2-extendability of toroidal polyhexes and Klein-bottle polyhexes

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A toroidal polyhex (resp. Klein-bottle polyhex) described by a string $\langle p, q, t \rangle$ arises from a $p \times q$-parallelogram of a hexagonal lattice by a usual torus (resp. Klein bottle) boundary identification with a torsion $t$. A connected graph $G$ admitting a perfect matching is $k$-extendable if $|V(G)| \geq 2k + 2$ and any $k$ independent edges can be extended to a perfect matching of $G$. In this paper, we characterize 2-extendable toroidal polyhexes and 2-extendable Klein-bottle polyhexes.

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1. Introduction

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A matching of $G$ is a set of independent edges. A matching $M$ of $G$ is perfect (or 1-factor) if $M$ covers every vertex of $G$. A connected graph $G$ is $k$-extendable if $|V(G)| \geq 2k + 2$ and any $k$ independent edges of $G$ belong to a perfect matching of $G$. A 2-extendable bipartite graph is also called a brace. Braces play a key role in matching theory [14]; for example, braces are fundamental blocks in the “tight edge cut decomposition” of 1-extendable graphs [13]. A generating method for all braces was given by McCuaig [16]. Robertson, Seymour and Thomas [23], and independently, McCuaig [17] presented a good characterization for the braces admitting a Pfaffian orientation (see also [18]).

The extendability of $G$ is the maximum integer $k$ such that $G$ is $k$-extendable. The extendability of a bipartite graph can be determined in polynomial time [12,28]. Let $G$ be a graph admitting a 2-cell embedding on a surface $\Sigma$. $G$ is a strong embedding (or closed 2-cell embedding [19]) if every face of $G$ is bounded by a cycle. If $\Sigma$ is a surface other than the sphere, Dean [3] proved that the extendability of $G$ is no more than $1 + \lfloor \sqrt{4 - 2\chi} \rfloor$, where $\chi$ is the Euler characteristic of $\Sigma$. According to Dean’s result, the extendability of a graph embeddable on the torus or the Klein-bottle is at most 3. For a planar graph $G$, its extendability is at most 2 [22]. A fullerene graph is a cubic 3-connected plane graph with 12 faces with size 5 and other faces of size 6. By a result of Holton and Plummer concerning cubic 3-connected plane graphs [5], Zhang and Zhang [30] pointed out that the extendability of a fullerene graph is 2.

A fullerene graph corresponds to a spherical fullerene molecule in chemistry. After the discovery of spherical fullerenes, the extension of fullerenes on other surfaces was considered [4]. The torus and the Klein bottle are the only two surfaces that can be tiled entirely with hexagons; the corresponding tilings are called toroidal polyhex and Klein-bottle polyhex [4,8], respectively, also “elemental benzenoids” [10]. We may refer to [7,9] for a comprehensive discussion on fullerene structures. A perfect matching of a graph coincides with a Kekulé structure of the corresponding organic molecule. Kekulé structures play an important role in resonance theory and valence bond theory. There is a series of work enumerating perfect matchings

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of toroidal polyhexes and Klein-bottle polyhexes [2,6,11]. Recently, k-resonant toroidal polyhexes and k-resonant Klein-bottle polyhexes were characterized [25,26].

A toroidal polyhex can be considered as a hexagonal tesselation (or dually triangulations) of the torus [20,27] determined by a unique string of three integers [1]. Toroidal polyhexes are bipartite and cover many interesting graphs, such as $K_{1,3}$, Cube ($Q_3$), Heawood graph, generalized Petersen graph $G(8, 3)$ [15,24] and some circulant graphs [29]. A Klein-bottle polyhex is treated as a hexagonal tesselation of the Klein bottle. Thomassen [27] classified Klein-bottle polyhexes into five classes. Klein-bottle polyhexes considered here are bipartite and can be analogously defined as toroidal polyhexes (detailed definitions are deferred to Section 2). Other Klein-bottle polyhexes are non-bipartite. An example of non-bipartite Klein-bottle polyhexes is shown in Fig. 1.

Both a toroidal polyhex and a Klein-bottle polyhex are 1-extendable [25,26]. But they are not 3-extendable since a k-extendable graph is $(k + 1)$-connected [21]. In this paper, we consider the 2-extendability of toroidal polyhexes and Klein-bottle polyhexes and obtain the following main theorem.

**Theorem 1.1.** A toroidal polyhex (resp. Klein-bottle polyhex) is a brace if and only if it is a strong embedding. □

Theorem 1.1 implies that all strong embeddable toroidal polyhexes and Klein-bottle polyhexes are 2-extendable.

