The induced path convexity, betweenness, and svelte graphs

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Abstract

The induced path interval \(J(u,v)\) consists of the vertices on the induced paths between \(u\) and \(v\) in a connected graph \(G\). Differences in properties with the geodesic interval are studied. Those graphs are characterized, in which the induced path intervals define a proper betweenness. The intersection of the induced path intervals between the pairs of a triple, in general, consists of a big chunk of vertices. The graphs, in which this intersection consists of at most one vertex, for each triple of vertices, are characterized by forbidden subgraphs. \(\textcopyright\) 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The notion of convexity in \(n\)-space carries over to graphs in a natural way. We say that vertex \(w\) is between vertices \(u\) and \(v\) in a graph \(G\) if \(w\) lies on a geodesic (i.e. a shortest path) between \(u\) and \(v\). The interval between \(u\) and \(v\) in \(G\) is the set of all vertices between \(u\) and \(v\). A geodesically convex set in \(G\) is then a set which contains with each pair of vertices the whole interval between the pair. This notion of interval in a graph probably belongs to the folklore in graph theory. It was first studied systematically in [9]. Although at first hand this notion is a natural extension of the interval in \(n\)-space, there are also striking differences. One of these is that the intervals between the pairs of a triple of distinct vertices may have a non-empty intersection in the graph case. Thus we find ourselves in the area of abstract convexity (cf. [14]). Here a convexity on a set \(X\) is just a family of subsets \(\mathcal{R}\) of \(X\) containing \(X\) as well

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as the empty set such that it is closed under arbitrary intersections and nested unions. The sets in \( \mathcal{R} \) are called the \textit{convex sets} of this convexity, the sets \( \emptyset \) and \( X \) being the \textit{trivial} convex sets. In the case of a finite set \( X \), the condition of nested unions is always satisfied, so only two conditions remain: that the convexity contains the trivial convex sets and that it is closed under arbitrary intersections. From this point of view, various sensible convexities on a graph can be defined. One of these, the induced path convexity, is the focus of this paper.

A well-studied case for the geodesic convexity is the class of graphs, in which, for each triple of vertices, the intersection of the intervals between the pairs of the triple consists of a unique vertex. Such a graph is called a \textit{median graph}. By now a rich structure theory is available for median graphs, cf. [9] and [7]. Median graphs have algebraic and geometric counterparts, and have applications in location theory and consensus theory, see [8], see [7] for a survey. They can be described intuitively as the graphs on which the graph distance is precisely the city block norm.

The aim of this paper is to study similar conditions on the \textit{induced path convexity}, cf. Duchet [4]. The situation here turns out to be quite different. First, the nice properties of betweenness for the geodesic case are not present in general. We propose an abstract notion of betweenness, and we characterize the graphs, for which the induced path intervals form such a proper betweenness. Second, the intersection of the induced path intervals between the pairs of a triple of vertices, in general, consists of a big chunk of vertices. So here the ‘extremal’ case is when this intersection is small. We call a graph \textit{svelte} if this intersection is always at most one vertex. We characterize these graphs by a list of forbidden subgraphs. On these graphs, the induced path intervals define a proper betweenness in the above sense.

The geodesic interval and the induced path interval are instances of so-called \textit{transit functions} on a graph. The idea of a transit function is introduced in [12] as a general notion to model how to move around in a graph. It is closely related to convexity and betweenness. This paper fits into the scheme developed in [12] to study these transit functions.

In Section 2, we discuss what the intersection of the geodesic intervals between the pairs of a triple of vertices may look like. Thus, we are able to point out the similarities and differences with the induced path intervals. In Section 3, we discuss the intersection of the induced path intervals between the pairs of a triple of vertices. In Section 4, we propose an abstract notion of betweenness defined by some quite natural properties, inspired by the early work of Sholander on median betweenness [13]. We characterize the graphs, for which the induced path intervals satisfy the conditions of such a betweenness. In Sections 5 and 6, we prove the main theorem of this paper characterizing the svelte graphs, i.e. the graphs in which, for each triple of vertices, the intersection of the induced paths intervals between the pairs of the triple always consists of at most one vertex. In the last section, we discuss the situation for graphs, in which the geodesic convexity and the induced path convexity coincide.

All graphs in this paper are finite, simple, and connected.
2. The geodesic interval

Let $G = (V, E)$ be a finite connected graph. The **geodesic interval** between the vertices $u$ and $v$ is the set

$$I(u, v) = \{w \in V \mid w \text{ lies on some } u, v\text{-geodesic}\}.$$ 

This notion probably already existed for many years in the folklore of graph theory. It was studied systematically for the first time in [9].

A subset $W$ of $V$ is said to be **geodesically convex** in $G$ if, for any two vertices $u$ and $v$ in $W$, we have $I(u, v) \subseteq W$. The family of geodesically convex sets in a graph forms a true convexity in the sense of abstract convexity theory: it contains the two trivial convex sets $\emptyset$ and $V$, and it is closed under arbitrary intersections. As an aside, note that, in the infinite case, a convexity also needs to be closed under nested unions. So, also in the infinite case, the geodesically convex sets in a connected graph form a true convexity.

We mention some simple facts on geodesic intervals, which we need below (cf. [9]). For any vertex $x$ in the interval $I(u, v)$, we have $I(u, x) \subseteq I(u, v)$, and $I(u, x) \cap I(x, v) = \{x\}$. For any three vertices $u$, $v$, $w$, we can find a vertex $y$ in $I(u, v) \cap I(u, w)$ such that $I(y, v) \cap I(y, w) = \{y\}$: simply take a vertex $y$ in the intersection at maximal distance from $u$. We say that $w$ is **geodesically between** vertices $u$ and $v$ if $w$ is in $I(u, v)$. Thus, the geodesic interval defines a ‘proper betweenness’ in the following sense: if $w$ is between $u$ and $v$ and $w$ is distinct from $u$ and $v$, then $v$ is not between $u$ and $w$.

