( p[R'_i] = p[R_i] \Rightarrow \psi_i(R'_i, R_{-i}) = \psi_i(R) \): my allocation depends only upon the peak of my preferences.

**Invariance:** A rule \( \psi \) is invariant if for all \( R \in R^M, i \in M, \) and \( R'_i \in R^i \)

\[
\{p[R_i] < \psi_i(R) \text{ and } p[R'_i] \leq \psi_i(R)\} \Rightarrow \psi_i(R'_i, R_{-i}) = \psi_i(R) \quad (15)
\]

\[
\{p[R_i] > \psi_i(R) \text{ and } p[R'_i] \geq \psi_i(R)\} \Rightarrow \psi_i(R'_i, R_{-i}) = \psi_i(R)
\]

**Strategyproofness:** A rule \( \psi \) is strategyproof if for all \( R \in R^M, i \in M, \) and \( R'_i \in R^i \)

\[
\psi_i(R)R_i \psi_i(R'_i, R_{-i})
\]

The next Lemma connects these three properties and Pareto optimality.

**Lemma 3:** Monotonicity and invariance
i) If a rule is monotonic and invariant, it is strategy-proof;
ii) A Pareto optimal and strategyproof rule is monotonic and invariant.

**Proof:** We omit for brevity any reference to capacity constraints, that play no role below. Statement i) is proven just as in the one resource model (see [6]).

Statement ii) Fix a Pareto optimal and strategyproof rule \( \psi \). We show first that the mapping \( R_i \rightarrow \psi_i(R_i, R_{-i}) \) is peak-only. Fix \( R_{-i} \) and consider two preferences \( R_i, R'_i \) such that \( p[R_i] = p[R'_i] \). The GE decomposition (Lemma 2) is the same in \( R \) and \( (R'_i, R_{-i}) \), so by efficiency agent \( i \)'s allocations \( x_i \) and \( x'_i \) are on the same side of \( p[R_i] \). Now strategyproofness implies peak-only.
Next we prove monotonicity. We fix \( i, R \) and \( R'_i \) such that \( p'_i = p[R'_i] \leq p[R_i] = p_i \), and let \( p,p' \) be the profile of peaks at \( R \) and \( (R'_i, R_{-i}) \) respectively. We also set \( x_i = \psi_i(R), x'_i = \psi_i(R'_i, R_{-i}) \). We distinguish two cases.

**Case 1:** \( i \in M_-(p) \). Assume first \( p'_i > x_i \). Then the decomposition is unchanged, in particular \( M_-(p) = M_-(p') \), so by efficiency \( x'_i \leq p'_i \). Assume \( x_i < x'_i \); then we have \( x_i < x'_i \), hence a contradiction of SP for agent \( i \) at profile \( R \). Assume next \( p'_i \leq x_i \). Then \( x_i < x'_i \) would give \( p'_i \leq x_i < x'_i \), hence a violation of SP for agent \( i \) with preference \( R'_i \).

**Case 2:** \( i \in (M_0 \cup M_+)(p) \). Then \( p_i \leq x_i \), so \( x_i < x'_i \) would give \( p'_i \leq p_i \leq x_i < x'_i \), hence a violation of SP for \( i \) at \( R'_i \).

We show finally that \( \psi \) is invariant. Under the premises of property (15), if \( \psi_i(R'_i, R_{-i}) > \psi_i(R) \) we have \( p[R'_i] \leq \psi_i(R) < \psi_i(R'_i, R_{-i}) \), hence a violation of SP for agent \( i \) at \( R'_i \). If \( \psi_i(R'_i, R_{-i}) < \psi_i(R) \) we can find a preference \( R^*_i \) such that \( p[R^*_i] = p[R_i] \) and \( \psi_i(R'_i, R_{-i}) \succ p[R^*_i] \psi_i(R) \). By peak-onliness, \( \psi_i(R'_i, R_{-i}) = \psi_i(R) \), so agent \( i \) with preferences \( R^*_i \) benefits by reporting \( R'_i \). The proof of the second property is identical.■

We now show that the egalitarian rule is both monotonic and invariant.