2. Preliminaries

Let $P$ be a $p \times q$-parallelogram in a hexagonal lattice as illustrated in Fig. 2: every corner of $P$ lies at the center of a hexagon, the top side is parallel to the bottom side across $p$ vertical edges and the two parallel lateral sides pass through $q$ edges perpendicular to them. A **toroidal polyhex** $H(p, q, t)$ is obtained from $P$ with a torus boundary identification: first identify two lateral sides along the same direction and then identify the bottom side to the top side along the same direction with a torsion $t$ (see Fig. 2 (left)). Analogously, a **Klein-bottle polyhex** $K(p, q, t)$ is obtained from $P$ by the following boundary identification: first identify two lateral sides along the same direction and then identify the bottom side to the top side along the reverse directions with a torsion $t$ (see Fig. 2 (right)).

For convenience, we adopt the affine coordinate system $XOY$ for $H(p, q, t)$ and $K(p, q, t)$ as introduced in [25,26]: take the bottom side as x-axis and one lateral side as y-axis such that $P$ lies on the non-negative region and the origin $O$ is the intersection of x-axis and y-axis, and define the distance between a pair of parallel edges of a hexagon to be the unit length. According to this affine coordinate system $XOY$, label each hexagon by its center coordinates $(x, y)$ and denote it by $(x, y)$ or $h_{x,y}$ where $x \in \mathbb{Z}_p, y \in \mathbb{Z}_q$ (for any integer $n, \mathbb{Z}_n := \{0, 1, \ldots, n - 1\}$). Let $e$ be the upper one of the pair of edges in $h_{x,y}$ perpendicular to y-axis. Color its up-left end by black and another end by white. Then such a 2-coloring gives a bipartition of $H(p, q, t)$ and $K(p, q, t)$. Denote the black end and the white end of $e$ by $b_{x,y}$ and $w_{x,y}$, respectively (see Fig. 3). By this notation, $w_{0,y}$ is adjacent to $b_{0,y}$ and $w_{x,0}$ is adjacent to $b_{x-1,0}$, where $x = x + t + 1$ for $H(p, q, t)$ and $x = p - x + t + 1$ for $K(p, q, t)$. For example, $H(6, 3, 2)$ and $K(6, 3, 2)$ arise from the $6 \times 3$-parallelogram of a hexagonal lattice in Fig. 3 with torsion $t = 2$. The even cycle $w_{0,y}b_{1,y} \cdots w_{t-1,y}b_{y}w_{y} \cdots b_{t-1,y}w_{0,y}$ of $H(p, q, t)$ (resp. $K(p, q, t)$) is also called the yth layer of $H(p, q, t)$ (resp. $K(p, q, t)$). Note that, in each yth layer, the vertices incident with an upward vertical edge are black, and the ones incident with a downward vertical edge are white.

Both $H(p, q, t)$ and $K(p, q, t)$ have $2pq$ vertices. None of them with less than 6 vertices is a strong embedding. Clearly, $H(1, 1, 0)$ is the unique toroidal polyhex with 2 vertices; $H(2, 1, 0), H(2, 1, 1)$ and $H(1, 2, 0)$ are the only three toroidal
polyhexes with 4 vertices. Note that $H(1, q, 0)$, $H(p, 1, 0)$ and $H(p, 1, p - 1)$ contain a hexagon (i.e. the hexagon $(0, 0)$) which is not bounded by a cycle. So they are not strong embeddings. But a toroidal polyhex $H(p, q, t)$ different from $H(1, q, 0)$, $H(p, 1, 0)$ and $H(p, 1, p - 1)$ is clearly a strong embedding. Analogously, $K(1, 1, 0)$ is the unique Klein-bottle polyhex with 2 vertices; $K(2, 1, 0)$, $K(2, 1, 1)$ and $K(1, 2, 0)$ are the only three Klein-bottle polyhexes with 4 vertices. It can be verified that a Klein-bottle polyhex $K(p, q, t)$ is a strong embedding if and only if $\min\{p, q\} \geq 2$. (The hexagon $(0, 0)$ in $K(1, q, 0)$ and the hexagon $(\frac{i+1}{2}, 0)$ in $K(p, 1, t)$ are not bounded by cycles.)

Let $M_1 := \{w_{i,-1}, b_{i,0} | j \in \mathbb{Z}_q \text{ and } i \in \mathbb{Z}_p\}$, $M_2 := \{w_{i,0}, b_{i,0} | j \in \mathbb{Z}_q \text{ and } i \in \mathbb{Z}_p\}$ and $M_3 := \{\text{vertical edges of } H(p, q, t) \text{ (resp. } K(p, q, t))\}$. Then $M_1$, $M_2$ and $M_3$ of $H(p, q, t)$ (resp. $K(p, q, t)$) are pairwise disjoint perfect matchings and compose its edge set.