For any triple of vertices $u$, $v$, $w$ in $G$, we denote

$$I(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, u).$$

In general, the set $I(u, v, w)$ will be empty. For example, if $G$ is a cycle of length at least 5, then we may choose $u$, $v$ and $w$ so that none of the three is on a geodesic between the other two. In a non-bipartite graph, we may choose a vertex $u$ on any shortest odd cycle and choose $v$ and $w$ to be the two vertices on the cycle at maximal distance from $u$.

If we require $I(u, v, w)$ to be non-empty for each triple $u$, $v$, $w$, then we have a so-called **modular graph**. Such graphs must be bipartite, as is shown by the last argument in the previous paragraph. Examples are the trees, the complete bipartite graphs, and the covering graphs of modular lattices, from which they derive their name.

A special case is formed by the graphs, in which the intersection $I(u, v, w)$, for each triple $u$, $v$, $w$, consists of one vertex. Such graphs are called **median graphs**. Prime examples are the trees, the hypercubes and the covering graphs of distributive lattices. In a way, one may say that they are the proper common generalization of trees and hypercubes, cf. [10,11]. By now, a rich structure theory for median graphs is available, cf. [9,11,7]. Also there exist relations with various discrete structures from different areas in discrete mathematics, algebra, geometry and computer science. For a survey of these, the reader is referred to [7]. Furthermore, median graphs have found
applications in location theory and consensus theory, see e.g. [8], and in the theory of dynamic search, see e.g. [2]. At first sight, these graphs may look quite exotic, but recently a one-to-one correspondence between the class of triangle-free graphs and a subclass of the median graphs of diameter 2 was established, see [6]. Hence the density of the median graphs in the Universe of Graphs is as high as that of the triangle-free graphs.

This short survey on geodesic intervals between pairs of a triple poses the question what happens in the case of the induced path intervals.

3. Betweenness and the induced path interval

The induced path interval between vertices \( u \) and \( v \) is the set

\[
J(u,v) = \{ w \in V \mid w \text{ lies on some induced } u,v\text{-path} \}.
\]

A subset \( W \) of \( V \) is said to be induced-path convex if, for any two vertices \( u \) and \( v \) in \( W \), we have \( J(u,v) \subseteq W \). The family of induced-path convex sets in a graph forms a true convexity in the sense of abstract convexity theory: it contains the two trivial convex sets \( \emptyset \) and \( V \), and it is closed under arbitrary intersections (and, in the infinite case, it is also closed under nested unions).

From the viewpoint of convexity, the term induced path interval is quite appropriate. On the other hand it is still in a way an abuse of language, because the term interval has the connotation of betweenness. Let us make this more precise, and let us determine under which conditions the induced path interval provides us with a proper betweenness.

In the sequel a long cycle is a cycle of length at least 5.

A betweenness relation \( B \subseteq X \times X \times X \) on a set \( X \) is a relation satisfying the following conditions:

(b1) \((u,u,v) \in B\), for any \( u \) and \( v \),
(b2) if \((u,w,v) \in B\), then \((v,w,u) \in B\),
(b3) if \((u,w,v) \in B\), and \( w \) is distinct from \( u \) and \( v \), then \((u,v,w) \notin B\),
(b4) if \((u,w,v) \in B\) and \((u,x,w) \in B\), then \((u,x,v) \in B\).

We say that \( w \) is between \( u \) and \( v \) if \((u,w,v) \in B\). The first three axioms are, in a way, just the translation of the word between into mathematics. The fourth axiom is a kind of transitivity: if \( x \) is between \( u \) and \( w \) and \( w \) is between \( u \) and \( v \), then \( x \) is also between \( u \) and \( v \).

Clearly, the geodesic betweenness \( B_I \) defined by \((u,w,v) \in B_I\) if \( w \in I(u,v) \), for any graph \( G \), satisfies the above conditions. On the other hand, the relation \( B_J \), defined by \((u,w,v) \in B_J\) if \( w \in J(u,v) \), in general is not a betweenness in the above sense. Trivially, \( B_J \) satisfies (b1) and (b2). But \( B_J \) in general does not satisfy (b3). In the long cycles, the house, and the domino (see Fig. 1) we can take \( vw \) to be an edge of vertices of degree two and \( u \) to be a vertex not adjacent to \( v \) or \( w \). Then we have \( w \in J(u,v) \) and \( v \in J(u,w) \). So, if any of these graphs is an induced subgraph of \( G \), then \( B_J \) does
not satisfy (b3). It is also easy to produce graphs, for which $B_J$ does not satisfy (b4): take, for instance, the graph consisting of the 5-cycle $u \to y \to w \to x \to z \to u$ and an extra vertex $v$ adjacent to $z$ and $w$ only. Then $w$ lies in $J(u,v)$, all vertices of the 5-cycle are in $J(u,w)$, but $x$ is not in $J(u,v)$.

