Kinematic mappings of plane affinities

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Abstract

In 1911 W. Blaschke and J. Grünwald described the group $\mathcal{G}$ of proper motions of the euclidean plane $\mathcal{E}$ in the following way: Let $(P, \mathcal{G})$ be the real three-dimensional projective space, let $\mathcal{E} \subset P$ be an isomorphic image of $\mathcal{E}$, and let $U \subset \mathcal{G}$ such that $\mathcal{E} \cup U$ is the projective closure of $\mathcal{E}$ in $P$. Then there is a bijection $\kappa : \mathcal{G} \to P' := P \setminus U$ called the kinematic mapping and an injective mapping $\mathcal{E} \times \mathcal{E} \to \mathcal{G}$, $(u, v) \mapsto [u, v]$ called the kinematic line mapping such that $[u, v] := \{\beta \in P' ; \beta(u) = v\}$ where the operation is defined by conjugation. A principle of transference is valid by which statements on group operations of $(\mathcal{G}, \mathcal{E})$ correspond with statements on incidence in the trace geometry of $P'$.

Following Rath (1988) I will show that a similar concept holds for the group of affinities of the real plane where $(P, \mathcal{G})$ is part of and spans the six-dimensional real projective space.

1. General kinematic spaces

Generalizing the classical ideas, Karzel gave the following definitions (see [3]): Let $(P, \mathcal{G})$ be a linear space and $(P, \cdot)$ be a group. Denote for $a \neq b$ by $a, b$ the line through $a, b \in P$ and for $T \subset P$ by $(T, \mathcal{G}(T))$ the trace geometry in $(P, \mathcal{G})$. $(P, \mathcal{G}, \cdot)$ is called an incidence group if $x \mapsto ax$ is a collineation for all $a \in P$ and is called a two-sided incidence group if $x \mapsto axb$ is a collineation for all $a, b \in P$. A kinematic space is a two-sided incidence group in which $1, a$ is a subgroup for all $a \in P \setminus \{1\}$.

Now I call an incidence group $(P, \mathcal{G}, \cdot)$ a general kinematic space (see [2]) if furthermore the mapping $x \mapsto x^{-1}$ is a collineation. $(P, \mathcal{G}, \cdot)$ is a general kinematic space if and only if it is a two-sided incidence group in which any line $G \in \mathcal{G}$ with $|G| \geq 3$ and $1 \in G$ is a subgroup.

A construction of Marchi and Zizioli [4] can be applied to find new examples of such spaces: Let $(P, \mathcal{G}, +)$ be a general kinematic space, $U$ a subgroup of $Aut(P, \mathcal{G}, +)$.
and $V$ a set of subgroups of $U$ such that the following condition holds:

(K) Let $V, V_1, V_2 \in V$, $\alpha, \beta \in U \setminus \{id\}$. Then

(a) $\alpha, \beta \in V \implies Fix \alpha = Fix \beta$,

(b) $\alpha V \alpha^{-1} \in V$,

(c) $V_1 \neq V_2 \implies V_1 \cap V_2 = \{id\}$.

Then we have

(1) (a) If $\mathcal{G}_U$ is the set of all cosets of subgroups of $V$ together with the set of all 2-sets of $U$ which are not contained in one of these cosets then $(U, \mathcal{G}_U, \cdot)$ is a two-sided incidence group.

(b) In the semidirect product $(M, \cdot) = P \times U$ with the multiplication $(a,\alpha)(b,\beta) = (a + \alpha(b), \alpha\beta)$, let $\mathcal{F}_M := \{G \times \{id\}; \ G \ni G \ni 0\} \cup \{(a - \alpha(a), \alpha); a \in P, V \in V\}$ and let $\mathcal{G}_M$ be the set of cosets of these subgroups of $\mathcal{F}_M$ together with the set of all 2-sets of $M$ which are not included in one of these cosets. Then $(M, \mathcal{G}_M, \cdot)$ is a general kinematic space [2] which we will call the connected space and denote by $(P, \mathcal{G}; U, V)$.

(c) For the subgroup $M_0 := \{0\} \times U$, $(M_0, \mathcal{G}_M(M_0))$ is a space which is isomorphic to $(U, \mathcal{G}_U)$.

(d) $(P, \mathcal{G}; U, V)$ is a kinematic space if and only if $\cup V = U$ and for all $\alpha \in U \setminus \{id\}$, $P = \{x - \alpha(x); x \in P\}$.