**Theorem 2.1.** Both $H(p, q, t)$ and $K(p, q, t)$ with at least 4 vertices are 1-extendable. □

An isomorphism between two graphs $G_1$ and $G_2$ is a bijection $\phi : V(G_1) \rightarrow V(G_2)$ such that, for any $u, v \in V(G_1), uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. An automorphism of a simple graph $G$ is an isomorphism from $G$ to itself. An automorphism $\phi$ of $G$ induces a permutation on $E(G) : \phi(uv) = \phi(u)\phi(v)$ for $uv \in E(G)$.

For $H(p, q, t)$ and $K(p, q, t)$, define two shifts on their vertex sets as introduced in [25,26]: the $r$–$l$ shift $\phi_r$ moves every vertex one unit along the reverse direction of the $x$-axis, i.e.,

$$\phi_r(w_{x,y}) = w_{x-1,y} \quad \text{and} \quad \phi_r(b_{x,y}) = b_{x-1,y};$$

and the $t$–$b$ shift $\phi_t$ moves every vertex one unit along the reverse direction of the $y$-axis, i.e.,

$$\phi_t(w_{x,y}) = w_{x,y-1} \quad \text{and} \quad \phi_t(b_{x,y}) = b_{x,y-1} \quad \text{for } 1 \leq y \leq q - 1,$$

$$\phi_t(w_{x,0}) = w_{x,q-1} \quad \text{and} \quad \phi_t(b_{x,0}) = b_{x,q-1},$$

where $x' = x+t$ for $H(p, q, t)$ and $x' + 1 = x'$ for $K(p, q, t)$.

For $H(p, q, t)$, let $\langle \phi_{h}, \phi_{t} \rangle$ be the subgroup of the automorphism group generated by $\phi_h$ and $\phi_t$. For each pair of edges $e, e' \in M_i$, $e$ can be transferred to $e'$ by shifts $\phi_h$ and $\phi_t$; that is, there is $\phi \in \langle \phi_{h}, \phi_{t} \rangle$ such that $\phi(e) = e'$. We also say that $\langle \phi_{h}, \phi_{t} \rangle$ acts transitively on $M_i$ for $i = 1, 2, 3$.

**Lemma 2.2.** For each $M_i$, of $H(p, q, t)$, $\langle \phi_{h}, \phi_{t} \rangle$ acts transitively on $M_i$, and $\phi(M_i) = M_i$ for each $\phi \in \langle \phi_{h}, \phi_{t} \rangle$. □

Two graphs embedded on a surface are equivalent if there exists a face-preserving isomorphism between them.

**Lemma 2.3** ([26]). $\phi_t$ and $\phi_{h}$ are two hexagon-preserving isomorphisms from $K(p, q, t - 2)$ to $K(p, q, t - 2)$, respectively. □

**Theorem 2.4** ([26]). All Klein-bottle polyhexes $K(p, q, t)$ for $t = 0, 1, \ldots, p - 1$ are equivalent. □

By Theorem 2.4, $K(p, q, t)$ can be abbreviated as $K(p, q)$. $K(p, q, t)$ is regarded as a representation of $K(p, q)$.

Let $S$ be a subgraph of $H(p, q, t)$ (resp. $K(p, q)$) such that each component is either a hexagon or an edge with end vertices. Then $S$ is called an ideal configuration [25,26] if $S$ is alternately incident with white and black vertices along any direction of each layer. An ideal matching is an ideal configuration $S$ without hexagons as components. For example, in Fig. 4, the non-ideal matching on the right side is incident with two consecutive white vertices in the 0th layer. By Lemma 3.1 of Ref. [25] and Lemma 3.2 of Ref. [26], we have the following result.

**Lemma 2.5.** An ideal matching $S$ of $H(p, q, t)$ (resp. $K(p, q)$) can be extended to a perfect matching. □
3. Toroidal polyhexes $H(p, q, t)$

In this section, we consider the 2-extendability of $H(p, q, t)$.

**Lemma 3.1.** For $\min(p, q) \geq 2$, $H(p, q, t)$ is a brace.

**Proof.** Let $H(p, q, t)$ be a toroidal polyhex with $\min(p, q) \geq 2$. Then $H(p, q, t)$ has at least 8 vertices. Let $e_1$, $e_2$ be any two independent edges of $H(p, q, t)$. Suppose $e_1 \in M_i$, $e_2 \in M_j$ $(i, j \in \{1, 2, 3\})$. If $i = j$, then $M_i$ is a perfect matching containing both $e_1$ and $e_2$. So we may assume that $i \neq j$. In the following, we will construct an ideal matching $\delta$ containing both $e_1$ and $e_2$. Then by **Lemma 2.5**, $H(p, q, t)$ has a perfect matching $M$ containing $e_1$, $e_2 \in \delta$. That is $H(p, q, t)$ is a brace.

Case 1. $e_1 \in M_1$ and $e_2 \notin M_1$. According to **Lemma 2.2**, we may assume $e_1 = w_{0,0}b_{1,0}$.

