Design of Low Order Robust Controllers for a VSC HVDC Power Plant Terminal

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This paper describes the design of controllers for a Voltage Source Converter High Voltage Direct Current (VSC HVDC) power terminal, a MIMO plant that must operate over a wide range and that experiences disturbances due to Pulse Width Modulation (PWM) switching. The controllers are designed using two approaches: the Glover–McFarlane loop shaping procedure (LSDP), and a novel implementation of a multiple operating point version of the LSDP using the ellipsoidal set approach, in which low order controllers are obtained using an iterative sequence of LMI programs and a convexifying potential function formulation. The LSDP provides controllers that give robust performance in the neighborhood of the nominal plant but poorer behavior at distant operating points. The novel approach is demonstrated to be practical on a rigorous plant model including the switching behavior. Both full and low order controllers are designed whose overall behavior is better than that of the standard LSDP.

Keywords: LMIs, robust control, power system control

1. Introduction

This paper describes the design of robust, low order controllers for VSC HVDC (Voltage Source Converter High Voltage Direct Current transmission), a technology used to transmit power between two AC networks, that has received considerable attention in the power transmission industry in recent years due to the development of high power switching transistor technology [21]. A VSC HVDC system consists of two AC–DC terminals and a DC link, the plant considered here is the terminal used to control power flow. It is a plant for which a controller is required that provides good performance over a wide range of operating conditions, is of as low order as possible without compromising robust performance, and is able to minimize the effects of PWM switching that is integral to the technology. There are no suitable measurements for scheduling the controller to compensate for the key change in operating conditions, the impedance of the attached AC system. This application thus provides an interesting and industrially relevant case study for practical application of robust control techniques. In this paper linear controller design is explored.

The Glover–McFarlane Loop Shaping Design Procedure (LSDP) [14] is a controller design method that has been successfully demonstrated on examples in different application fields, for example [17], [32]. Among its desirable properties are an intuitive design procedure and guaranteed bounds on all the input and output sensitivity functions. For the VSC HVDC plant, the selection of LSDP design weights to provide
robust control in the neighborhood of a nominal operating point is relatively straightforward. However, the performance at ‘distant’ operating points is not satisfactory and cannot be easily improved by changing the structure or parameters of the weights; indeed providing good performance at distant plants is not an explicit feature of the LSDP. A further drawback of the standard LSDP is that the order of the resulting controllers is large, the order of the plant plus twice the order of the shaping filters, which can be undesirable for practical implementation.

Several approaches may be taken to extend the LSDP to take into account plant variation across its operating range. These include \( \mu \) synthesis [31] and Robust \( \mathcal{H}_\infty \) design using generalized multipliers [8], [20]. Both approaches impose a common Lyapunov function across an uncertainty set constructed to cover the operating range, providing guaranteed quadratic performance across the uncertainty set considered. However, if this uncertainty set is generated using linearizations of a nonlinear plant model, the performance across the nonlinear model is not guaranteed. Furthermore, the quadratic performance requirement and the presence of plants in the uncertainty set that do not represent true plant behavior can lead to significant conservatism. These approaches also lead to controllers of relatively high order. To avoid the conservatism and high order controllers of these approaches, the described approach finds low or full order controllers that minimize worst case \( \mathcal{H}_\infty \) performance across a finite set of operating points each furnished with its own Lyapunov function, and can be considered as an extension of the LSDP for multiple operating points and low order controllers. The approach uses the convexifying potential function of [4] to find controllers in the ellipsoidal set formulation of [15], [16]. In this approach there is no a priori guarantees regarding stability or performance at any operating points other than those explicitly considered in the design. Instead, these features are established after design, both by analysis of individual operating points and by a \( \nu \)-gap metric analysis [28] around the operating points used in the design. On the other hand, the method does not suffer from the conservatism of methods that impose a common Lyapunov function across a set of operating points. As recognized, in, e.g., [7], [27], such an approach can, in practice, provide better performance across an operating range than approaches which do provide a priori guarantees. It is noted that two recent developments in low order design, the sufficient LMI formulation of [18] and the HIFOO package [3], do not provide multiple operating point formulations; however, the nonsmooth optimization approach introduced in [1] provides a possible alternative approach to the one described here.

The contribution of this paper is the introduction of the convexifying approach for making the ellipsoidal set formulation tractable, and its application to provide an extension to the LSDP that enables this procedure’s attractive features to be retained. The practical effectiveness of the proposed approach is demonstrated by the successful design of low order controllers for a realistic plant which has stringent performance requirements across a range of operating conditions.

The paper is organised as follows. Section 3 describes the model used for the controller designs and its validation against a rigorous model developed in the power system modeling tool PSCAD [19]. Control objectives are set out and a number of measures of nominal and robust performance are proposed in Section 4. In Section 5 the results of a nominal loop shaping design that provides good performance at the nominal operating point are presented. Section 6 summarizes the ellipsoid design formulation for multiple operating point performance optimization, and describes the technique for using convexifying potentials to find controllers using this formulation. Section 7 compares the performance of controllers designed using this technique with the nominal controller on both the linearized models and the PSCAD model.

2. Nomenclature

The control nomenclature used is mostly standard. \( x, z, y, w \) and \( u \) are the state vector, exogenous output, measured output, exogenous input and manipulated input respectively; their dimensions are denoted by \( n_x, n_z, n_y, n_w, n_u \) etc. The set of symmetric matrices of dimension \( n \) is represented by \( \mathbb{S}^n \). The identity matrix of dimension \( n \) is represented by \( I_n \). The controllers for a shaped plant derived using the LSDP are termed LSDP controllers and have the subscript \( s \). LFT refers to the Linear Fractional Transform. The physical units used throughout are normalized to ‘per unit’ (pu) as is standard practice for power systems.

3. The VSC HVDC Terminal Models

An eighth order physical nonlinear model of a VSC HVDC terminal attached to an AC network in which switching effects are neglected has been developed as detailed in [5]. Here details of this model and the results of a validation against a PSCAD model of a four level converter including realistic PWM switching
behavior and auxiliary circuits are briefly presented. The four level converter topology is described in more detail in [30].