First, we introduce some notation. If $S$ is an induced $u,v$-path and $x$ and $y$ are vertices of $S$, then we denote the subpath of $S$ between $x$ and $y$ by $x \to \cdots S \cdots \to y$, in other words, the induced path from $x$ to $y$ along $S$. If $S$ and $T$ are induced paths sharing only an end-vertex $a$, say $S$ is a path between $a$ and $b$ and $T$ is a path between $a$ and $c$, then by $S \to T$ we denote the $b,c$-path consisting of the concatenation of the paths $S$ and $T$ at $a$. If we want to stress that $a$ is on this path, then we write $S \to a \to T$, and if we want to stress that $a$ is on $S$, we write $a \to S$, or $S \to a$, depending on whether we start or end at $a$. Here we always assume that we take the complete paths $S$ and $T$.

If $S$ and $T$ are internally disjoint paths, then any edge with one end on $S$ and the other on $T$ will be called a chord between $S$ and $T$. Such edges play a central role in all the proofs below.

**Lemma 1.** Let $G=(V,E)$ be a connected graph without the long cycles, the house, or the domino as induced subgraph. Let $u$ and $v$ be vertices of $G$. If $w \in J(u,v)$ with $u \neq w \neq v$, then $v \notin J(u,w)$.

**Proof.** Assume the contrary, and let $P=u \to \cdots P_1 \cdots \to w \to \cdots P_2 \cdots \to v$ be an induced $u,v$-path containing $w$, and let $Q=u \to \cdots Q_1 \cdots \to v \to \cdots Q_2 \cdots \to w$ be an induced $u,w$-path containing $v$. We may choose $w$ such that $w$ is the first vertex on $Q_2$ after $v$ that lies on $P$. Note that $w$ may be the predecessor of $v$ on $P$, in which case it is the successor of $v$ on $Q$. Now $Q_2$ and $P$ have only $v$ and $w$ in common. Then we may choose $u$ to be the last vertex of $Q_1$ on $P_1$. Now $P_1$ and $Q_1$ only have $u$ in common. Finally, if $Q_1$ contains a vertex of $P_2$ before $v$, then we may replace $v$ by the first vertex of $Q_1$ on $P_2$, thus shortening $P_2$. Now we are in the following situation: $P$ and $Q_2$ have only $v$ and $w$ in common, and $P_1$ and $Q$ have only $u$ and $w$ in common. Since $P$ is induced, it follows that $u$ is not adjacent to $v$, whence $Q_1$ has length at least 2 and $Q$ has length at least 3. Since $Q$ is induced, $P_1$ is of length at least 2. Thus, we are in the following situation: $P_1$ and $Q$ are two internally disjoint induced $u,w$-paths with $v$ an internal vertex of $Q$ without any chord to internal vertices of $P_1$.  

![The long cycles, the house, the domino.](image-url)
To avoid an induced long cycle, there must be a chord between an internal vertex of \( P_1 \) and an internal vertex of \( Q \). First, assume that there is no chord from an internal vertex of \( Q_2 \) to \( P_1 \). Then there must be a chord from an internal vertex of \( Q_1 \) to an internal vertex of \( P_1 \). Let \( x \) be a vertex on \( Q_1 \) closest to \( v \) with a chord to \( P_1 \). Let \( y \) be the vertex on \( P_1 \) such that \( xy \) is a chord with \( y \) closest to \( w \). Then \( x \to y \to \cdots P_1 \to w \to \cdots Q_2 \to v \to \cdots Q_1 \to x \) is an induced cycle. This is only possible if it is the 4-cycle \( C = x \to y \to w \to v \to x \). Any other chord from \( y \) to \( Q_1 \) would produce either an induced long cycle or an induced domino or house together with \( C \). So \( y \to x \to \cdots Q_1 \to u \) and \( y \to \cdots P_1 \to u \) are two internally disjoint \( y,u \)-paths. To avoid the forbidden subgraphs, there must be a chord between these two paths. Let \( q \) be the vertex on \( y \to \cdots P_1 \to u \) closest to \( y \) with a chord to \( x \to \cdots Q_1 \to u \), and let \( p \) be the vertex closest to \( x \) on \( x \to \cdots Q_1 \to u \) such that \( pq \) is a chord. Note that \( y \neq q \), but that we may have \( x = p \). Then \( y \to x \to \cdots Q_1 \to p \to q \to \cdots P_1 \to y \) is an induced cycle, whence a triangle or a 4-cycle. But now this cycle together with \( C \) induce a house or domino.

Thus, we conclude that there must be a chord between an internal vertex of \( P_1 \) and an internal vertex of \( Q_2 \). Let \( x \) be the vertex of \( Q_2 \) closest to \( v \) with a chord to \( P_1 \). Let \( y \) be the vertex on \( P_1 \) such that \( xy \) is a chord with \( y \) closest to \( u \). Then there are no chords from the part \( x \to \cdots Q_2 \to v \) to \( y \to \cdots P_1 \to u \). Moreover, \( x \to y \to \cdots P_1 \to u \) is an induced \( x,u \)-path. Recall that \( v \) has no chords to any internal vertex of \( x \to y \to \cdots P_1 \to u \). Now, if we replace \( w \) by \( x \), the path \( x \to \cdots Q_2 \to v \to \cdots Q_1 \to u \), and the path \( P_1 \) by \( x \to y \to \cdots P_1 \to u \), then we are in the previous situation, so that again we arrive at a contradiction. \( \square \)

**Theorem 2.** Let \( G = (V,E) \) be a connected graph. Then the relation \( B_J \subseteq V \times V \times V \) defined by \( (u,w,v) \in B_J \) if \( w \in J(u,v) \) is a betweenness relation if and only if \( G \) does not contain long cycles, the house or the domino as induced subgraph.

**Proof.** It was already observed above that \( B_J \) satisfies (b1) and (b2), and that, if the long cycles, the house, and the domino are not forbidden, then (b3) is not satisfied.