**Subcase 1.1.** $e_2 = w_{0,0}b_{1,0}$, $y 
eq 0$. Then $E_0 := \{w_{0,y}b_{1,0} | y \in \mathbb{Z}_q\}$ and $E_y := \{w_{x,y}b_{1,y} | x \in \mathbb{Z}_p\}$ for $y \neq 0$. Then $M = \cup_{y \in \mathbb{Z}_q} E_y$ is a perfect matching containing $e_1$ and $e_2$.

If $y_2 = 0$, then $2 \leq x_2 \leq p - 1$ since $e_1$ and $e_2$ are disjoint. Choose a series of vertical edges (see Fig. 5):

$$w_{1,0}b_{2+1,0} \quad \text{and} \quad w_{x,y}b_{2+1,y} \quad \text{for} \quad y = 1, 2, \ldots, q - 1.$$

Let $\delta := \{w_{x,y}b_{2+1,y} | y \in \mathbb{Z}_q \setminus \{0\}\} \cup \{w_{1,0}b_{2+1,0}, e_1, e_2\}$. Then $\delta$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_{0,0}, b_{1,0}, w_{1,0}, b_{2,0}, w_{2,0}, b_{2+1,0}$, and $w_{2+1,0}, b_{2+1,0}, b_{2+1,y}$ in the $y$th layer ($1 \leq y \leq q - 2$), and $w_{2+1,q-1}, b_{2+1,q-1}$ in the $(q - 1)$th layer.

**Subcase 1.2.** $e_2 \notin M_1$. Assume that $e_2 = w_{x_2,y}b_{x_2+1,y} - 1$ if $y_2 \neq 0$, and $e_2 = w_{x_2,0}b_{x_2+1,0} - 1$, otherwise.

Choose a series of vertical edges (see Fig. 6):

$$w_{1,0}b_{2+1,0} \quad \text{for} \quad y = 1, 2, \ldots, y_2 - 1,$$

$$w_{x,y}b_{2+1,y} \quad \text{for} \quad y = y_2 + 1, \ldots, q - 1.$$

Let $E := \{w_{1,0}b_{2+1,y} | 1 \leq y \leq y_2 - 1\} \cup \{w_{x_2,y}b_{x_2+1,y} - 1 | y_2 + 1 \leq y \leq q - 1\}$.

If $y_2 = 0$, then $x_2 \neq 0$ since $e_1$ and $e_2$ are disjoint. Note that $E = \{w_{x_2,y}b_{x_2+1,y} - 1 | y_2 + 1 \leq y \leq q - 1\}$. Then $\delta = E \cup \{e_1, e_2\}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_{0,0}, b_{1,0}, w_{1,0}, b_{2,0}, w_{2,0}, b_{2+1,0}$, and $b_{2+1,y}$ in the $(q - 1)$th layer.

Suppose $y_2 \neq 0$. Note that $x_2 \neq 0$ if $y_2 = 1$. Let $\delta := E \cup \{w_{1,0}b_{2+1,0} - 1, e_1, e_2\}$. Then $\delta$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_{0,0}, b_{1,0}, w_{1,0}, b_{2,0}, b_{2+1,0}$ ($x = 2$ if $y_2 \neq 1$, and $x = x_2$, otherwise), and two vertices with different colors in all other $y$th layers (see Fig. 6 (right)).
Case 2. $e_1 \in M_2$ and $e_2 \in M_3$. By Lemma 2.2, we may assume that $e_1 = w_{1,0}b_{1,0}$. Let $e_2 = w_{x_2,y_2}b_{x_2+1,y_2−1}$ if $y_2 \neq 0$, and $e_2 = w_{x_2,0}b_{x_2+t+1,q−1}$, otherwise.

Choose a series of vertical edges (see Fig. 7):

$w_{1,y}b_{y−1}$ for $y = 1, 2, \ldots, y_2 − 1, y_2 + 1, \ldots, q − 1$.

If $y_2 = 0$, then $x_2 \neq 1$. Let $\delta = \{w_{1,y}b_{y−1} \mid y \in \mathbb{Z}_q \setminus \{0\}\} \cup \{e_1, e_2\}$. Then $\delta$ is an ideal matching since it is incident with $b_{1,0}, w_{1,0}, b_{1,0}$ and $w_{x_2,0}$ in the 0th layer, and two vertices with different colors in all other $y$th layers (see Fig. 7 (left)).

Suppose $y_2 \neq 0$. Note that $x_2 \neq 0$ if $y_2 = 1$. Let $\delta = \{w_{1,y}b_{y−1} \mid y \in \mathbb{Z}_q \setminus \{0, y_2\}\} \cup \{w_{0,0}b_{1,q−1}, e_1, e_2\}$. Then $\delta$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $b_{1,0}, w_{1,0}, b_{1,0}$ and $w_{0,0}$ ($x = 2$ if $y_2 \neq 1$, and $x = x_2$, otherwise), and two vertices with different colors in all other $y$th layers (see Fig. 7 (right)). \hfill \qed

Lemma 3.2. Suppose $\min(p, q) = 1$ and $H(p, q, t)$ has at least 6 vertices. Then $H(p, q, t)$ is a brace if and only if $p \geq 3$ and $t \neq 0, p − 1$.