The components of the physical model are shown in Fig. 1. The model inputs are the VSC control voltage and relative angle, \( V_{in} \) and \( \delta_{in} \). The model outputs are the converter terminal voltage magnitude \( V_l \) and the power delivered \( P \). The 3-phase AC system is represented by an AC source in series with an impedance \( Z_c = R_c + jX_c \), with states and inputs represented in dq orthogonal components relative to the rotating reference frame of the AC voltage source \( V_s \). This dq representation is commonly used for describing the dynamics of 3-phase AC systems [11]. In this paper the impedance is parameterized by \( \alpha \) where \( Z_c = 0.1 \alpha + j \alpha \) and \( 0 \leq \alpha \leq 1 \). The circuit behavior is linear in the reference frame of the AC voltage source \( V_s \), and affine in \( 1/\alpha \). The behavior of the phased locked loop (PLL) phase measuring device is represented by a second order linear system. The plant model has three distinct static nonlinearities: the angle \( \delta_i \) of the converter terminal voltage relative to the \( V_s \) reference frame, a function of the circuit states \( x_c \), represented by \( f_s(x_c) \) (a rectangular to polar co-ordinate transform); the circuit inputs in the \( V_s \) reference frame as functions of the plant inputs and the measured converter terminal angle \( \delta_{lin} \) (containing a polar to rectangular co-ordinate transform P2C); and the model outputs \( V_l \) and \( P \) as a function \( f_l(x_c) \) of the circuit states. Hence the complete model can be represented in the following form:

\[
\dot{x} = \tilde{f}_s(x,u,\alpha), \quad y = \tilde{f}_l(x)
\]

where \( y = [V_l \ P]^T \) and \( u = [V_{in} \ \delta_{in}]^T \). As \( \alpha \) increases from zero, the behavior of the nonlinear elements deviates increasingly from linear behavior. Further details of the model are provided in Appendix 1.

This model is numerically linearized across a range of power and \( \alpha \) operating points using the Simulink linearization tool \texttt{linmod} [22] to provide a basis for validation and design, and for determination of suitable robustness and performance measures. A fine grid of 94 models covering the range of power and network impedance operating points (parameterized through \( \alpha \)) is denoted as the set \( P \) with members \( P_1 \ldots P_{94} \). The set of six operating points \( V \) in Table 1 around the edge of the operating range in the power-\( \alpha \) operating space are denoted by \( v_1 \ldots v_6 \) and capture the variation in behavior across the operating range. The lower power limit at high impedance arises because at such high impedance the capacity for transferring power while maintaining \( V_l = 1.0 \) pu becomes limited.

Open loop singular value plots of the linearized models at the two operating points \( v_1 \) and \( v_6 \) are presented in Fig. 2 to indicate the range of variation of the models. While all the models exhibit a resonant peak at the system electrical frequency of 50 Hz (314 rad/s) they exhibit variations in steady-state gain, dynamic characteristics and condition number across frequency. At larger impedances a right half plane zero arises which moves increasingly leftward as the impedance increases.

In Fig. 3 the singular value plot of the linearized model for the operating point \( \alpha = 1.0, P = 0.3 \) pu is compared with the results of a frequency sweep on the PSCAD model, generated with small input perturbations around the operating point. The correspondence between the linearized models and the rigorous model is good up to a frequency of approximately 500 rad/s, after which the effects of the PWM switching process begins to dominate the response. The correspondence between the phases in the bode plots is also good up to 500 rad/s. The correspondence for models with lower

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**Table 1.** Set of operating points \( V \)

| \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) |
| \( \alpha = 0.05, P = 0.1 \) | \( \alpha = 0.05, P = 1.0 \) | \( \alpha = 0.25, P = 0.1 \) | \( \alpha = 0.25, P = 1.0 \) | \( \alpha = 1.0, P = 0.1 \) | \( \alpha = 1.0, P = 0.5 \) |

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**Fig. 1.** VSC circuit.
values of $\alpha$ is as good or better than this. The standard deviation of the PWM ‘noise’ on the output channels of the PSCAD model is approximately 0.003 pu.

4. Control Objectives

The outputs $y = [V_i, P]^T$ are to be controlled using inputs $u = [V_{in}, \delta_{in}]^T$. The control objectives are categorized into requirements on stability, setpoint tracking, disturbance rejection and robustness as described below. In Section 7 the quality of the controllers designed are evaluated against these objectives.

Closed loop stability and zero steady-state error are required for $V_i$ and $P$ in response to step setpoint changes and disturbances across the grid of operating points $P$. Performance-wise, rapid responses to power setpoint changes (typically less than 0.5 pu) and voltage setpoint changes (typically less than 0.05 pu) without excessive overshoot or under-damping are desirable across this operating range with a bandwidth of at least 10 rad/s.

The response to step changes in system impedance as parameterized by $\alpha$ should also be responded to quickly and with reasonable damping. By inspection of linearized models including the changes in $\alpha$ as disturbances, it can be seen that the resulting frequency responses at the plant output have a resonant peak at 50 Hz, and a roll off beyond this frequency. Hence a good response to such disturbances requires relatively low output sensitivity in this range of frequencies.

The PWM process can at its simplest be represented as a set of periodic disturbances entering directly at the plant input, and also, due to various nonlinear effects, indirectly at the plant output. The time and frequency responses of the PSCAD model suggest that the PWM effects are significant beyond about 500 rad/s. In addition, from the harmonic analysis in [29], the demanded VSC voltage is reproduced without significant distortion for a four level converter for input signals bandlimited to about 10% of the switching frequency. For the PWM frequency of 450 Hz used here this suggests that the control sensitivity should roll off beyond about 50 Hz (314 rad/s).

In summary, to avoid the amplification of the PWM switching disturbances around the closed loop, the various sensitivities of the loop, in particular input sensitivity and plant output sensitivity to input disturbances, should be small beyond 50 Hz across the operating range; as this frequency range is beyond the bandwidth, ‘small’ means not much greater than one.

There is inevitable plant uncertainty in a power system arising from deviation in the system characteristics from the ideal present in the model. In particular, this includes deviation of the system impedance from the simple series resistance and reactance model used here. Furthermore, high frequency dynamics, such as those of high frequency AC filters, measurement devices, and internal dynamics of the VSC are not included in the model.