**Proposition 4:** The egalitarian rule is monotonic and invariant, hence strategyproof as well.

**Proof:** We fix \( (G, \omega, c) \), an agent \( i \) and a benchmark profile of peaks \( p \), with corresponding egalitarian allocation \( x \). We consider a change of peak by agent \( i \) only, to \( p'_i \), and we write \( p'_j = p_j \) for all \( j \neq i \), so that \( p' = (p'_i, p_{-i}) \). Given a profile \( p \), we write \( M_+(p), M_0(p), Q_+(p), \cdots \) for the elements of the GE decomposition (Lemma 2).

**Step 1** We prove monotonicity for shifts in \( p_i \) inside \( M_+(p) \) or inside \( M_-(p) \).

**Step 1a** Consider a change of peak from \( p_i \) to \( p'_i \) such that \( i \in M_+(p) = M_+(p') \).

Suppose first \( p'_i < p_i \). We show \( x'_i \leq x_i \) by distinguishing two cases. Write in both cases \( S^k, \lambda^k \) for the partition and corresponding parameters of the ascending algorithm at \( p \), and let \( i \in S^j, x_i = \lambda^k \lor p_i \).

First case: \( p_i < \lambda^k = x_i \). Then the partition and corresponding parameters are unchanged at \( p' \) so that \( x'_i = x_i \).

Second case: \( p_i = x_i \geq \lambda^k \). Then \( S^k, \lambda^k \) are unchanged for \( 1 \leq k \leq \ell - 1 \), but \( S^\ell, \lambda^\ell \) may change. However for \( \lambda = p_i \)

\[
\sum_{S^j} \lambda \lor p'_j \geq \sum_{S^j} \lambda^\ell \lor p'_j = \omega_{T^i}
\]

(where we write \( T^i = f(S^j) \setminus f(S^1 \cup \cdots \cup S^{l-1}) \)), therefore if \( S^\ell \) changes, the new set \( \tilde{S}_\ell \) contains \( i \) and \( \lambda^\ell \leq p_i \), hence \( x'_i \leq p_i = x_i \).

Suppose next, until the end of step 2a, \( p_i < p'_i \). If \( p'_i \geq x_i \) notice that \( i \in M_+(p') \) implies \( x'_i \geq p'_i \) so we are done. We are left with the case \( p_i < p'_i < x_i = \lambda^\ell \), that requires more work. We prove by induction on \( \ell \) that the first \( \ell \) terms \( S^k, \lambda^k, 1 \leq k \leq \ell \), of the partition and corresponding parameters are unchanged at \( p' \). We write \( S^k, \lambda^k \) for the latter.

Suppose \( \ell = 1 \), then \( \sum_{j \in S} \mu^1 \lor p_j = \sum_{j \in S} \mu^1 \lor p'_j \) for all \( S \subseteq M_+(p) \), so the claim holds.
Next suppose $\ell \geq 2$. Assume $S^1 \neq \tilde{S}^1$ and derive a contradiction. This implies there exists a coalition $S \subseteq M_+(p)$ such that $S \nsubseteq S^1$ and

$$\sum_{j \in S} \lambda^j \lor p_j^* \geq \omega_f(S)$$

(16)

Indeed suppose (16) fails for all $S \nsubseteq S^1$; as $p$ and $p'$ coincide inside $S^1$, we would get $S^1 = \tilde{S}^1$. Fix a coalition $S$ as in (16), that must contain $i$, hence $S \cap S^\ell$ is non empty. By definition of the ascending algorithm, the sets $T^1 = f(S \cap S^1), \ldots, T^k = f(S \cap S^k) \setminus (T^1 \cup \cdots \cup T^{k-1}), \ldots$, are pairwise disjoint and $\sum_{S \cap S^k} \lambda^k \lor p \leq \omega_{T^k}$ for all $k$, therefore

$$\sum_{1 \leq k \leq K} \left[ \sum_{S \cap S^k} \lambda^k \lor p_j \right] \leq \omega_f(S)$$