Proof. For convenience, we omit the second label 0’s of all vertices of $H(p, 1, t)$ ($p \geq 3$), i.e. $w_x = w_{x,0}, b_x = w_{x,0}$. Then $w_x$ is adjacent to $b_{x+t+1}$. For $a, b \in \mathbb{Z}_p$ and $a < b$, let $[a, b] \equiv [a, a + 1, a + 2, \ldots, b − 1, b]$ be the interval of $\mathbb{Z}_p$ between $a$ and $b$ with the increasing order. Then $[b, a] \equiv [b, b + 1, b + 2, \ldots, a − 1, a] = \mathbb{Z}_p \setminus [a + 1, b − 1]$.

Necessary: It suffices to prove that $H(1, q, 0)$ ($q \geq 3$) and $H(p, 1, t)$ ($p \geq 3$) with $t = 0, p − 1$ are not 2-extendable. For $H(1, q, 0)$ ($q \geq 3$), choose two independent edges $e_1 = w_{0,0}b_{0,0}$ and $e_2 = w_{0,0}b_{0,1}$. Then $w_{0,1}$ is an isolated vertex in $H(1, q, 0) = \{w_{0,0}, w_{0,0}, b_{0,0}, b_{0,1}, w_{0,2}\}$. For $H(p, 1, t)$ ($p \geq 3$) with $t = 0, p − 1$, choose two independent edges $e_1 = w_{0,0}b_{1}$ and $e_2 = b_{2,2}$. Then $w_{2,1}$ is an isolated vertex in $H(p, 1, t) = \{w_{0,0}, b_{0,1}, w_{2,1}\}$. Hence $H(1, q, 0)$ and $H(p, 1, t)$ with $t = 0, p − 1$ are not 2-extendable.

Sufficiency: It suffices to prove that $H(p, 1, t)$ with $1 \leq t \leq p − 2$ is 2-extendable. Let $e_1, e_2$ be any two independent edges. Assume $e_1 \in M_1, e_2 \in M_1 (i, j \in \{1, 2, 3\})$. If $i = j$, then $e_1, e_2 \in M_1$. If $i \neq j$, we will construct an ideal matching $\delta$ such that $e_1, e_2 \in \delta$. By Lemma 2.5, $H(p, 1, t)$ has a perfect matching $M$ such that $\delta \subseteq M$. Further $e_1, e_2 \in M$.

Case 1. $e_1 \in M_1$ and $e_2 \in M_3$ or $e_2 \in M_2$. By Lemma 2.2, we may assume $e_1 = w_{0,0}b_{1}$.

Subcase 1.1. $e_2 = w_{x_2}b_{x_2}$. Then $2 \leq x_2 \leq p − 1$ since $e_1$ and $e_2$ are disjoint.

Consider a vertical edge $w_{x_2,b_{x_2+t+1}}$ such that $1 \leq x_2 \leq x_2 − 1$. Then $t + 2 \leq x + t + 1 \leq x_2 + t$. Since $2 \leq t \leq p$ and $x_2 + 1 \leq x_2 + t$, $[t + 2, x_2 + t] \cap [x_2 + 1, p] \neq \emptyset$. Let $x_0 + t + 1 \in [t + 2, x_2 + t] \cap [x_2 + 1, p]$. Then $x_0 \in [x_2 + 1, t = 1$ and $x_0 + t + 1 \in [x_2 + 1, p]$. Choose the additional edge $w_{x_2,b_{x_2+t+1}}$. Then $\delta = \{e_1, e_2, w_{x_2,b_{x_2+t+1}}\}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_0, b_1, w_{x_2}, b_{x_2}$ and $b_{x_2+t+1}$.

Subcase 1.2. $e_2 = w_{x_2}b_{x_2+t+1} \in M_3$. Then $1 \leq x_2 \leq p − 1$ and $x_2 + t + 1 \neq 1 (\mod p)$.