Finally, for a discrete time implementation with a sampling frequency of 1 kHz, the control demands should have a bandwidth limited to, at most, 500 Hz.

Using the linearized models, measures of performance and robustness against the control objectives are now discussed. Setpoint tracking performance is defined in terms of a time domain measure, while all other aspects of performance are covered by the reciprocal, $\gamma$, of the generalized stability margin, $\epsilon$, [14] which provides a composite measure of robustness and performance that is readily applicable when the LSDP is applied.
4.1. Setpoint Tracking

The measure \( J_{td} \) defined below captures the important characteristics of the response to power and voltage setpoint changes in terms of rise time, settling time, overshoot and cross coupling, normalized to desirable values. The nomenclature is explained in Table 2.

\[
J_{td} = \max_i j_i
\]

\[
j_1 = \frac{t_{rP} + t_{sP}}{2}, \quad j_2 = \frac{t_{rV} + t_{sV}}{2}, \quad j_3 = |P_s|, \quad j_4 = |V_s|, \quad j_5 = |P_o|, \quad j_6 = |V_o|
\]

Each component \( j_i \) of \( J_{td} \) in Eq. (2) represents an aspect of time domain performance and is normalized to its maximum desired value. Hence \( J_{td} \) represents the worst normalized deviation for an operating point, providing a quantification of setpoint response performance. \( J_{td}^{\text{max}} \), the maximum of \( J_{td} \) over \( P \), provides a measure of worst case time domain setpoint following performance across the operating range.

4.2. Use of the Generalized Stability Margin as a Performance and Robustness Measure

Among other properties, the generalized stability margin (and its reciprocal \( \gamma \)) neatly captures the quality of disturbance rejection, minimization of PWM effects, robustness to unmodeled dynamics and bandwidth requirements for a discrete implementation of a continuous controller:

- The combined requirements of disturbance rejection and PWM effect minimization can be summarized as requiring the closed loop input and output sensitivities, output sensitivity to input disturbances and input sensitivity to output disturbances all to be small in a frequency band around and immediately beyond the bandwidth. The requirements for a discrete-time implementation put bounds on the control sensitivity beyond about 500 Hz. Functions of the generalized stability margin and the LSDP design weights upper bound each of the sensitivity functions [14]. For example, in the critical area around the bandwidth the bound on the sensitivity at input and output is close to \( \frac{1}{\gamma} \).
- Robustness: the generalized stability margin has interpretations in terms of stabilization of plants nearby in terms of the Vinnicombe gap metric [26]. It is also related to bounds on, e.g., gain and phase margin with respect to different types of uncertainty [9].

The smaller \( \gamma \) is, the better performance and robustness are in these respects. Hence \( \gamma_{\text{max}} \), the maximum of \( \gamma \) over \( P \), provides a worst case composite measure of performance and robustness.

5. Glover–McFarlane Loop Shaping Design

In the LSDP the scaled open-loop plant is first shaped using frequency dependent design weights to reflect the desired open loop response, then the optimal LSDP controller \( K_s \) for the shaped plant is found by solving a \( \mathcal{H}_\infty \) minimization problem. One formulation of this optimization is the following four block \( \mathcal{H}_\infty \) problem.

\[
\begin{bmatrix}
  z_1 \\
  z_2 \\
  y_s
\end{bmatrix} = P_G \begin{bmatrix}
  w_1 \\
  w_2 \\
  u_s
\end{bmatrix},
\]

\[
P_G = I \begin{bmatrix}
  W_o PW_i & W_o PW_i \\
  0 & I
\end{bmatrix},
\]

\[
\in \mathbb{C}(j\omega)^{(n_s+n_o+n_i+n_k)(n_s+n_o+n_i+n_k)}
\]

Table 2. Time response abbreviations

| \( t_{rP}, t_{rV} \) | \( 90\% \) rise time (s) of \( P, V \) in response to unity \( P, V \) setpoint steps respectively normalized to 0.1 s |
| \( t_{sP}, t_{sV} \) | \( 95\% \) settling time (s) of \( P, V \) in response to unity \( P, V \) setpoint steps respectively normalized to 0.1 s |
| \( P_o, V_o \) | \( P \) overshoot (%) following unit \( P \), \( V \) setpoint steps normalized to 0.1 s |
| \( V_p, P_p \) | \( \max \) \( V, P \) deviation following \( 0.5 \) pu \( P \), \( V \) and \( 0.05 \) \( V \) setpoint steps respectively each normalized to 0.05 pu |

Fig. 4. Generalized plant for loop shaping design with controller.
The LSDP controller $K_s$ is the solution of

$$
\min_{K_s} \gamma : \quad \gamma = \| \text{LFT}(P_0, K_s) \|_\infty
$$

(4)

The VSC HVDC plant model is scaled so unity inputs approximately correspond to unity outputs at intermediate values of $\alpha$ and power demand. Following standard guidelines, e.g., [23], design weights to shape the VSC HVDC terminal plant are then readily found if only performance in the neighborhood of an operating point is of interest. Such designs are discussed after considering the selection of the nominal operating point.

### 5.1. Selection of Nominal Operating Point and Nominal Design

The approach taken here for selecting the nominal operating point is to find the operating point in the operating range whose maximum distance from the other operating points is minimized. One advantage of using the LSDP is that through its intimate connection with the Vinnicombe gap metric [26], an appropriate way of measuring the distance between open loop plants that takes into account desirable closed loop behavior is provided. That is, if the design weights are $W_i$ and $W_o$, an appropriate measure of distance is the following weighted gap between two plants $P_i$ and $P_j$

$$
\nu_{wij} = \nu(W_i P_j W_i^*, W_o P_j W_i),
$$

(5)

where $\nu$ is the Vinnicome gap metric, as calculated with the Matlab command nugap [13]. The nominal operating point is selected as follows. Let

$$
\tilde{p} = \arg\max_{p \in P} \nu_{wij},
$$

(6)

i.e., $\tilde{p}$ is the index of the operating point in the grid for which the largest weighted gap to all the other plants in the grid is smallest; $\tilde{p}$ is readily determined after calculating $\nu_{wij}$ for each pair of plants in $P$. The corresponding transfer function $P_0(s) \equiv P_{\tilde{p}}(s)$ is used as the nominal operating point. The operating point with $P = 0.3 \text{ pu}, \alpha = 0.3$ is the optimal one for the design weights $W = \{W_i, W_o\}$ that are focussed on in the remainder of the paper:

$$
W_i(s) = \frac{20}{s} I_2, \quad W_o(s) = \frac{1}{(s + 0.01)} I_2
$$

(7)

for which the maximum $\nu$-gap to other plants in the grid is 0.53.