2. Kinematic groups

Let $(P, \mathcal{G}, \cdot)$ be a two-sided incidence group and $T \subset P$ an invariant subset. Then for each $a \in P$, $\tilde{a} : P \rightarrow P; x \rightarrow axa^{-1}$ is a collineation of $(T, \mathcal{G}(T))$ and $\tilde{P} := \{\tilde{a}; a \in P\} \leq Aut(T, \mathcal{G}(T))$. If $\kappa : P \rightarrow \tilde{P}$; $a \rightarrow \tilde{a}$ is injective then $(P, \mathcal{G}, T)$ is called a kinematic group and $\kappa^{-1}$ a kinematic map. The kinematic group is transitive if $\tilde{P}$ acts transitively on $T$.

Now let $(M, \mathcal{G}_M, \cdot) := (P, \mathcal{G}; U, V)$ be a connected space (where the group operation of $P$ is written by $+$) such that $\iota : P \rightarrow P; x \rightarrow -x$ is contained in $U$ and has no fixed elements. Then $2^* : P \rightarrow P; x \rightarrow 2x := x + x$ is injective and since $U \leq Aut(P, +)$, $(P, +)$ is abelian. Since $\mathcal{G}_M =$ \{\[\tilde{a} := (a, 1) \in M ; a \in P\] is an invariant subset of $M$ and the centralizer of $I$ in $M$ is the identity. Therefore

(2) The maps $- : P \rightarrow I; p \rightarrow \tilde{p} := (p, 1), \sim : I \rightarrow \tilde{I}; \tilde{p} = (p, 1) \rightarrow \tilde{p} = (p, 1)$ are injective and is a collineation of $(P, \mathcal{G})$ onto $(I, \mathcal{G}_M(I))$ (i.e. $(P, \mathcal{G})$ and $(I, \mathcal{G}_M(I))$ are isomorphic) and so $(M, \mathcal{G}_M, \cdot)$ is a kinematic group.

For $u, v \in P$ let $[\tilde{u}, \tilde{v}] := \{\Theta \in M; \Theta(\tilde{u}) = \tilde{v}\} = \{(a, x) \in M; 2a = v - \alpha(u)\}$. Then

(3) (a) $\Theta^{-1}[\tilde{u}, \tilde{v}] = [\tilde{u}, \tilde{v}] = [\Theta(\tilde{u}), \tilde{v}]$ in particular $[\tilde{u}, \Theta(\tilde{u})] = \Theta[\tilde{u}, \tilde{u}]$,

(b) $M_0 = [0, 0] \cong U$ and $[2u, 2v] = (v, 1) \cdot M_0 \cdot (u, 1)$,
(c) The following statements are equivalent:
   (i) \((M, \mathcal{G}_M, \cdot, I)\) is transitive,
   (ii) \(\forall u, v \in P: [\bar{u}, \bar{v}] \neq \emptyset\),
   (iii) \(\forall u \in P: [\bar{0}, \bar{u}] \neq \emptyset\),
   (iv) \(2^*\) is a permutation.

By (1)(c), (3)(b) and (3)(c) we obtain

4) \(\forall u, v \in P\) we have:
   (a) \([\bar{u}, \bar{v}]\) is a subspace of \((M, \mathcal{G}_M)\),
   (b) \([2u, 2v], \mathcal{G}_M([2u, 2v]) \) \(=\) \((U, \mathcal{G}_U)\) and \([2u, 2v] \cap I = \{(u + v, i)\},
   (c) The following statements are equivalent:
      (i) \([\bar{0}, \bar{u}], \mathcal{G}_M([\bar{0}, \bar{u}]]) \cong (U, \mathcal{G}_U),
      (ii) \([\bar{0}, \bar{u}] \neq \emptyset\),
      (iii) \(\exists x \in P: 2x = u,\)
      (d) \(\mathcal{D}_{2u} := \{[2u, 2x]; x \in P\} = \{(x, i) \cdot M_0 \cdot (u, i); x \in P\} \) and \(\mathcal{D}_{2u} := \{[2x, 2u]; x \in P\} = \{(u, i) \cdot M_0 \cdot (x, i); x \in P\}\) are spreads in \((M, \mathcal{G}_M)\).