If $x_2 + t + 1 \leq p$, then $\delta = \{e_1, e_2\}$ is an ideal matching since $\delta$ is incident with the vertices in the 0th layer as ordered $w_0, b_1, w_{x_2}, b_{x_2+t+1}$. If $x_2 + t + 1 > p$, then it is obvious that $\{e_1, e_2\}$ is not an ideal matching. Choose a series of vertical edges:

$w_{x_2+j}b_{x_2+j} \cdots w_{x_2+(j−1)}b_{x_2+(j−1)+t}$,

such that $j$ is maximal subject to $x_2 + j + t + 1 \neq x_2$ (see Fig. 8, where $j = 3$). Then $x_2 + (j + 1)t + 1 − p \leq x_2$. If $x_2 + (j + 1)t + 1 − p \leq p$, let $\delta = \{w_{x_2+(t−1)}b_{x_2+(t−1)+1}|t = 1, \ldots, j + 1\} \cup \{e_1, e_2\}$. Then $\delta$ is incident with the vertices in the 0th layer as ordered $w_0, b_1, \ldots, w_{x_2+t+1}, b_{x_2+t+1}, \ldots, w_{x_2}$ and $b_{x_2+(j+1)+t+1}$ since $x_2 + j + t + 1 \leq p \leq x_2 + (j + 1)t + 1 − p \leq p$.

Hence $\delta$ is an ideal matching.

So suppose $x_2 + (j + 1)t + 1 − p > p$. Let $\eta : = x_2 + (j + 1)t + 1 − 2p$ (see Fig. 8, where $\eta = 1$). Then $\eta < x_2 + t + 1 − p < t = 1$. Choose the edge $w_{x_2+\eta}b_{x_2+(j+1)p}$. Then $x_2 + (j + 1)t + 1 − \eta = 2p$ and $x_2 + (j + 1)t + 1 = x_2 + j + t − \eta$. Let

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7.png}
\caption{Illustration for Case 2 in the proof of Lemma 3.1.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{Illustration for Subcase 1.2 in the proof of Lemma 3.2.}
\end{figure}
Lemma 2.2. For the vertex $w_0, b_1, b_2, \cdots$ ordered such that $t$ and $w_1$ and $w_2$ and $b_0$. Hence $\delta$ is incident with vertices in the 0th layer as ordered $w_0, b_1, b_2, \cdots$. By Lemma 2.2 we may assume that $e_1 = (a_1, b_1, b_2)$. Note that $x_2 \neq 1$ and $y_2 + t + 1 \neq p + 1$ since $e_1$ and $e_2$ are disjoint (see Fig. 9).

If $x_2 + t + 1 > p + 1$, then $\delta = (e_1, e_2)$ is an ideal matching since it is incident with $b_1, w_1, b_2, x_2$, and $w_2$ in the 0th layer. Suppose $x_2 + t + 1 \leq p$. Consider a vertical edge $w_1, b_2 + t + 1$ with $2 \leq x + t + 1 \leq x \leq x_2 + x + p + t - 1$. For $x_2 + p - t - 1 \geq x_2 + t + 1$ (i.e. $t \leq p - t - 1$), let $z_1 := x_2 + p - t - 1$. Choose the edge $e_2, b_2 + t + 1 = w_1, b_2 + t$. Then $\delta = (e_1, e_2, w_2, b_2)$ is an ideal matching since it is incident with vertices in the 0th layer as ordered $b_1, w_1, b_2, x_2$, and $w_2$.

So suppose $t \geq \frac{p - t}{2}$. Clearly, $(e_1, e_2, w_2, b_2)$ is not an ideal matching. So we will construct a series of additional vertical edges to obtain an ideal matching.

Let $\varepsilon := p - 1 - t$ and $N$ be the minimal positive integer to guarantee that $(p - 1) - \frac{N-1}{N}(p - 1) < \varepsilon$. Then $t < \frac{N-1}{N}(p - 1)$ and $t \geq \frac{N}{N^2}(p - 1)$. Choose a series of vertical edges:

$$w_2, b_2 + t + 1, \cdots, w_2, b_2 + t + 1, \cdots,$$

such that $z_1 = x_2 + p - t + 1 \leq p$. Clearly, we also have inequality $x_2 + (N - 1)(p - 1) \geq x_2 + p + t + 1$ since $t < \frac{N-1}{N}(p - 1)$. So we can choose $z_1 \in \{z_1 - 1, x_2 + (N - 1)(p - 1) \cap \{x_2 + t + 1, p \} \} = \emptyset$. Then $z_1 \in \{z_1 - 1, x_2 + (N - 1)(p - 1) \cap \{x_2 + t + 1, p \} \} = \emptyset$. Then $z_1 \in \{z_1 - 1, x_2 + (N - 1)(p - 1) \cap \{x_2 + t + 1, p \} \} = \emptyset$.

Hence $z_1 < \cdots < z_2 < \cdots < z_{N-2} < x_2 + t + 1 \leq p$.

Now, consider a vertical edge $w_2, b_2 + t + 1$ with $x + t + 1 \in \{z_1 + 1, x_2 + (N - 1)(p - 1) \cap \{x_2 + t + 1, p \} \} = \emptyset$. Then $z_1 \in \{z_2 - 1, x_2 + (N - 1)(p - 1) \cap \{x_2 + t + 1, p \} \} = \emptyset$.

Hence $z_1 < \cdots < z_2 < \cdots < z_{N-2} < x_2 + t + 1 \leq p$.