The design weights $W$ provide integral action and unity crossover of the largest singular value of the loop transfer function of about 30 rad/s on the nominal model. The additional lag provides attenuation of the resonant peak and a high rate of roll off beyond 300 rad/s. A flattening of the open loop transfer function around the bandwidth to improve robustness is provided by the zero in the input filter of a second candidate set of weights, $W_z = \{W_{iz}, W_{oz}\}$:

$$
W_{iz}(s) = \frac{15(s + 0.02)}{s} I_2, \quad W_{oz}(s) = \frac{1}{(s + 0.003)} I_2.
$$

(8)

The $\nu$-optimal plant for the weights $W_z$ is close to that for $W$. The weights $W$ and $W_z$ generate the full order nominal controllers $K_{nF}$ and $K_{nF_z}$ for the plant $P_0(s)^1$. For these controllers the shaped plant $P_s$ and the loop gains resulting from the LSDP at the plant input and output, $L_i = K P_0$ and $L_o = P_0 K$, are shown in Fig. 5.

These controllers have good performance and fulfill the design requirements for plants in the neighborhood of the nominal plant. For example, the step responses of the nominal plant $v_o$ to unit step point steps introduced at point $r$ with each of the two controllers is smooth and has little overshoot as shown in Figs 6 and 7.

When the full operating range is considered, the weights $W_z$ provide a lower variation of $\epsilon$ across the operating range than $W$. However, this is at the expense of poorer attenuation around 1000 rad/s, as shown in

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1Subscripts $n$ for nominal, $F$ for full order controller, $z$ for zero.
the loop gain plots on the right hand graph of Fig. 5. This exhibits itself, for example, in poorer output sensitivity and complementary sensitivity across the operating range around 1000 rad/s than the weights $W$. Thus a key conflict in designing controllers for this plant is that between achieving a desirable bandwidth and avoiding too much excitation of the resonance across the operating range.

The step responses of neither controller is satisfactory at operating points ‘distant’ from $P_0(s)$. For $K_{nF}$ there is large overshoot at $v_6$ to power setpoints, while for $K_{nFS}$ there is a more pronounced oscillation early in the power response for $v_1$ and $v_6$, as indicated in Figs 6 and 7. This latter behavior can be alleviated by introducing the setpoint at the shaped plant input $u_s$ through the filter $K_s(0) W_s(0)$ [23] at the expense of more cross-coupling at high values of $\alpha$ and a slower response. While modification of the loop shapes or increasing their order may improve the behavior at distant operating points, such modification will involve
either trial and error, parameter gridding or a direct search to get an acceptable compromise between performance at nominal and distant operating points.

6. Ellipsoidal Set Approach to Controller Design

The application of this approach to multi-objective design (including multiple operating point performance) is described in [15]. Here the method is applied to formulate an upper bound for the maximum $H_\infty$ norm across a number of distinct operating points, each furnished with its own Lyapunov function. It is also possible to use the same formulation to impose a common Lyapunov function, in which case the formulation may be used to optimize quadratic performance across a set of operating points.

First the theorem for a single operating point is cited from [15], adapted for notational purposes to suit the LSDP formulation. The LSDP generalized plant $P_G$, partitioned into performance and input–output channels

$$P_G = \left( \begin{array}{ccc} A & B_w & B \\ C & D_{zw} & D_{zu} \\ C & D_{yw} & D_{yu} \end{array} \right)$$

and its dynamic controller $K_s$ are replaced by the equivalent plant $P_G^s$ and static controller $K_s^s$ [25] as detailed in Appendix 2; the replacement of the corresponding partitioned matrices of $P_G$ is denoted by the superscript $s$.

**Theorem 1. The Ellipsoidal Set $H_\infty$ Synthesis Formulation**

With the matrices $N_1$, $N_2$ and $N_3$ defined as:

$$N_1 = \begin{bmatrix} I & 0 & 0 \\ A^s & B^w & B^o \end{bmatrix}, \quad N_2 = \begin{bmatrix} C^s & D^s_{zw} & D^s_{zu} \\ 0 & I & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} C^s & D^s_{yw} & D^s_{yu} \\ 0 & 0 & I \end{bmatrix},$$

if and only if there exist $P \in S^{n_p}$, $X \in S^{n_y}$, $Y \in \mathbb{R}^{n_y \times n_x}$, and $Z \in S^{n_w}$ and the scalar $\tau_p$ that simultaneously satisfy the nonlinear constraint

$$X \leq YZ^{-1}YT$$

and the following LMI constraints

$$\tau_p > 0$$

$$Z > 0, \quad P > 0$$

then $\|\text{LFT}(G^s_w, K^s)\|_\infty < \gamma$ for each controller in the set $K^s$ is defined by

$$K^s = \left\{ K^s : Z > 0, [I \quad K^{sT} \quad Y \quad YT \quad Z] [I \quad K^s] \leq 0 \right\}.$$

The central member of this set is $K^s = -Z^{-1}YT$. $\square$

For the case of a single operating point, $\tau_p$ is 1 without loss of generality: any solution with $\tau_p = \tau_p \geq 0$ equates to a solution with the same controller with

$$W = \frac{W}{\tau_p}, \quad P = \frac{P}{\tau_p}.$$

Each additional operating point adds a constraint of the form (13). No conservatism is introduced into the multiple operating point formulation if each operating point has its own value of $\tau_p$ and its own Lyapunov matrix $P_G$ [15].

Two possible solution methods have been reported for dealing with the nonlinear constraint (10). The method proposed in [15] involves a reformulation to an equivalent problem that can be solved by the ‘cone complementarity’ algorithm of [6] while the method proposed in [10] uses a different iterative scheme involving a penalty function to reduce the rank of $W$ and hence provide equality in (10). In this paper a novel formulation using convexifying potential functions [4] is used. This was found to give results more quickly and with better numerical stability than the two methods described above. It is first described in detail for application to a single operating point, where it may be applied to the problem of improving the $H_\infty$ performance of an initial low order controller. The formulation is based on the following theorem.