In view of (16), we get

$$\sum_{1 \leq k \leq K} \left[ \sum_{S \cap S^k} \lambda^k \lor p_j \right] \leq \sum_{j \in S} \lambda^j \lor p_j^*$$

For all $k \neq \ell$, we have $\lambda^k \geq \lambda^j$ and $p = p'$ in $S \cap S^k$, implying $\sum_{S \cap S^k} \lambda^k \lor p_j \geq \sum_{S \cap S^k} \lambda^j \lor p_j'$. As $\lambda^j$ is larger than $\lambda^1, \lambda^2, \lambda^3$ and $p_i$, and $S \cap S^\ell$ is non empty, we get $\sum_{S \cap S^\ell} \mu^\ell \lor p_j > \sum_{S \cap S^\ell} \mu^j \lor p_j'$. The desired contradiction follows and we conclude $S^1 = \tilde{S}^1$.

To show next $S^2 = \tilde{S}^2$, we replicate the above argument as follows. If $\ell = 2$, then $\sum_{S \cap S^2} \lambda^2 \lor p_j = \sum_{S \cap S^2} \lambda^2 \lor p_j'$ for all $S \subseteq M_+(p) \setminus S^1$, because $p_1, p_2 < \lambda^2$, and the claim holds. If $\ell \geq 3$ and $S^2 \neq \tilde{S}^2$, we can pick a coalition $S \subseteq M_+(p) \setminus S^1$ such that $S \nsubseteq S^2$ and

$$\sum_{j \in S} \lambda^2 \lor p_j' \geq \omega_f(S) \setminus S^1$$

and proceed as above by decomposing $S$ along $S^k, 2 \leq k \leq K$. The induction step is now clear.

**Step 1b** For a change of peak from $p_i$ to $p_i'$ such that $i \in M_-(p) = M_-(p')$, the parallel argument is omitted for brevity.

**Step 2** We examine the critical peaks at which the GE decomposition change.

**Step 2a** Suppose $i \in M_+(p)$. If $p_i' < p_i$, the decomposition does not change, so $i \in M_+(p')$. Consider the critical report $p_i^*, p_i^* > p_i$, if any, at which the GE decomposition and the status of agent $i$ change. By Lemma 2, $i, M_+(p) = M_+(p')$ as long as $p_i^* < \omega_{f(S) \cap Q_-(p)}$ for all $S \subseteq M_+(p)$. Thus $p_i^*$ is the smallest number such that

$$p_i^* + p_i^* = \omega_f(S \cap Q_-(p))$$

(17)

for some subset $S$ of $M_+(p)$ containing $i$. If $p_i^* > c_i^+$, then in fact the decomposition never changes when $p_i'$ varies in the relevant interval, and we set $p_i^* = \infty$ to fix ideas. Assume from now on $p_i^* \leq c_i^+$, and let $S^*$ be the largest $S$ satisfying (17) (well defined by the usual submodularity argument). Recall from the proof of Lemma 2 that $(M_- \cup M_0)(p)$ is the largest solution of $\arg \max_{S \subseteq M} \{p_S - \omega_f(S)\}$. At $p^*$ we have $\max_S \{p_S^* - \omega_f(S)\} = \max_S \{p_S - \omega_f(S)\}$ and the largest
solution of \( \text{arg max}_{S \subseteq M} \{ p_S - \omega_f(S) \} \) is now \((M_+ \cup M_0)(p) \cup S^* = (M_+ \cup M_0)(p^*)\); moreover \( M_-(p) \) is still a solution of \( \text{arg max}_{S \subseteq M} \{ p_S^* - \omega_f(S) \} \), therefore it is the smallest. So \( i \in M_0(p^*) \).