By Lemma 1, it suffices to show that, if \( G \) does not contain the long cycles, the house, or the domino as induced subgraph, then we have \( J(u,x) \subseteq J(u,v) \), for any \( x \) in \( J(u,v) \).

First, we show that, for any neighbor \( x \) of \( v \) in \( J(u,v) \), we have \( J(u,x) \subseteq J(u,v) \). Assume that this is not true, and let \( w \) be a vertex in \( J(u,x) \) not in \( J(u,v) \) with \( x \) a neighbor of \( v \) in \( J(u,v) \). Let \( P \) be an induced \( u,v \)-path containing \( x \). Note that \( w \) is not on this path, and that \( x \) is the last vertex on \( P \) before \( v \). Let \( Q \) be an induced \( u,x \)-path containing \( w \). Then \( Q \to v \) cannot be induced, so that there are chords between \( v \) and internal vertices of \( Q \). Let \( vz \) be the chord from \( v \) to \( Q \) with \( z \) closest to \( u \). Then \( u \to \cdots Q \to z \to v \) is an induced \( u,v \)-path, whence \( w \) cannot be on this path. We may choose \( u \) to be the common vertex of \( P \) and \( u \to \cdots Q \to z \to v \) are two internally disjoint
In general, the set \( C \) convexity. If mutually non-adjacent vertices on \( 4 \). The intersection of intervals between pairs of a triple \( J \) we infer that \( \text{such vertices} \) can even be much more vertices in the set \( \text{BJ} \) \( v \) in \( \text{an induced} \) \( z;u \) \( \text{path} \) \( u \) \( \text{connected graph} \) \( G \) \( \text{is a big, chubby chunk of vertices.} \) If all the sets \( \text{at most one,} \) then one could say that the graph is \( \text{vertices} \) \( u \). Then \( v \to x \to y \to \cdots \to p \to q \to \cdots \to z \to v \) together with the chord \( \text{y}z \) either induce a house or domino or contain an induced long cycle. This impossibility concludes the proof that \( J(u,x) \subseteq J(u,v) \), for any neighbor \( x \) of \( v \) in \( J(u,v) \). Now let \( y \) be any vertex in \( J(u,v) \). Choose an induced \( u,v \)-path \( P \) containing \( y \), say \( P = u \to \cdots \to y \to y_1 \to y_2 \to \cdots \to y_k \to v \). Then \( y_k \) is a neighbor of \( v \) in \( J(u,v) \), so that, by the previous argument, we have \( J(u,y_k) \subseteq J(u,v) \). Similarly, we infer that \( J(u,y) \subseteq J(u,y_1) \subseteq \cdots \subseteq J(u,y_k) \subseteq J(u,v) \). This concludes the proof that \( B_J \) is a betweenness relation on \( V \). \( \Box \)

4. The intersection of intervals between pairs of a triple

For any triple of vertices \( u, v, w \) in a graph \( G \), we denote

\[
J(u,v,w) = J(u,v) \cap J(v,w) \cap J(w,u).
\]

In general, the set \( J(u,v,w) \) is non-empty, so we have a sharp contrast with the geodesic convexity. If \( C \) is any induced cycle of length at least six in \( G \), and \( u, v, w \) are three mutually non-adjacent vertices on \( C \), then all vertices of \( C \) lie in \( J(u,v,w) \). And there can even be much more vertices in the set \( J(u,v,w) \). So, in general, the set \( J(u,v,w) \) is a big, chubby chunk of vertices. If all the sets \( J(u,v,w) \) are small, that is, of size at most one, then one could say that the graph is nicely shaped. Therefore, we call a connected graph \( G \) a svelte graph if \( |J(u,v,w)| \leq 1 \), for all triples \( u, v, w \) in \( G \).

In this section, we consider the cases, where \( |J(u,v,w)| = 0 \), for all triples of distinct vertices \( u, v, w \) in \( G \), or where \( |J(u,v,w)| = 1 \), for all triples \( u, v, w \) in \( G \).

Proposition 3. Let \( G \) be a connected graph. Then \( |J(u,v,w)| = 0 \), for any triple of distinct vertices \( u,v,w \), if and only if \( G \) is a complete graph.

Proof. Assume that \( G \) is not complete. Since \( G \) is connected, we can find non-adjacent vertices \( u \) and \( w \) having a common neighbor \( v \). Then \( u \to v \to w \) is an induced path, whence \( J(u,v,w) = J(u,w) \cap \{u,v\} \cap \{v,w\} = \{v\} \). \( \Box \)

Recall that a block in a graph \( G \) is a maximal 2-connected subgraph of \( G \).
Proposition 4. Let $G = (V, E)$ be a connected graph. Then $|J(u, v, w)| = 1$, for any triple of vertices $u, v, w$, if and only if every block in $G$ is a $K_2$ or a 4-cycle.

Proof. First, assume that every block in $G$ is a $K_2$ or a 4-cycle. Then it follows immediately that for any triple of vertices $u, v, w$, we have $|J(u, v, w)| = 1$.

Conversely, assume that $|J(u, v, w)| = 1$, for any triple of vertices $u, v, w$. If $G$ contains a triangle on $u, v, w$, then we have $J(u, v, w) = \emptyset$. So $G$ is triangle-free. If $G$ contains an induced long cycle or an induced domino or an induced $K_{2,3}$, then we can find three vertices $u, v, w$ on this induced subgraph with $|J(u, v, w)| \geq 2$. So $G$ does not contain an induced long cycle or an induced domino or an induced $K_{2,3}$.