**Theorem 2. A Feasible LMI for the Ellipsoidal Set $H_\infty$ Formulation**

The feasibility of the LMI

$$N_1^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} N_1 - N_2^T \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 \end{bmatrix} N_2 - N_2^T \begin{bmatrix} 0 & Y \\ YT & Z \end{bmatrix} N_3 + N_3^T \begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{G}(Y,Z,Y_o,Z_o)[I \quad 0] N_3 < 0$$

where

$$W = \begin{bmatrix} X & Y \\ YT & Z \end{bmatrix}$$

and its own Lyapunov function. It is also possible to use the same formulation to impose a common Lyapunov function, in which case the formulation may be used to optimize quadratic performance across a set of operating points.

First the theorem for a single operating point is cited from [15], adapted for notational purposes to suit the LSDP formulation. The LSDP generalized plant $P_G$, partitioned into performance and input–output channels

$$P_G = \left( \begin{array}{ccc} A & B_w & B \\ C & D_{zw} & D_{zu} \\ C & D_{yw} & D_{yu} \end{array} \right)$$

and its dynamic controller $K_s$ are replaced by the equivalent plant $P_G^s$ and static controller $K_s^s$ [25] as detailed in Appendix 2; the replacement of the corresponding partitioned matrices of $P_G$ is denoted by the superscript $s$.

**Theorem 1. The Ellipsoidal Set $H_\infty$ Synthesis Formulation**

With the matrices $N_1$, $N_2$ and $N_3$ defined as:

$$N_1 = \begin{bmatrix} I & 0 & 0 \\ A^s & B^w & B^o \end{bmatrix}, \quad N_2 = \begin{bmatrix} C^s & D^s_{zw} & D^s_{zu} \\ 0 & I & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} C^s & D^s_{yw} & D^s_{yu} \\ 0 & 0 & I \end{bmatrix},$$

if and only if there exist $P \in S^{n_p}$, $X \in S^{n_y}$, $Y \in \mathbb{R}^{n_y \times n_x}$, and $Z \in S^{n_w}$ and the scalar $\tau_p$ that simultaneously satisfy the nonlinear constraint

$$X \leq YZ^{-1}YT$$

and the following LMI constraints

$$\tau_p > 0$$

$$Z > 0, \quad P > 0$$

then $\|\text{LFT}(G^s_w, K^s)\|_\infty < \gamma$ for each controller in the set $K^s$ is defined by

$$K^s = \left\{ K^s : Z > 0, [I \quad K^{sT} \quad Y \quad YT \quad Z] [I \quad K^s] \leq 0 \right\}.$$

The central member of this set is $K^s = -Z^{-1}YT$. $\square$

For the case of a single operating point, $\tau_p$ is 1 without loss of generality: any solution with $\tau_p = \tau_p \geq 0$ equates to a solution with the same controller with

$$W = \frac{W}{\tau_p}, \quad P = \frac{P}{\tau_p}.$$
where matrices $X_o$, $Y_o$ and $Z_o$ have the same dimensions as $X$, $Y$ and $Z$, $X_o$ and $Z_o$ are symmetric, and

$$
\mathcal{G}_L(Y,Z,Y_o,Z_o) = Y_oZ_o^{-1}ZZ_o^{-1}Y^T - Y_oZ_o^{-1}Y^T - YZ_o^{-1}Y^T
$$

implies the feasibility of the LMI (13) and the non-linear matrix inequality (10).

Proof: The constraint

$$
X = YZ^{-1}Y^T
$$

(19)

corresponding to equality in the constraint (10) is used to eliminate $X$ from (13), leading to

$$
N_1^{-1} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} N_1 - N_2^{-1} \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 \end{bmatrix} N_2 - N_3^{-1} \begin{bmatrix} 0 & Y \\ Y^T & Z \end{bmatrix} N_3
- N_3^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} YZ^{-1}Y^T[I 0]N_3 < 0
$$

(20)

The function

$$
\mathcal{G}(Y,Z,Y_o,Z_o) = N_2^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} (Y_oZ_o^{-1} - YZ^{-1})
\times Z(Y_oZ_o^{-1} - YZ^{-1})^T[I 0]N_3
= N_3^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} (\mathcal{G}_L + YZ^{-1}Y^T)[I 0]N_3
$$

(21)

is a so-called convexifying potential for the nonlinear system of inequalities in Theorem 1 in terms of constant matrices $X_o$, $Y_o$ and $Z_o$ as a result of the properties detailed in [14] for functions of this form.

The addition of $\mathcal{G}$ to the left hand side of (13) causes cancellation of the nonlinear terms to form the LMI (17). The feasibility of this LMI implies that of the non-linear matrix inequality (10) because $\mathcal{G}$ is a convexifying potential.

The algorithm sketched below is proposed for synthesising the ‘ellipsoidal’ controller, which is in the form of the generic algorithm presented in [4]. Starting from an initializing controller $K^o_0$ the algorithm generates a converging sequence of feasible solutions $\{X,Y,Z,P,\gamma\}$ to the inequalities in Theorem 1 with decreasing values of $\gamma$.

Algorithm 1. ‘Ellipsoidal’ Controller Synthesis Using a Convexifying Function

Initialization Set $j = 0$, determine a static controller $K^o_0$ which provides $P^o_0$ with a finite closed loop $\mathcal{H}_\infty$ norm, set $Y^0 = -K^o_0^T$ and $Z^0 = I$

Step 1 Set $j = j + 1$

Step 2 Set $Y^j = Y^{j-1}$ and $Z^j = Z^{j-1}$; determine $\gamma^j$, the minimal value of $\gamma$ over the LMIs (11), (12) and (17), and $Y^j$ and $Z^j$, the corresponding values of $Y$ and $Z$

Step 3 If $\gamma^j - \gamma^{j-1}$ is small, calculate $K^j_0 = -(Z^j)^{-1}(Y^j)^T$ and Exit; otherwise go to Step 1

The initializing controller must be an LSDP controller of the required order that provides a finite $\mathcal{H}_\infty$ norm in closed loop. At each step the convexifying potential becomes smaller, and hence the inequality ‘gap’ becomes smaller. One advantage of this approach is that it provides an optimization program in $\gamma$, rather than a feasibility program in $\gamma$ as provided by the two published approaches.