Now the restriction of \( x \) to \( M_+ \) is in \( \mathcal{A}(G(M_+), \omega) \cap [p, c^+] \) and \( S^* \subseteq M_+(p) \); these two facts imply

\[
x^{S^*} \leq \omega_f(S^*) - Q_-(p)
\]

In \( M_+ \) we have \( p \leq x \), so \( p_i^* < x_i \), would give \( p_S^* < x_{S^*} \), and a contradiction of (17) for \( S^* \).

Therefore

\[
p_i^* \geq x_i \geq p_i
\] (18)

**Step 2b** Suppose \( i \in M_-(p) \). By entirely symmetric arguments we can show that one of two cases arises. If \( M_-(c_i, p-i) = M_-(p) \), the decomposition never changes when \( p_i' \) varies in the relevant interval, and we set \( p_i^* = -\infty \). Otherwise there is a critical peak \( p_i^* \) below \( p_i \) at which the decomposition changes for the first time. The details of the decomposition at \( p^* \) are similar and they only matter to prove \( i \in M_0(p^*) \) and

\[
p_i^* \leq x_i \leq p_i
\] (19)

**Step 3** We consider now a move from \( p_i \) to \( p_i' \) when \( i \in M_0(p) \). If \( p_i' > p_i \), we have \( i \in M_-(p') \). Then in the downward shift starting at \( p_i' \), \( p_i \) is the critical value (described in step 2b) at which the status of \( i \) changes, so by (19) \( x_i' \geq p_i = x_i \) as desired. Symmetrically \( p_i' < p_i \) gives \( i \in M_+(p') \) and \( p_i \) is the critical value starting from \( p_i' \) described in step 1a, so (18) gives \( x_i' \leq p_i = x_i \).

It remains to look at a shift from \( p_i \) to \( p_i' \) such that \( i \in M_+(p) \) and \( i \in M_-(p') \). This requires \( p_i' > p_i \); clearly the critical value \( p_i^* \) for \( p_i \) described in step 2a is the same as the critical value for \( p_i' \) in step 2b. Therefore (18) and (19) imply

\[
p_i \leq x_i \leq p_i^* \leq x_i' \leq p_i'
\]

**Step 4** The invariance property is clear from (18) and (19) and the arguments of steps 1a and 1b. □

**7** Characterization result

For our last result, we restrict attention to the case where all capacity constraints are identical: \( c_i = c_j \) for all \( i, j \in M \).

**Theorem 1**: Let the capacity constraints be identical for all agents. Then the egalitarian rule \( E \) is characterized by Pareto optimality, strategyproofness and equal treatment of equals.

**Proof**: Beyond ensuring that (2) holds true, the common capacity constraints play no role any more in the proofs. So we assume below \( c^- = 0, c^+ = \infty \), and let the reader check that the proof is unchanged when we add arbitrary common constraints. We fix \( G, \omega \) and a rule \( \psi \) meeting the three properties.

**Step 1** We fix in this step two partitions \( M_{+, -0} \) of \( M \) and \( Q_{+, -0} \) of \( Q \), that coincide with the GE decomposition for some profile of peaks \( p^* \). Then we choose a profile of preferences \( \bar{R} \) with peaks \( \bar{p} \) such that

\[
\bar{p} \ominus \mathcal{A}(G(M_+), \omega) \cap [\bar{p}, c^+] \subseteq Q_0.
\]

\[ g^{-1}(M_0) \supseteq Q_0. \]

\[ 21 \]
\[ \tilde{p} \text{ such that} \]
\[ \tilde{R}_i = \tilde{R}_j \text{ if } i, j \in M_+ \text{ and if } i, j \in M_- \]
\[ \tilde{p}_i = 0 \text{ if } i \in M_+; \quad \tilde{p}_j > \omega_Q \text{ if } j \in M_-; \quad (G(M_0, Q_0), \omega, \tilde{p}) \text{ is balanced} \]

We show that \( \psi(\tilde{R}) = \mathcal{E}(\tilde{R}) \). Setting \( y = \psi(\tilde{R}), x = \mathcal{E}(\tilde{R}) \), by Proposition 2 it is enough to check that the \( M_+ \)-component of \( y \) (resp. its \( M_- \)-component) is Lorenz dominant in the corresponding component of \( \mathcal{PO}(G, \omega, \tilde{R}) \).