Theorem 3.3. Let $H(p, q, t)$ be a toroidal polyhex with at least 6 vertices. Then $H(p, q, t)$ is a brace if and only if one of the following cases appears:

1. $\min(p, q) \geq 2$.
2. $q = 1, p \geq 3$ and $t \neq 0, p - 1$.

Theorem 3.3 implies that Theorem 1.1 holds for toroidal polyhexes.

4. Klein-bottle polyhexes $K(p, q)$

We now turn to discuss the 2-extendability of $K(p, q)$. Note that every white vertex $w_0, b_0$ in the 0th layer is adjacent to a black vertex $b_{x+1}, q-1$ in the $(q - 1)$th layer in $K(p, q, t)$, a representation of $K(p, q)$.

Lemma 4.1. For $\min(p, q) = 1, K(p, q)$ is not a brace.

Proof. Let $K(p, q)$ be a Klein-bottle polyhex with at least 6 vertices and $\min(p, q) = 1$. Then $p \geq 3$ if $q = 1$ and $q \geq 3$ if $p = 1$.

Suppose that $q = 1$ and $p \geq 3$. It suffices to consider $K(p, 1)$ by Theorem 2.4. The neighbors of $w_0, b_0, b_1, b_2$ and $b_3$. Since $b_1 \cdots b_3 = b_0, w_1, w_2, b_0$ is an isolated vertex in $K(p, q) - \{w_0, b_0, b_1, b_2, b_3\}$ (see Fig. 10). Hence two disjoint edges $w_0, b_0, b_1, b_2$ cannot be extended to a perfect matching of $K(p, 1, 1)$. So $K(p, 1)$ is not 2-extendable.

Now suppose that $p = 1$ and $q \geq 3$. Consider two disjoint edges $w_0, b_0, b_1, b_2$ and $w_0, b_0, b_1, b_2, b_3$. Since $K(p, q) - \{w_0, b_0, w_2, b_2, b_3\}$ has an isolated vertex $w_0, b_1, K(1, q)$ is not 2-extendable.
Lemma 4.2. For \( \min(p, q) \geq 2 \), \( K(p, q) \) is a brace.

Proof. Let \( K(p, q) \) be a Klein-bottle polyhex with \( \min(p, q) \geq 2 \). It suffices to prove that any two independent edges \( e \) and \( e' \) of \( K(p, q, 0) \) can be extended to a perfect matching. If \( \{e, e'\} \subseteq M_r \), it is true trivially. Otherwise, by applying repeatedly the shifts \( \phi_{q,r} \) and \( \phi_{r,q} \) if necessary, \( e \) and \( e' \) of \( K(p, q, 0) \) can be transformed to independent edges \( e_1 \) and \( e_2 \) of some \( K(p, q, t) \) for \( t \in \mathbb{Z}_q \) such that \( e_1 \) is incident with \( b_{1,0} \). So it needs to show that \( e_1 \) and \( e_2 \) can be extended to a perfect matching for every representation \( K(p, q, t) \). By Lemma 2.5, we will construct an ideal matching \( \delta \) containing \( e_1 \) and \( e_2 \). There are three cases to be considered.

Case 1. Let \( \epsilon_1 = w_{0,0}b_{1,0} \in M_1 \) and \( \epsilon_2 = w_{2,2}b_{2,2} \in M_2 \).

- If \( y_2 \neq 0 \), let \( E_0 := \{w_{x,0}b_{y,1} \mid x \in \mathbb{Z} \} \) and \( E_y := \{w_{x,0}b_{y,1} \mid x \in \mathbb{Z} \} \) for \( y \neq 0 \). Then \( e_1 \in E_0 \) and \( e_2 \in E_{y_2} \). So \( M = \bigcup_{y \in \mathbb{Z}_q} E_y \) is a perfect matching containing \( e_1 \) and \( e_2 \).

Now suppose \( y_2 = 0 \). Note that \( 2 \leq x_2 \leq p - 1 \) since \( e_1 \) and \( e_2 \) are disjoint. Choose a series of vertical edges:

\[
\{w_{1,0}b_{i,y-1} \mid 1 \leq y \leq q - 1 \} \quad \text{and} \quad \{w_{2,2}b_{2,1,y-1} \mid y = 1, 2, \ldots, q - 1 \}.
\]

Let \( \delta := \{w_{s,0}b_{s+1,1,0} \mid 1 \leq y \leq q - 1 \} \cup \{w_{1,0}b_{i,y-1} \mid 1 \leq y \leq q - 1 \} \cup \{w_{2,2}b_{2,1,y-1} \mid y = 1, 2, \ldots, q - 1 \} \) (see Fig. 11). Then \( \delta \) is an ideal matching since it is incident with the vertices in the 0th layer as ordered \( w_{0,0}, b_{1,0}, w_{1,0}, b_{2,2}, w_{2,0}, b_{2,1,0} \) and \( w_{2,2}, b_{2,1,0} \) in the 0th layer (\( y \neq 0, p - 1 \)), and \( w_{2,2}, b_{2,1,0} \) in the \( (q - 1) \)th layer.