Take any block $B$ with at least three vertices. Since $G$ is triangle-free, it follows that $B$ even contains at least four vertices. Then we can find non-adjacent vertices $u$ and $w$ having a common neighbor $v$ in $B$. Since $B$ is 2-connected, we can find an induced $u, w$-path $P$ in $B$–$v$. Now consider the subgraph $H$ induced by $v$ together with $P$. Since $G$ is triangle-free and does not contain induced long cycles or an induced domino, it turns out that $H$ must be a 4-cycle, say $u \rightarrow v \rightarrow w \rightarrow x \rightarrow u$. If $H$ is not $B$, then there must be a vertex $z$ in $B$–$H$ adjacent to a vertex of $H$, say $u$. Note that, to avoid $K_{2,3}$ and triangles, $z$ has no other neighbor in $H$. In particular, $z$ is not adjacent to $v$. Take an induced $v,z$-path in $B$–$u$. Again to avoid the forbidden subgraphs, this path must be of length 2, say $v \rightarrow y \rightarrow z$. Then, $G$ being $K_{2,3}$-free and triangle-free, $y$ has no other neighbors in $H$. But now $H$, $y$, $z$ induce a domino in $G$. Hence $H = B$, and we are done. □

5. Internally disjoint induced paths

In the next section, we will characterize the svelte graphs as those graphs for which the graphs in Figs. 1–3 are the forbidden induced subgraphs. But before we can harvest, we have to prepare the ground. First, we study the structure of the chords between two internally disjoint $u,v$-paths in graphs, for which the graphs of Figs. 1 and 2 are forbidden. Such paths form a cycle, and by the forbidden figures there must be chords

![Fig. 2. The 4-fan, $K_{2,3}$ and $K_{1,1,1}$.](image-url)
The 3-sun

The bonnet

Fig. 3. The 3-sun and the bonnet.

between the two paths. We need to know the structure of these chords. A *k-fan* consists of a path of length *k* and an extra vertex adjacent to all vertices of the path. Thus a *k-fan* contains *k* triangles.

Throughout this section, \( G = (V, E) \) is a connected graph without the house, the domino, the 4-fan, the long cycles \( C_{\geq 5} \), and the multipartite graphs \( K_{2,3} \) and \( K_{1,1,3} \) as induced subgraph.

Let \( P = u \rightarrow x_1 \rightarrow \cdots \rightarrow x_p \rightarrow v \) and \( Q = u \rightarrow y_1 \rightarrow \cdots \rightarrow y_q \rightarrow v \) be two internally disjoint induced \( u,v \)-paths such that \( P \cup Q \) does not induce a 4-cycle. Note that \( p, q \geq 1 \). Let \( C \) be the cycle induced by \( P \cup Q \). Then, necessarily, \( C \) must contain a chord, which is of the form \( x_iy_j \) with \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \). Below a *chord* will always be such an edge joining a vertex of one induced path to a vertex of another induced path. If the chords from \( x_i \) are to consecutive vertices of \( Q \), then we will call them *consecutive chords*. For convenience, we also call the edge \( x_1u \) a *chord* from \( x_1 \) to \( Q \). Likewise, the edges \( y_1u, x_pv, y_qv \) are called *chords*. The length of a path \( R \) is denoted by \( l(R) \).

**Claim 1.** The neighbors of \( u \) on \( P \) and \( Q \) are adjacent.

**Proof.** We prove the claim by induction on \( l(P) + l(Q) = p + q + 2 \).

Clearly, we have \( l(P) + l(Q) \geq 4 \). If \( l(P) + l(Q) = 4 \), then the claim follows from the fact that \( P \cup Q \) does not induce a \( C_4 \).

Let \( l(P) + l(Q) = 5 \). Assume that \( x_1y_1 \) is not a chord and let \( x_iy_j \) be a chord between \( P \) and \( Q \) that minimizes \( i + j \). Then \( u \rightarrow \cdots \rightarrow x_i \rightarrow y_j \rightarrow \cdots \rightarrow u \) is an induced cycle in \( G \) of length \( i + j + 1 \geq 4 \). So \( i + j = 3 \), say \( i = 1 \) and \( j = 2 \). Since \( P \cup Q \) does not induce a 4-cycle, we have \( y_2 \neq v \). If there would be a chord \( x_1y_j \), with \( j > 2 \) as small as possible, then we would get an induced house, in case \( j = 3 \), or an induced domino, in case \( j = 4 \), or an induced long cycle, in case \( j \geq 5 \). So there are no other chords from \( x_1 \) to \( Q \). In particular \( x_1v \) is not a chord, so that \( v \neq x_2 \) and \( p, q \geq 2 \). Hence \( x_1 \rightarrow y_2 \rightarrow \cdots \rightarrow y_q \rightarrow v \) and \( x_1 \rightarrow \cdots \rightarrow x_p \rightarrow v \) are internally disjoint \( u,v \)-paths with length sum \( p + q \). If they induce a 4-cycle, then \( P \cup Q \) induces a domino. So, by induction, \( x_2 \) and \( y_2 \) are adjacent.

To avoid the house on the vertices \( u, x_1, x_2, y_1, y_2 \), we must have the chord \( x_2y_1 \).