As explained immediately after the statement of Proposition 1, the \( M_+ \)-component of \( \mathcal{PO}(G, \omega, \tilde{R}) \) contains \( z \geq 0 \) iff \( z \) is in the upper core of the submodular game \( (M_+, v^+) \), where
\[ v^+(S) = \omega_{f(S) \cap Q_-} \text{ for all } S \subseteq M_+; \quad v^+(M_+) = \omega_Q \]

Similarly the \( M_- \)-component of \( \mathcal{PO}(G, \omega, \tilde{R}) \) contains \( z \) iff \( z \) is in the lower core of the supermodular game \( (M_-, w^-) \) where
\[ w^-(S) = \max\{\omega_T|T \subseteq Q_+, g(T) \cap M_- \subseteq S\} \text{ for all } S \subseteq M_-; \quad w^-(M_-) = \omega_Q \]

(note that, by our choice of \( \tilde{p} \), the constraints \( z \leq \tilde{p} \) are not binding).

We use only ETE and Pareto optimality to show \( y = x \) on \( M_+ \). We omit for brevity the similar argument establishing this equality on \( M_- \). Set \( m_+ = |M_+| \) and recall that \( y^{*m_+} \geq y^{*(m_+ - 1)} \geq \cdots \geq y^1 \) is the order statistics of \( y \).

**Claim 1** Fix an agent \( i_1 \in M_+ \), such that \( y_{i_1} = y^{*m_+} \), then
\[ y_{i_1} = x_{i_1} = y^{*m_+} = x^{*m_+} \quad (20) \]

Because \( x \) Lorenz dominates \( y \), we have \( y^{*m_+} \geq x^{*m_+} \). If \( y_{i_1} = y^{*m_+} \) for all \( i \in M_+ \) then \( y = x \) at once and we are done. If \( y_{i_1} = 0 \) (recall in \( M_+ \) all peaks are 0) then \( x_{i_1} \geq 0 \) implies \( x_{i_1} = y_{i_1} \) and (20) is again proven. From now on we assume \( y_{i_1} > 0 \), and that there is at least one agent such that \( y_i < y^{*m_+} \). We show there exists a subset \( S(i) \subset M_+ \) such that
\[ i_1 \notin S(i), \quad i \in S(i), \text{ and } y_{S(i)} = v^+(S(i)) \quad (21) \]

Otherwise \( y_S < v^+(S) \) for all \( S \subset M_+ \) containing \( i \) but not \( i_1 \). Choosing \( \varepsilon > 0 \) smaller than the smallest such difference \( v^+(S) - y_S \), we see that an \( \varepsilon \)-transfer from agent \( i_1 \) to agent \( i \) (a Pigou-Dalton transfer) preserves the core property (inequalities \( y_S \leq v^+(S) \) for \( S \) containing \( i \) are automatically satisfied), and \( y_{i_1} > 0 \) ensures the new allocation is non negative. This contradiction of (14) proves (21).

Set \( S^* = \bigcup_{y_{i_1} < y^1} S(i) \). Submodularity of \( v^+ \) implies \( y_{S^*} = v^+(S^*) \), so
\[ x_{S^*} \leq v^+(S^*) = y_{S^*} \Rightarrow x_{M_+ \setminus S^*} \geq y_{M_+ \setminus S^*} \]

But by construction \( y_j = y^{*m_+} \geq x_j \) for all \( j \in M_+ \setminus S^* \), therefore \( x_j = y^{*m_+} \) for all \( j \in M_+ \setminus S^* \). Combining this with \( y^{*m_+} \geq x^{*m_+} \) proves (20).