To extend the method to multiple operating points, the LMI below is included in the program in place of (20) for each operating point $\rho \in \mathcal{P}$ to be considered:

$$
N_1^{-\rho} \begin{bmatrix} 0 & P^\rho \\ P^\rho & 0 \end{bmatrix} N_1 - N_2^{-\rho} \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 \end{bmatrix} N_2 - N_3^{-\rho} \begin{bmatrix} 0 & Y \\ Y^T & Z \end{bmatrix} N_3
- N_3^{-\rho} \begin{bmatrix} I \\ 0 \end{bmatrix} YZ^{-1}Y^T[I 0]N_3 < 0
$$

(22)

A key difference between this and the single point formulation in (17) is that each LMI has its own value of $\tau^\rho$, a variable, and the variable $\gamma$ is replaced by a constant target value $\gamma_{\text{max}}$, if the LMIs are feasible then $\gamma_{\text{max}}$ is an upper bound of $\max_{\rho \in \mathcal{P}}(\gamma^\rho)$. Hence for a multiple operating point synthesis the optimization program in $\gamma$ is replaced by a program of reducing $\gamma_{\text{max}}$ in small steps from an initial value until the LMIs become unfeasible, at each step updating the values of $Y_0$ and $Z_0$. To implement this, Algorithm 1 is modified by extending the initialization and replacing Steps 2 and 3 as follows:

Initialization extension Calculate $\gamma^0_{\text{max}}$, the value of $\max_{\rho \in \mathcal{P}}(\gamma^\rho)$ corresponding to the initializing controller (this must have a finite value of $\gamma_{\text{max}}$)

Step 2 Set $Y_0 = Y^{j-1}$, $Z_0 = Z^{j-1}$ and $\gamma_{\text{max}} = \gamma^{j-1}_{\text{max}}$ and determine feasibility of the LMIs (11), (12) and (17).

If the LMIs are feasible, set $\gamma^j_{\text{max}} = \gamma^{j-1}_{\text{max}}(1 - \gamma^j_{\text{min}})$, where $0 < \gamma^j_{\text{min}} < 1$; determine $Y^j$ and $Z^j$, the corresponding values of $Y$ and $Z$ that satisfy the LMIs.

If the LMIs are not feasible, set $\gamma^j_{\text{max}} = \gamma^{j-1}_{\text{max}}(1 + \delta^j_{\text{min}})$, where $\delta^j_{\text{min}} < \gamma^j_{\text{min}}$.

Step 3 If no further reduction in feasible $\gamma^j_{\text{max}}$ is achievable then Exit; otherwise go to Step 1
7. Results

The full order LSDP controller for the nominal operating point with the design weights $W$ was first reduced, in order, to orders 4, 2 and 1 (full controllers of order 8, 6 and 5) using the minimal multiplicative $H_\infty$ approach implemented in the Matlab Robust Control Toolbox\[13\]. From these starting points the reduced order, $H_\infty$ norm minimizing LSDP controllers were found using the single operating point formulation of the ellipsoidal set method. The resulting controllers have $\gamma$ equal to 2.27, 2.35 and 2.41 compared to the full order value of 2.24. The deterioration of performance in terms of $\gamma$ through reducing the order down to 2 through the ellipsoidal technique is hence quite small. To reach each of these values took less than 20 iterations and less than 1 minute of CPU time on a 2.4 GHz Pentium 4 processor using the SeDuMi solver\[24\] with the YALMIP interface\[12\]; similar final controllers were synthesized using the Matlab Robust Control Toolbox\[2\] but the improvement in each iteration was more gradual, requiring up to 100 iterations to settle, and the total synthesis time was longer, up to 5 minutes.

The multiple operating point formulation was then used to generate ‘multi-performance’ LSDP controllers at the full order (12th) and at 2nd and 1st order using the operating points in $V$; in this application the initializing controllers, the controllers from the low order synthesis, have finite values of $\gamma_{\text{max}}$. The low order designs required approximately 20 iterations for both the Matlab solver and SeDuMi to settle to similar controllers, but again the calculation time was shorter for SeDuMi, approximately 1 minute compared to 5 minutes for the Matlab solver. Full order designs required up to 4 hours for the Matlab solver while numerical problems prevented effective operation of the algorithm using SeDuMi.

The convexifying algorithm clearly depends on the initializing controller. Nonetheless, the sensitivity to the initial starting point does not appear to be excessive as demonstrated on the graphs in Fig. 8, which show the progress of the algorithm for two different initializing controllers, the 2nd order LSDP controller used for initializing the nominal design $K_{n20s}$ and the simple, low gain 2nd order LSDP controller

$$K_{2s} = 0.2 \begin{bmatrix} -1 & 0 \\ s + 0.02s & 1 \\ 0 & s + 0.02s \end{bmatrix}. \quad (23)$$

In each case the target multi-performance $H_\infty$ norm $\gamma_{\text{max}}$ is also shown. The sequences converge to similar final controllers even though they are starting from different points.

At each iteration of the convexifying algorithm the LMIs directly yield a feasible controller. In comparison, the ‘cone complementarity’ and ‘rank minimization’ approaches to solving the inequalities of the ellipsoidal approach, described in [15] and [10] respectively, each involve iterative algorithms in which particular conditions must be reached to guarantee generation of a feasible controller: in the case of the cone complementarity approach for example, the objective function must reach zero for a feasible controller to be guaranteed. In this application the use of the cone complementarity approach often generates feasible controllers without these conditions being fulfilled, but this is not always the case, leading the controller synthesis algorithm to be more complicated and to require more LMI problems to be solved than the convexifying algorithm. A potential advantage of the cone complementarity algorithm is that if an associated initializing

---

\[\text{Fig. 8. Progress of convexifying algorithm in multi-performance design: initializing controllers } K_{n20s} \text{ (left) and } K_{2s} \text{ (right).}\]
feasibility problem (an LMI problem) is indeed feasible, no a priori initializing controller is required; however, in this application this initializing problem is usually either infeasible or gives poor initialization. If on the other hand the same initializing controller is used as is used for the convexifying algorithm, the cone complementary algorithm provides little improvement; for example, when initialized by the same controllers as used in generating the graphs in Fig. 8, this algorithm settled to controllers with values of $\gamma_{\text{max}}$ of 6.3 and 19.5 respectively. In this application the progress of the ‘rank minimization’ algorithm is also less satisfactory than that of the convexifying algorithm: little improvement is provided by the algorithm if a priori initializing controllers are used, while if the algorithm is initialized by setting the penalty function to zero, the resulting controllers have values of $\gamma_{\text{max}}$ at least five times the values achieved using the convexifying algorithm.