As above, to avoid the house, the domino and the long cycles, \( y_1 \) has no other chords.
to $P$. If there were a chord from $x_2$ to $y_3 \rightarrow \cdots Q \rightarrow y_q \rightarrow v$, then we would get either a $K_{1,1,3}$, or a house or a long cycle. So $x_2 \rightarrow y_2 \rightarrow \cdots Q \rightarrow y_q \rightarrow v$ is an induced path between $x_2$ and $v$. Similarly, $y_2 \rightarrow x_2 \rightarrow \cdots P \rightarrow x_p \rightarrow v$ is an induced path. But now we get a contradiction with Lemma 1. This final impossibility settles Claim 1. □

**Claim 2.** Each internal vertex of $P$ and $Q$ is incident with a chord.

**Proof.** Assume the contrary, and let $a$ be an internal vertex of $P$ without any chord to $Q$. Note that $a$ is not adjacent to $u$ or $v$. Hence, by Claim 1, there is a chord between the neighbors of $u$ on $P$ and $Q$. Let $x$ be the vertex on $u \rightarrow \cdots P \rightarrow a$ nearest to $a$ with a neighbor on $Q$, and let $y$ be the neighbor of $x$ on $Q$ nearest to $v$. Note that $y \neq v$. Then $x \rightarrow \cdots P \rightarrow v$ and $x \rightarrow y \rightarrow \cdots Q \rightarrow v$ are two internally disjoint induced paths, where $x \rightarrow \cdots P \rightarrow v$ is of length at least 3. So, by Claim 1, $y$ is adjacent to the neighbor of $x$ on $x \rightarrow \cdots P \rightarrow v$, contradicting the choice of $x$. □

**Claim 3.** Each internal vertex of $P$ and $Q$ is incident with consecutive chords.

**Proof.** Assume the contrary, and let $x_i$ be adjacent to non-consecutive vertices $a$ and $b$ on $Q$ such that $x_i$ is not adjacent to any vertices on $Q$ between $a$ and $b$. If $a = u$ and $b = v$, then $P \cup Q$ induces a 4-cycle. So we may assume that $b \neq v$. If $a = u$, then we necessarily get an induced house domino or long cycle. So we have also $a \neq u$. Since long cycles do not occur in $G$, the ‘gap’ between $a$ and $b$ must be of length two, that is, $a = y_{j-1}$ and $b = y_{j+1}$, for some $1 < j < q$. Because of the forbidden house, domino and long cycles, we infer that there are no other chords from $x_i$. In particular, we have $1 < i < p$.

Then $x_i \rightarrow \cdots P \cdots \rightarrow u$ and $x_i \rightarrow y_{j-1} \rightarrow \cdots Q \cdots \rightarrow u$ are two internally disjoint induced paths. If they induce a 4-cycle, then $i = j = 2$. To avoid the domino, $x_1$ must have a chord to $y_2$ or $y_2$. But a chord to $y_2$ would produce an induced $K_{2,3}$ on $u, x_1, y_1, x_2, y_2$, whereas a chord to $y_3$ would produce an induced house on $x_1, x_2, y_1, y_2, y_3$. Hence $x_i \rightarrow \cdots P \rightarrow u$ and $x_i \rightarrow y_{j-1} \rightarrow \cdots Q \rightarrow u$ cannot induce a 4-cycle. Therefore, by Claim 1, $x_{i-1}y_{j-1}$ is a chord. Similarly, $x_{i+1}y_{j+1}$ is a chord. Since $y_{j+1}$ already has two consecutive chords, it follows, by the above argument, that all chords from $y_{j+1}$ must be consecutive. Similarly, all chords from $y_{j-1}$ are consecutive.

By Claim 1, there must be a chord from $y_j$ to some vertex on $y_{j+1} \rightarrow x_i \rightarrow \cdots P \rightarrow u$ (viz. to the vertex closest to $u$ that still has a chord to $y_{j+1}$). Similarly, there must be a chord from $y_j$ to some vertex on $y_{j-1} \rightarrow x_{i+1} \rightarrow \cdots P \rightarrow v$. Since the chord $y_jx_i$ is missing, it follows as above that the only chords from $y_j$ are $y_jx_{i-1}$ and $y_jx_{i+1}$. Hence also $x_{i-1}$ has consecutive chords.

If either $x_{i-1}y_{j-2}$ or $y_{j-1}x_{i-2}$ is a chord, then we get an induced $K_{1,1,3}$. Therefore, both paths $y_{j-1} \rightarrow x_{i-1} \rightarrow \cdots P \rightarrow u$ and $x_{i-1} \rightarrow y_{j-1} \rightarrow \cdots Q \rightarrow u$ must be induced. But this is impossible by Lemma 1. This settles Claim 3. □
Claim 4. Each internal vertex of $P$ and $Q$ is incident with at least two chords.

Proof. Assume the contrary, and let $x_i y_j$ be the unique chord incident with $x_i$. By Claim 1, we have $1 < i < p$. So $x_i \to \cdots P \cdots \to u$ and $x_i \to y_j \to \cdots Q \cdots \to u$ are internally disjoint induced $x_i u$-paths. Since $p \geq 3$, $P \cup Q$ does not induce a 4-cycle, so that the neighbors of $u$ on $P$ and $Q$ are joined by a chord. Therefore, $x_i \to \cdots P \cdots \to u$ and $x_i \to y_j \to \cdots Q \cdots \to u$ cannot induce a 4-cycle. Hence, by Claim 1, $y_j x_{i-1}$ is a chord. Similarly, $y_j x_{i+1}$ is a chord. To avoid a 4-fan, at least one of the chords $y_j x_{i-2}$ and $y_j x_{i+2}$ is missing, say $y_j x_{i-2}$. Then, by Claim 3, the path $y_j \to x_{i-1} \to \cdots P \cdots \to u$ is induced, so, from Claim 1, we infer the existence of the chord $y_{j-1} x_{i-1}$. Note that, since $x_i y_{j-1}$ is missing, it follows from Claim 3 that $y_{j-1}$ has no chords to $x_i \to \cdots P \cdots \to v$.