**Claim 2** Fix an agent \( i_2 \in M_+ \), such that \( i_2 \neq i_1 \) and \( y_{i_2} = y^{*(m_+ - 1)} \), then
\[ y_{i_2} = x_{i_2} = y^{*(m_+ - 1)} = x^{*(m_+ - 1)} \quad (22) \]

As \( x \) Lorenz dominates \( y \), we have \( y^{*m_+} + y^{*(m_+ - 1)} \geq x^{*m_+} + x^{*(m_+ - 1)} \Rightarrow y^{*(m_+ - 1)} \geq x^{*(m_+ - 1)} \) (by Claim 1). If \( y_i = y^{*(m_+ - 1)} \) for all \( i \in M_+ \setminus i_1 \) then \( y \geq x \) so \( y = x \) by \( y_{M_+} = x_{M_+} \), and we are
done. If $y_{i_2} = 0$ then $x_{i_2} \geq 0$ implies $x_{i_2} = y_{i_2}$ and (22) is again proven. From now on we assume $y_{i_2} > 0$, and that there is at least one agent $i \in M_+$ such that $y_i < y^{(m+1)}$. For any such agent, we claim there is a subset $S(i) \subset M_+$ such that

$$i_2 \notin S(i), \quad i \in S(i), \quad \text{and } y_{S(i)} = v^+(S(i))$$

Otherwise, we can construct as above a Pigou-Dalton transfer from agent $i_2$ to agent $i$, in contradiction of (14). Set $S^* = \bigcup_{i: y_i < c_i^+, y_{i_2} S(i)}$, then submodularity of $v^+$ gives $y_{S^*} = v^+(S^*)$, hence

$$x_{S^*} \leq v^+(S^*) = y_{S^*} \Rightarrow x_{M_+ \setminus S^*} \geq y_{M_+ \setminus S^*} \Rightarrow x_{M_+ \setminus \{i_1\}} \geq y_{M_+ \setminus \{S^* \cup \{i_1\}\}}$$

But by construction $y_j \geq x_j$ for all $j \in M_+ \setminus \{S^* \cup \{i_1\}\}$ (as $y_j \geq y^*$), and $M_+ \setminus \{S^* \cup \{i_1\}\}$ contains $i_2$. Combining this with $y^{(m+1)} \geq x^{(m+1)}$ proves (20).

The inductive argument establishing $y = x$ is now clear.

**Step 2** (inspired by Ching (1994)) We fix an arbitrary profile $R^* \in R^M$ with peaks $p$, and associated GE decomposition $M_{+, - 0}$, $Q_{+, - 0}$ of $(G, \omega, p^*)$. We choose $\tilde{R}$ with peaks $\tilde{p}$ as in step 1, and the additional requirement $p^* \leq \tilde{p}$ on $M_-$ and $p^* = \tilde{p}$ on $M_0$; we also have $\tilde{p} = 0 \leq p^*$ on $M_+$.

For $S \subset M$, we write $(R^*_{|S^+}, \tilde{R}_{|M_+ \setminus S^+})$ the profile equal to $R^*$ for agents in $S$ and to $\tilde{R}$ for agents in $M_+ \setminus S$. For any integer $n, 0 \leq n \leq m$, consider the following subset of preference profiles

$$D_n = \{(R^*_{|S^+}, \tilde{R}_{|M_+ \setminus S^+})| \text{for some } S \subset M : |S| \leq n\}$$

We prove by induction on $n$ the property $\mathcal{H}^+(n)$: $\psi = \mathcal{E}$ on $D_n$. This is enough because step 1 establishes $\mathcal{H}^+(0)$ and $\mathcal{H}^+(m)$ $\psi(R^*) = \mathcal{E}(R^*)$ for an arbitrary $R^*$.