We now consider the following example. Let \(K\) be a field of characteristic \(\neq 2\), \(n \in \mathbb{N}\) and \((P, K) = (K^n, K)\) the \(n\)-dimensional vectorspace. Let \(\mathcal{L}\) resp. \(\mathcal{H}\) be the set of all 1- resp. \(n - 1\)-dimensional vectorsubspaces of \((P, K)\) then \(\mathcal{G} := \{a + L; a \in P, L \in \mathcal{L}\}\) is the line set of the corresponding affine space \(A(K^n, K)\) and \((P, \mathcal{G}, +)\) is a commutative kinematic space. For \(U := GL(n, K)\), \((M, \cdot, +) := P \times U\) is the group of all linear affinities of the affine space \(A(K^n, K)\) and \(M\) can be identified with the subset \(\{(x) \in M_{n+1, n}; \det X \neq 0\}\) of the set of all \((n + 1) \times n\)-matrices of \(K\), hence with the pointset of the \((n + 1) \cdot n\)-dimensional affine space \(A(M_{n+1, n}, K)\) where the points of the manifold \(Q := \{(x) \in M_{n+1, n}; \det X = 0\}\) are omitted.

Now let \(E := (\delta_{ij}) \in U; U_H := \{a \in U; H \subset \text{Fix} a\}\) with \(H \in \mathcal{H}\) and \(V := \{U_H; H \in \mathcal{H}\} \cup \{K^*E\}\). Then \(U \leq \text{Aut} (P, \mathcal{G}, +)\) and \(V\) is a set of subgroups of \(U\) fulfilling (K), where (a) is fulfilled because an affinity that fixes more than a hyperplane pointwise is the identity. Then the connected space \((P, \mathcal{G}, U, V)\) together with \(I := P \times \{1\}\), \(i = -E\) is a transitive kinematic group such that \(2^* : K^n \rightarrow K^n; x \rightarrow 2x\) is a permutation (cf. (3)(c)) and the set \(\mathcal{F} := \{[\bar{u}, \bar{v}]; u, v \in P\}\) consists of isomorphic subspaces. \((P, \mathcal{G}, U, V)\) is not a kinematic space because \(M\) contains noninvolutory elements which do not fix a hyperplane pointwise.

With \(M\) each subset \(Y\) of \(M\) is a subset of \(\mathcal{M}_{n+1, n}(K)\). Let \(< Y >_a\) denote the smallest affine subspace of \(A(\mathcal{M}_{n+1, n}, K)\) which contains the set \(Y\). If \(Y\) is a subspace of \((M, \mathcal{G}_M)\) then \(\dim_a(Y) := \dim(< Y >_a)\) shall be called the affine dimension of \(Y\). Then we have \(\dim_a(M_0) = n^2\). If we extend the multiplication \(\cdot\) of \(M = K^n \times GL(n, K)\) onto \(\mathcal{M}_{n+1, n} = K^n \times \mathcal{M}_{n,n}\) by \((a, A) \cdot (b, B) := (a + Ab, AB)\) then for each \((a, A) \in M\) the maps

\[
(a, A)_l : \mathcal{M}_{n+1, n} \rightarrow \mathcal{M}_{n+1, n}; (x, X) \rightarrow (a + Ax, AX),
\]

\[
(a, A)_r : \mathcal{M}_{n+1, n} \rightarrow \mathcal{M}_{n+1, n}; (x, X) \rightarrow (x + Xa, XA)
\]
are bijective affinities of the affine space \( A(\mathcal{M}_{n+1,n}, K) \) and the \((a,E)\) are even translations fixing the subspace \( I \). Therefore, by (4)(d) we have:

(5) Each \( u \in P \) defines a spread \( \mathcal{S}_u \) and for the elements \([\tilde{u}, \tilde{x}]\), \([\tilde{u}, \tilde{y}]\) of \( \mathcal{S}_u \) we have:

(a) \( < [\tilde{u}, \tilde{x}] >_a \parallel < [\tilde{u}, \tilde{y}] >_a \). (\( \parallel \) denotes the parallelism of the affine space \( A(\mathcal{M}_n, K) \)),

(b) \( < [\tilde{u}, \tilde{x}] >_a = < [\tilde{u}, \tilde{y}] >_a \iff x = y \),

(c) \( \forall \theta \in M \exists ! [\tilde{u}, \tilde{x}] \in \mathcal{S}_u : \theta \in [\tilde{u}, \tilde{x}] \) namely \( x = \theta(u) \) hence \( [\tilde{u}, \tilde{x}] = [\tilde{u}, \Theta(u)] \).