Now the chord $y_j x_{i+2}$ would produce a 4-fan, but the absence of that chord would force the existence of the chord $x_{i+1} y_j + 1$. As above, there are no chords from $y_j + 1$ to $x_i \to \cdots P \cdots \to u$. Hence we have an induced 4-fan on $x_{i-1}, x_i, x_{i+1}, y_j, y_j$, and $y_j + 1$. This impossibility settles Claim 4.

Claim 5. Let $x_i y_j$ and $x_k y_m$ be two crossing chords, with $i < k$ and $m < j$. Then the chords between the sets $\{x_i, x_{i+1}, \ldots, x_k\}$ and $\{y_m, y_{m+1}, \ldots, y_j\}$ induce a complete bipartite graph on these vertices.

Proof. We prove the assertion by induction on $k-i$.

First assume that $k = i + 1$. Note that $P \cap Q$ cannot induce a 4-cycle, so that the neighbors of $u$ on $P$ and $Q$ are joined by a chord. We may assume that $y_m$ is the vertex on $Q$ closest to $u$ with a chord to $x_i$, and that $y_j$ is the vertex on $Q$ closest to $v$ with a chord to $x_i$. Then $x_{i+1} \to \cdots P \cdots \to u$ and $x_{i+1} \to y_m \to \cdots Q \cdots \to u$ are two internally disjoint induced paths. Since the neighbors of $u$ on $P$ and $Q$ are joined by a chord, these paths cannot induce a 4-cycle. Hence, by Claim 1, $x_i y_m$ is a chord, so that, by Claim 4, all chords from $x_i$ to $y_m, y_{m+1}, \ldots, y_j$ exist. Similarly, all chords from $x_{i+1}$ to $y_m, y_{m+1}, \ldots, y_j$ exist, and we are done.

Now assume that $k > i + 1$. Again, let $y_m$ be the vertex on $Q$ closest to $u$ with a chord to $x_k$. Then, by Claim 1, $x_k y_m$ is a chord. Now the chords $x_i y_j$ and $x_{k-1} y_m$ are crossing chords, and by induction all chords between $x_i \to \cdots P \cdots \to x_{k-1}$ and $y_m \to \cdots Q \cdots \to y_j$ are present, and then also those between $x_k$ and $y_m \to \cdots Q \cdots \to y_j$. □

6. Svelte graphs and forbidden subgraphs

Using Claims 1–5 above, we are now able to prove a basic lemma. Yet another forbidden subgraph is necessary: the 3-sun in Fig. 3. The other graph of Fig. 3, the bonnet, only occurs in Theorem 6. First note the following: if $u \to \cdots P \cdots \to z$ and $u \to \cdots Q \cdots \to z$ are two induced $u, z$-paths, then any induced $u, z$-path $R$ con-
tained in \((u \rightarrow \cdots P \cdots \rightarrow z) \cup (u \rightarrow \cdots Q \cdots \rightarrow z)\) consists of pieces alternately from \(P\) and \(Q\).

**Lemma 5.** Let \(G = (V, E)\) be a connected graph without the long cycles, the house, the domino, the 4-fan, the multipartite graphs \(K_{2,3}\) and \(K_{1,1,3}\), and the 3-sun as induced subgraph. Let \(u, v\) and \(w\) be three mutually non-adjacent vertices, and let \(P\) be an induced \(u, v\)-path and \(Q\) be an induced \(u, w\)-path with a common internal vertex \(z\). Then there exists an induced \(u, z\)-path \(R\) contained in \((u \rightarrow \cdots P \cdots \rightarrow z) \cup (u \rightarrow \cdots Q \cdots \rightarrow z)\) such that \(R\) contains \(P \cap Q\) and \(u \rightarrow \cdots R \cdots \rightarrow z \rightarrow \cdots P \cdots \rightarrow v\) and \(u \rightarrow \cdots R \cdots \rightarrow z \rightarrow \cdots Q \cdots \rightarrow w\) are induced paths in \(G\).

**Proof.** If \(u \rightarrow \cdots P \cdots \rightarrow z\) and \(u \rightarrow \cdots Q \cdots \rightarrow z\) coincide, then there is nothing to prove. So we may assume that \(P\) and \(Q\) differ between \(u\) and \(z\), that is, they diverge and converge between \(u\) and \(z\), maybe even more than once. If \(P\) and \(Q\) diverge at a vertex \(x\) between \(u\) and \(z\), then let \(y\) be the first vertex on \(P\) after \(x\) that is again on \(Q\). Then \(x \rightarrow \cdots P \cdots \rightarrow y\) and \(x \rightarrow \cdots Q \cdots \rightarrow y\) are two internally disjoint induced \(x, y\)-paths. Let \(p\) be the predecessor of \(y\) on \(P\) and let \(q\) be the predecessor of \(y\) on \(Q\). If these two paths induce a 4-cycle, then \(p\) is not on \(Q\) and \(q\) is not on \(P\). Otherwise, by Claim 1, \(pq\) is a chord, so again \(p\) is not on \(Q\) and \(q\) is not on \(P\). This implies that \(P\) and \(Q\) continue in the ‘same direction’ after \(y\), that is, either \(P\) and \(Q\) diverge again at \(y\), or they both continue with the same neighbor \(y'\) of \(y\), or \(y = z\). In other words, every edge on \(P \cap Q\) is traversed in the same direction by \(P\) and \(Q\). This implies that any induced \(u, z\)-path contained in \((u \rightarrow \cdots P \cdots \rightarrow z) \cup (u \rightarrow \cdots Q \cdots \rightarrow z)\) consists of \(P \cap Q\) together with either the \(P\)-part or the \(Q\)-part between consecutive points of divergence and convergence, for any common vertex \(z\) of \(P\) and \(Q\).