Assume $\mathcal{H}^+(n - 1)$ is true, and fix $R = (R^*_{|S^+}, \tilde{R}_{|M_+ \setminus S^+})$ with $|S| = n$. We pick an agent $i \in S \cap M_+$, so by Pareto optimality $p_i^* \leq \psi_i(R), \mathcal{E}_i(R)$. To prove $\psi_i(R) = \mathcal{E}_i(R)$ we consider the profile $R' = (R^*_{|S \setminus \{i\}}, \tilde{R}_{|M_+ \setminus S^+ \cup \{i\}}) \in D_{n-1}$ where the inductive assumption gives $\psi_i(R') = \mathcal{E}_i(R') = z_i$. We compare $\psi_i(R), \mathcal{E}_i(R)$ and $z_i$ by distinguishing two cases.

If $p_i^* \leq \psi_i(R) < \mathcal{E}_i(R)$ then $z_i \leq \psi_i(R)$ by monotonicity of $\psi$ (Lemma 3) and $\tilde{p}_i \leq p_i^*$ in $M_+$. As $\mathcal{E}$ is invariant (Proposition 4) and $\tilde{p}_i, p_i^* < \mathcal{E}(p_i^*, p_{-i})$, we have $\mathcal{E}(\tilde{p}_i, p_{-i}) = \mathcal{E}(p_i^*, p_{-i})$, i.e., $z_i = \mathcal{E}_i(R)$. This is a contradiction. If $p_i^* \leq \mathcal{E}_i(R) < \psi_i(R)$ the same contradiction obtains by exchanging the role of $\psi$ and $\mathcal{E}$.

We just proved $\mathcal{E}_i(R) = \mathcal{E}_i(R)$ for $i \in S \cap M_+$. We check it next for $M_+ \setminus S$. Write $\psi(R) = y$, $\mathcal{E}(R) = x$, both vectors in $M_+$ and $\tilde{y}, \tilde{x}$ their restrictions to $M_+ \setminus S$, and $\pi$ their common restriction to $S \cap M_+$. Consider the set

$$\mathcal{C}(R) = \{z \in \mathbb{R}^{M_+ \setminus S}_+| (z, x_{|S \cap M_+}) \text{ in the upper core of } (M_+ \setminus S^+\}$$

that contains $y$. Clearly $\tilde{x}$ is still Lorenz dominant in $\mathcal{C}(R)$, hence we can mimic the proof of Step 1 to show that ETE and Pareto optimality imply the desired equality of $\tilde{y}$ and $\tilde{x}$. The key is that the restriction of the profile $R$ to $M_+ \setminus S$ consists of pairwise identical preferences, therefore we can apply ETE to any pair of agents in $M_+ \setminus S$. To copy the proof of step 1, observe that $\mathcal{C}(R)$ is defined, besides the constraints $z \geq 0$, by the inequalities

$$z_A \leq \tilde{v}^+(A) = v^+ (A \cup (S \cap M_+)) = x_{S \cap M_+} \text{ for all } A \subset M_+ \setminus S$$

and the equality $z_{M_+ \setminus S} = x_{M_+ \setminus S}$. Thus $\mathcal{C}(R)$ is the upper core of the concave game $(M_+ \setminus S, \tilde{v}^+)$ and the proof proceeds exactly as in Step 1. We omit the details.

We also omit the entirely similar proof that $\psi_i(R) = \mathcal{E}_i(R)$ on $M_-$.■

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8 Concluding comments

1) Summary of our results: Our model generalizes to a considerable extent the standard division model introduced by Sprumont [18]. This extension generates several hurdles because of additional feasibility constraints imposed by the bipartite graph. The division of the graph in three submarkets in which there is overdemand, balancedness, and underdemand respectively, gives the structure of the Pareto optimal allocations. Then the feasibility constraints are captured by a system of submodular upper bounds on coalitional shares in the underdemanded segment of the market, and a system of supermodular lower bounds in the overdemanded segment. Finally equal treatment of equals must be restricted to those equalizing transfers that do not affect the shares of agents not involved in the transfer. After those new features are properly incorporated, our Egalitarian solution is still characterized by the combination of efficiency, strategyproofness and equal treatment of equals.