(d) \( [\tilde{u}, \Theta(u)] \cap I = \{([1/2(u + \Theta(u)), t])\} \) (by (4)(b)), i.e. each \( u \in P \) defines via the spread \( \mathcal{S}_u \) a parallel projectivity on \( M \) onto \( I \), with the direction \( < [\tilde{u}, \tilde{u}] >_a \) which is given by

\[
(\mathcal{S}_u) : M \to I; \quad \theta \to ([1/2(u + \Theta(u)), t]).
\]

Instead of a kinematic line mapping we have in general the bijection

\[
[\cdot, \cdot] : I \times I \to \mathcal{S}; \quad ([\tilde{u}, \tilde{v}] \to [\tilde{u}, \tilde{v}]\]

which extends the 'principle of transference' between properties of the permutation group \((M,I)\) and incidence properties of \((M,\mathcal{S})\). Let \((u_1, u_2, u_3)\) and \((v_1, v_2, v_3)\) be two triples of points of \( P = K^2 \) and \( T_i := [\tilde{u}_i, \tilde{v}_i] \) \((i \in \{1,2,3\})\) the triple of corresponding subspaces of \((M,\mathcal{S}_M)\). Then \( T_1 \cap T_2 \cap T_3 \neq \emptyset \) if and only if there is an affinity \( \sigma \in M \) with \( \sigma(u_i) = v_i \) for \( i \in \{1,2,3\} \) and we have:

\[
|T_1 \cap T_2 \cap T_3| = 1 \iff \{u_1, u_2, u_3\} \text{ and } \{v_1, v_2, v_3\}
\]

are not collinear,

\[
T_1 \cap T_2 \cap T_3 = \emptyset \iff \{u_1, u_2, u_3\}
\]

not collinear and \( \{v_1, v_2, v_3\} \) collinear, or \( \{u_1, u_2, u_3\} \) collinear and \( \{v_1, v_2, v_3\} \) are not collinear or \(|\{u_1, u_2, u_3\}| \neq |\{v_1, v_2, v_3\}|\).

\[
|T_1 \cap T_2 \cap T_3| > 1 \iff \{u_1, u_2, u_3\} \text{ and } \{v_1, v_2, v_3\}
\]

are collinear and one of the conditions holds:

\[
|\{u_1, u_2, u_3\}| = |\{v_1, v_2, v_3\}| = 3, u_i = u_j \neq u_k \text{ and } v_i, v_j \neq v_k \text{ for } \{i,j,k\} = \{1,2,3\}, u_1 = u_2 = u_3 \text{ and } v_1 = v_2 = v_3.
\]

3. Plane affinities

In the example of Section 2, let \( K = \mathbb{R} \), and \( n = 2 \). Then \( Q \) is a cone in the six-dimensional affine space \( A(\mathcal{M}_{3,2}(\mathbb{R}), \mathbb{R}) \) with the two-dimensional vertex \( \{x(x); x \in \mathbb{R}^2\} \). In the connected space \((\mathbb{R}^2, \mathcal{S}; GL(2, \mathbb{R}), V)\) there are the following substructures
\( S := (\mathbb{R}^2, \mathcal{G}; U, V_U) \) which are also kinematic groups and which are determined by one of the following subgroups \( U \) of \( GL(2, \mathbb{R}) \):

1. Proper similarities:

\[
U := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} ; \ a, b \in \mathbb{R}, a^2 + b^2 \neq 0 \right\}.
\]

Here \( M_1 := \mathbb{R}^2 \times U \) is a four-dimensional affine space where a plane is omitted.

2. Equiaffine transformations: \( U := SL(2, \mathbb{R}) \). Here \( M_2 := \mathbb{R}^2 \times U \) is a hypercylinder.

3. Proper euclidean displacements: \( U := O_+(2, \mathbb{R}) \), hence \( M_3 := \mathbb{R}^2 \times U \) can be identified with the three-dimensional projective space where a line is omitted and \( M_3 = M_1 \cap M_2 \).

4. Isotropic similarities:

\[
U := \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} ; \ a, b, c \in \mathbb{R}, ac \neq 0 \right\}.
\]

\( M_4 := \mathbb{R}^2 \times U \) is a five-dimensional affine space where two hyperplanes \( \mathbb{R}^2 \times H_i \) with \( H_1 := \left\{ \begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix} ; \ b, c \in \mathbb{R} \right\} \) and \( H_2 := \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} ; \ a, b \in \mathbb{R} \right\} \) are omitted.

5. Equiaffine isotropic similarities:

\[
U := \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} ; \ a \in \mathbb{R} \right\}.
\]

\( M_5 := \mathbb{R}^2 \times U \) is a three-dimensional affine space.

6. Pseudo-euclidean displacements:

\[
U := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} ; \ a \in \mathbb{R} \setminus \{0\} \right\}.
\]

\( M_6 := \mathbb{R}^2 \times U \) is a hypercylinder in a four-dimensional affine space.

References