7.1. Controller Performance on Linearized Models

Table 3 tabulates the following measures of nominal and robust performance:

- the nominal value $\gamma_0$ value of $\gamma$ and its worst case across $\mathcal{P}$, $\gamma_{\text{max}}$;
- the nominal value $J_{0d}$ of the setpoint tracking measure $J^d$ and its worst case across $\mathcal{P}$, $J_{\text{max}}^d$ and
- the nominal and worst case values across $\mathcal{P}$ of the peaks of the input sensitivity $S_I$ and output complementary sensitivity functions $T_O$, as example sensitivity functions relating to performance and robustness across the operating range

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_0$</th>
<th>$\gamma_{\text{max}}$</th>
<th>$J_{0d}$</th>
<th>$J_{\text{max}}^d$</th>
<th>$T_{00}$</th>
<th>$T_{\text{max}}$</th>
<th>$S_{I0}$</th>
<th>$S_{I\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{nF}$</td>
<td>2.24</td>
<td>5.63</td>
<td>0.97</td>
<td>4.17</td>
<td>1.02</td>
<td>2.46</td>
<td>1.50</td>
<td>2.98</td>
</tr>
<tr>
<td>$K_{mF}$</td>
<td>3.08</td>
<td>3.62</td>
<td>1.13</td>
<td>3.53</td>
<td>1.08</td>
<td>1.38</td>
<td>1.74</td>
<td>2.34</td>
</tr>
<tr>
<td>$K_{n2}$</td>
<td>2.36</td>
<td>7.13</td>
<td>0.97</td>
<td>4.54</td>
<td>1.09</td>
<td>3.13</td>
<td>1.56</td>
<td>3.29</td>
</tr>
<tr>
<td>$K_{m2}$</td>
<td>3.22</td>
<td>3.74</td>
<td>1.54</td>
<td>3.55</td>
<td>1.04</td>
<td>1.51</td>
<td>1.76</td>
<td>2.46</td>
</tr>
</tbody>
</table>

The Table shows that for each controller there is a trade-off between nominal performance and performance at distant operating points, and the trade-off deteriorates as the controller order reduces. The setpoint step responses with the controllers $K_{nF}$ and $K_{m2}$ are shown in Figs 9 and 10. The improved performance over the nominal full order controller $K_{nF}$, particularly the power response for operating point $v_{\text{ref}}$, is clearly visible and does not have the rapid oscillations at the start of the response suffered with $K_{nF}$.

It is clear that no a priori guarantees of performance across the operating range are provided by the approach used other than at the operating points in $\mathcal{V}$ – and even this is only for the linearized operating points. Further post-design analysis can be used to provide a indication of stability across the operating range using the concept of the point-wise Vinnicombe gap [28], albeit only a heuristic one. There is a frequency dependent ball of stabilized linear plants around each operating point in $\mathcal{V}$, defined by the excess of the point-wise stability margin of each controller designed over the point wise $\nu$-gap. Each weighted member of $\mathcal{P}$ lies within at least one of these balls for the multi-performance controllers designed, but this is not the case for the nominal controllers. This provides some assurance that with the multi-performance controllers not only is each operating point in $\mathcal{P}$ stabilized (which is readily established), but that each is a member of a stabilized set associated with the operating points used in the design. This falls short of guaranteeing stability or performance of the nonlinear controlled plant but is arguably a more practical measure than many other possible ones, such as the size of the ball of linear or quasi-LPV plants that is quadratically stabilized, as it does not share their conservatism.

7.2. Controller Behavior on the Rigorous, Nonlinear PSCAD Model

The time responses of the PSCAD model with the controllers $K_{nF}$, $K_{mF}$ and $K_{m2}$ to a sequence of setpoint and circuit changes are generated by simulation. The sequence consists of a step up (at 0.2s) and step down (0.7s) in power setpoint from 0.2 pu to 0.4 pu at $\alpha = 0.05$, followed by a change to $\alpha = 1$ at 1.2s, followed by a power setpoint step up (2.2s) and down (3.2s) from 0.2 pu to 0.3 pu. The responses are shown
in Figs 11, 12 and 13. The controllers have a discrete
time implementation with a sampling frequency of
1000 Hz, and anti-aliasing filters at the plant output
with a bandwidth of 500 Hz. To improve clarity of the
responses the voltage signals have been filtered by a
first order filter with a time constant of 0.003 s.

It is noted (but not shown) that closing the loop
does not amplify the apparent noise on the power and
voltage signals. Second, the response of the full and
second order multi-performance controllers to all the
disturbances, and in particular the step change in $\alpha$, is smaller than that of the nominal controller, as predicted.

8. Conclusions

This paper demonstrates that the ellipsoidal set con-
troller synthesis approach can be successfully used to
synthesize low order ‘multi-performance’ controllers
for a practical application: the controllers synthesized provide better overall performance over a large operating range than standard LSDP controllers. Of particular benefit is the fact that the heuristic and theoretical features of the LSDP are naturally extended to include multiple explicit operating points. As separate Lyapunov functions for each operating point are used, the conservatism inherent in approaches such as robust $H_{\infty}$ is avoided. It is even possible that each of the operating points be represented by a model of different order, or different physical meaning associated with each state, as would be the case for example if identified models were used instead of physically derived ones. The approach is thus both a flexible one and one that has practical appeal.