2) Standard properties: The Uniform solution in the standard model satisfies many other desirable properties([20]), some of which are easily adapted to our model.

a) Resource Monotonicity requires that the share of every agent increases weakly when the amount of one of the resources (one of the numbers \( \omega_r \)) increases([19]). This property can be established by mimicking the proof of the analog “peak monotonicity” property in the companion paper([4]). We omit the tedious proof.

b) Consistency plays a central role in characterizing the parametric rules (that include the uniform rule and much more): see [20] and [21]. It can be adapted to our model as follows. Consider a solution \( \psi \) defined for all problems \((M, G, \omega, c, R)\), (in particular for problems involving smaller sets of agents), and pick an arbitrary \( G \)-flow \( \varphi \) implementing \( \psi(M, G, \omega, c, R) \). Suppose agent \( i \) leaves the problem with her share of \( \varphi \), resulting in a flow \( \varphi(-i) \) where all edges between \( i \) and \( Q \) are deleted, and the resources correspondingly reduced to \( \omega(-i) \): Consistency of \( \psi \) requires that \( \varphi(-i) \) implements the solution \( \psi(M\setminus i, G(-i), \omega(-i), c_{-i}, R_{-i}) \). It is clear that the Egalitarian solution is consistent in this sense.

c) Group-strategyproofness strengthens strategyproofness by ruling out profitable joint misreports by arbitrary subsets of agents (see e.g. [2]). The Uniform solution in the one resource model is groupstrategyproof, and we conjecture that the egalitarian solution is also groupstrategyproof.

3) Changing the bilateral constraints: We can also introduce new properties, not present in the one resource model. In the spirit of the axiomatic fair division literature (e.g., [20]), it is natural to ask what happens when a new link is added to the graph \( G \). In particular a rule would exhibit Link Monotonicity if the addition of a link from agent \( i \) to resource \( r \) always weakly benefits this agent; it would satisfy Link Solidarity if this addition always affects all agents other than \( i \) in the same direction (they are all weakly better off, or all weakly worse off). In fact the egalitarian rule violates both properties.

To check this claim, consider the example given by Figure 3 (just before Proposition 1 in Section 4), where the egalitarian solution is \((x_O, x_A, x_B, x_{AB}) = (13, 15, 9, 5)\). Now add a link from agent \( B \) to resource \( O \). In the new economy, the entire set of agents is overdemanded, and the new egalitarian allocation is \((10.5, 15, 10.5, 6)\). Agent \( O \) strictly benefits in the change, while \( AB \) is hurt. And if \( 9P_B 10.5 \), agent \( B \) is hurt as well.
Interestingly in the model of our companion paper [4], with agents on both sides of the links, we find that the analog egalitarian solution satisfies Link Monotonicity. Looking for link monotonic and link solidary solutions in the current model is an interesting avenue for future research.

4) Extensions: We mention finally two possible extensions of our work. First, as in [15], [9], we can think of a “discrete” variant where indivisible units have to be distributed. Both papers above offer a characterization of the randomized Uniform rule, and it is possible that their result could be adapted to include bilateral constraints.

We have considered only symmetric rules. In the standard model, the rich family of allotment rules ([2]) preserves the incentive properties of the Uniform rule while allowing a very different treatment of the agents. Similarly the family of fixed paths rules ([12]) is characterized by the combination of efficiency, strategyproofness, resource monotonicity and consistency. Both families can naturally be extended to our model, though the corresponding characterization results, if any, would require further research.

References