Case 2. Let \( \epsilon_1 = w_{0,0}b_{1,0} \in M_1 \) and \( \epsilon_2 = w_{2,2}b_{2,2} \in M_2 \). We may assume that \( e_2 = w_{2,2}b_{2,2+1,y-1} \) if \( y_2 \neq 0 \), \( e_2 = w_{2,2}b_{2,2+1,y+1} \), otherwise.

First suppose \( \gamma_2 \neq 0 \). Note that \( x_2 + 1 \neq 1 \) if \( y_2 = 1 \) since \( e_1 \) and \( e_2 \) are disjoint. Choose a series of vertical edges:

\[
\{w_{1,0}b_{i,y-1} \mid 1 \leq y \leq q - 1 \} \quad \text{and} \quad \{w_{2,2}b_{2,1,y-1} \mid y = 1, 2, \ldots, q - 1 \}.
\]

Let \( \delta := \{w_{1,0}b_{i,y-1} \mid y \in \mathbb{Z} \} \cup \{w_{1,0}b_{i,y-1} \mid i \in \mathbb{Z} \} \) (see Fig. 12). Then \( \delta \) is an ideal matching since it is incident with the vertices in the 0th layer as ordered \( w_{0,0}, b_{1,0}, w_{1,0}, b_{2,2}, w_{2,0}, b_{2,1,0} \) and \( b_{x,0} \) (\( x = 1 \) if \( y_2 \neq 1 \), and \( x = x_2 \), otherwise), and two vertices with two different colors in all other \( y \)th layers.

If \( y_2 = 0 \), then \( x_2 \neq 0 \) since \( e_1 \) and \( e_2 \) are disjoint. Choose a series of vertical edges:

\[
\{w_{2,2}b_{2,1,y-1} \mid y = 1, 2, \ldots, q - 1 \}.
\]

Let \( \delta := \{w_{2,2}b_{2,1,y-1} \mid y \in \mathbb{Z} \} \cup \{e_1, e_2 \} \) (see Fig. 12). Then \( \delta \) is an ideal matching since it is incident with \( w_{0,0}, b_{1,0}, w_{2,2}b_{2,2+1,0} \) in the 0th layer, and two vertices with different colors in all other \( y \)th layers.
Case 3. \( e_1 = w_{1,0}b_1,0 \in M_2 \) and \( e_2 \in M_3 \). Assume that \( e_2 = w_{x_2,y_2}b_{x_2+1,y_2-1} \) if \( y_2 \neq 0 \), and \( e_2 = w_{x_2,0}b_{x_2+1,y_2-1} \) if \( y_2 = 0 \). First choose a series of vertical edges:

\[
w_{1,0}b_{2,y_1-1} \quad \text{for} \quad y = 1, 2, \ldots, y_2 - 1, y_2 + 1, \ldots, q - 1.
\]

If \( y_2 = 0 \), then \( x_2 \neq 1 \). Let \( \delta := \{ w_{1,0}b_{2,y_1-1} \mid y \in \mathbb{Z}_p \setminus \{ 0 \} \} \cup \{ e_1, e_2 \} \) (see Fig. 13 (left)). Then \( \delta \) is an ideal matching since it is incident with \( b_{1,0}, w_{1,0}, b_{2,0}, w_{2,0} \) in the 0th layer, and two vertices with different colors in all other \( y \)th layers.

So suppose \( y_2 \neq 0 \). Note that \( x_2 \neq 0 \) if \( y_2 = 1 \). Let \( \delta := \{ w_{1,0}b_{2,y_1-1} \mid y \in \mathbb{Z}_p \setminus \{ 0, y_2 \} \} \cup \{ w_{0,0}b_{x_1+1,y_1-1}, e_1, e_2 \} \) (see Fig. 13 (right)). Then \( \delta \) is incident with the vertices in the 0th layer as ordered \( b_{1,0}, w_{1,0}, b_{2,0} \) and \( w_{0,0} \) (\( x = x_2 + 1 \) if \( y_2 = 1 \), and \( x = 2 \), otherwise), and two vertices with different colors in all other \( y \)th layers. So \( \delta \) is a desired ideal matching.

By Lemmas 4.1 and 4.2, we immediately have the following theorem.

**Theorem 4.3.** A Klein-bottle polyhex \( K(p, q) \) is a brace if and only if \( \min(p, q) \geq 2 \).

**Theorem 4.3** implies that **Theorem 1.1** is true for Klein-bottle polyhexes. Together with the result in Section 3, **Theorem 1.1** follows immediately.

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**References**