References

6. Farag A, Werner H. Robust $H_2$ controller design and tuning for the ACC benchmark problem and a real-time application. In Proc. 15th IFAC World Congress, Barcelona, Spain, 2002
Appendix 1

Further Model Details

The whole VSC terminal model from inputs $u = [V_{in} \delta_{in}]$ to outputs $y = [V_f P]^T$ can be represented by 2 linear elements and 3 non-linear elements with the interconnections shown in Fig. 14, as detailed below. The linear elements represent the circuit and the phase locked loop. The nonlinearities arise because of the changes in reference frame between the system voltage and the VSC terminal voltage, and the nonlinear relationship between the circuit and the voltage and power outputs.

Matrix and vector elements are indicated by square brackets in this Appendix.

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**Fig. 14.** Block diagram of the complete system with the non-linear elements $f_u$, $f_y$ and $f_c$. 
Linear elements

1) The linear circuit model \( S_c \) in the reference frame of the system voltage source, parameterized by \( \alpha \), represented by (24). The inputs to the model are the converter inputs \( v_{cd}, v_{cq} \); system voltage source orthogonal components \( v_{sd} \) and \( v_{sq} \), the two further inputs to the circuit, have the constant values of 1 and 0 respectively. The circuit outputs are \( v_{ld}, v_{lq} \).

\[
\dot{x}_c = A_c x_c + B_c \begin{bmatrix} v_{sd} \\ v_{sq} \\ v_{cd} \\ v_{cq} \end{bmatrix}, \quad \begin{bmatrix} v_{ld} \\ v_{lq} \end{bmatrix} = C_c x_c \tag{24}
\]

where

\[
A_c = 1000 \begin{bmatrix}
-0.945619 & 0.314159 & 0.942478 & 0 \\
-0.314159 & -0.945619 & 0 & 0.942478 \\
9.42478 & 0 & -9.45619 & 0.94159 \\
0 & 9.42478 & -0.314159 & -9.45619 \\
1.42857 & 0 & -1.42857 & 0 \\
0 & 1.42857 & 0 & -1.42857 \\
\end{bmatrix}
\]

\[
B_c = 1000 \begin{bmatrix}
0.314 & 0 & 0 & 0 \\
0 & 0.314 & 0 & 0 \\
0 & 0 & -3.14 & 0 \\
0 & 0 & 0 & -3.14 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
C_c = \begin{bmatrix} 3 & 0 & -3 & 0 & 1 & 0 \\ 0 & 3 & 0 & -3 & 0 & 1 \end{bmatrix}
\]

2) The linearised PLL model to measure the angle between the converter terminal and the voltage source represented by:

\[
\dot{x}_p = A_p x_p + b_p \delta_l, \quad \delta_l = c_p x_p \tag{26}
\]

where

\[
A_p = \begin{bmatrix} -200 & 1 \\ -2000 & 0 \end{bmatrix}, \quad b_p = \begin{bmatrix} 200 \\ 0 \end{bmatrix}, \quad c_p = [1 \ 0]
\]

Nonlinear Elements

1) The polar to rectangular conversion at the circuit input, \( f_d(V_{in}, \delta_{lm}) \)

\[
\begin{bmatrix} v_{cd} \\ v_{cq} \end{bmatrix} = \begin{bmatrix} V_{in} \cos(\delta_{lm} + \delta_{lm}) \\ V_{in} \sin(\delta_{lm} + \delta_{lm}) \end{bmatrix} \tag{28}
\]

2) The rectangular to polar conversion at the plant output, \( f_c (v_{lp}, v_{ld}) \)

\[
\delta_l = \arctan \left( \frac{v_{lp}}{v_{ld}} \right) \tag{29}
\]

3) The output voltage and Power as functions of the circuit states \( x_c \)

\[
\begin{bmatrix} V_l \\ P \end{bmatrix} = \begin{bmatrix} \sqrt{v_{ld}^2 + v_{ld}^2} \\ \sqrt{v_{ld}^2 + v_{ld}^2} \end{bmatrix} \tag{31}
\]

where \( v_{ld} \) and \( v_{ld} \) are functions of \( x_c \) as given in (30).
The Complete System

The complete system can hence be represented by the nonlinear system with the states \( x = [x_c, x_p]^T \)

\[
\dot{x} = \bar{f}_x(x, u, \alpha), \quad y = \bar{f}_y(x)
\]  

(32)

where \( y = [V_l, P]^T, \ u = [V_{in} \ \delta_{in}]^T \), with \( \bar{f}_x \) and \( \bar{f}_y \) as defined below:

\[
\bar{f}_x(x, u, \alpha) = \left[ \begin{array}{c} A_c(\alpha)x_c + B_c \left[ \begin{array}{c} 1 \\ 0 \\ V_{in} \cos(\delta_{in} + x_p[1]) \\ V_{in} \sin(\delta_{in} + x_p[1]) \\ A_p x_p + b_p \arctan \left( \frac{v_{ld}}{v_{id}} \right) \end{array} \right] \\ \sqrt{v_{id}^2 + v_{iq}^2} \\ v_{ld} x_c[3] + v_{iq} x_c[4] \end{array} \right]
\]

\[
\bar{f}_y(x) = \left[ \begin{array}{c} v_{id} x_c[3] + v_{iq} x_c[4] \end{array} \right]
\]

(33)

where \( v_{ld} \) and \( v_{iq} \) are functions of \( x_c \) as given in (30).

Appendix 2

Equivalence Between Static Feedback on an Augmented Plant and Dynamic Feedback

The generalized plant \( P_G \) from Eq. (3) is written as

\[
P_G = \frac{A_G B_G}{C_G D_G}
\]

(34)

The closed loop \( \text{LFT}(P_G, K_s) \) with the dynamic controller \( K_s = \frac{A_k B_k}{C_k D_k} \) is equal to the closed loop \( \text{LFT}(P^s_G, K_s) \). Where

\[
P^s_G = \frac{A^s_G B^s_G}{C^s_G D^s_G}
\]

\[
A^s_G = \frac{A 0}{0 0}, \quad B^s_G = \frac{B_w B 0}{0 0 1}
\]

\[
C^s_G = \frac{C 0}{0 I}, \quad D^s_G = \frac{D_{2w} D_{2u} 0}{D_{yw} D_{yu} 0}
\]

(35)

and \( K_s^s \) is the static controller

\[
K_s^s = \frac{D_k C_k}{B_k A_k}
\]

(36)