Let \(R_1\) be a maximal \(u, u'\)-path contained in \((u \rightarrow \cdots P \cdots \rightarrow z) \cup (u \rightarrow \cdots Q \cdots \rightarrow z)\) such that \(u \rightarrow \cdots R \cdots \rightarrow u' \rightarrow \cdots P \cdots \rightarrow v\) and \(u \rightarrow \cdots R \cdots \rightarrow u' \rightarrow \cdots Q \cdots \rightarrow w\) are induced paths in \(G\). If \(u' = z\), then we are done. So assume the contrary. Then \(P\) and \(Q\) diverge at \(u'\) and then converge again at, say, \(s\) before \(z\). Let \(t\) be the first vertex after \(s\) at which point \(P\) and \(Q\) again diverge (\(t\) exists, since \(v\) and \(w\) are distinct vertices), and let \(R_2 = s \rightarrow \cdots P \rightarrow \cdots \rightarrow t\) be this common subpath of \(P\) and \(Q\). Note that \(t\) may be a vertex on \(P \cap Q\) after \(z\). Let \(P_1 = s \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_p \rightarrow u'\), and \(Q_1 = s \rightarrow y_1 \rightarrow y_2 \rightarrow \cdots \rightarrow y_q \rightarrow u'\). Furthermore, let \(P_2 = t \rightarrow \cdots P \rightarrow \cdots \rightarrow v\) and \(Q_2 = t \rightarrow \cdots Q \rightarrow \cdots \rightarrow w\). Then \(P_1\) and \(Q_1\) are internally disjoint induced \(s, u'\)-paths, so that both are of length at least 2. By the maximality of \(R_1\), it follows that neither \(u' \rightarrow \cdots P_1 \cdots \rightarrow s \rightarrow R_2 \rightarrow Q_2\) nor \(u' \rightarrow \cdots Q_1 \cdots \rightarrow s \rightarrow R_2 \rightarrow P_2\) is induced. Hence there must be a chord joining an internal vertex of \(P_1\) to an internal vertex of \(Q_2\) and a chord joining an internal vertex of \(Q_1\) to an internal vertex of \(P_2\). Let \(x_s\) be the vertex on \(P_1\) closest to \(s\) having a chord to \(Q_2\), say to \(y_s\), with \(y_s\) closest to \(t\). Similarly, let \(x_p\) be the vertex on \(Q_1\) closest to \(s\) with a chord to \(P_2\), say to \(x_t\), with \(x_t\) closest to \(t\). Then \(C = x_s \rightarrow \cdots P_1 \cdots \rightarrow s \rightarrow R_2 \rightarrow t \rightarrow \cdots Q_2 \cdots \rightarrow y_1 \rightarrow x_s\) and \(C^* = y_s \rightarrow \cdots Q_1 \cdots \rightarrow s \rightarrow R_2 \rightarrow t \rightarrow \cdots P_2 \cdots \rightarrow x_t \rightarrow y_s\) are induced cycles.
of length at most 4. Therefore, \( R_2 \) has length at most 1, that is, either \( s = t \) or \( st \) is an edge.

If \( P_1 \cup Q_1 \) would induce a 4-cycle, then together with \( C \) we would have a house or a domino. Hence \( P_1 \cup Q_1 \) does not induce a 4-cycle, so that by Claim 1, \( x_1 \) and \( y_1 \) are adjacent.

If \( s \neq t \), then \( C \) is a 4-cycle and \( x_s = x_1 \), whence \( C \) together with \( y_1 \) induce a house. So we have \( s = t \).

If \( y_1 \) has distance 2 to \( s \), then \( x_s = x_1 \) and \( C \) is an induced 4-cycle. Now \( C \) together with \( y_1 \) induce a house, which is impossible. So \( y_i \) is adjacent to \( s \). Similarly, \( x_i \) is adjacent to \( s \).

If \( C \) is a 4-cycle, then \( x_s = x_2 \). To avoid a house induced by \( C \) and \( y_1 \), it follows that \( y_1 x_2 \) is also a chord. Then \( D = y_1 \to s \to y_i \to x_2 \to y_1 \) is an induced 4-cycle. If \( x_2 y_2 \) is a chord, then \( D \) together with \( y_2 \) induce a house. So \( x_2 y_2 \) is missing, whence, by Claims 3 and 1, there must be a chord \( y_1 x_3 \). To avoid a house on \( D \) and \( x_3 \), there must be a chord \( y_3 x_3 \). Now \( D^* = y_1 \to s \to y_i \to x_3 \to y_1 \) is an induced 4-cycle. By the same argument, we deduce the existence of the chord \( y_1 x_4 \). But now we are in trouble, because we have produced an induced 4-fan on \( y_1, s, x_1, x_2, x_3, x_4 \). Thus, we conclude that \( C \) is a triangle. Similarly, \( C^* \) must be a triangle.

By Claim 1, \( x_2 y_1 \) or \( x_1 y_2 \) is a chord, say, \( x_2 y_1 \) is a chord. If we would have \( x_2 = u' \), then \( x_2, x_1, y_1, s, x_3, y_i \) would induce a 3-sun (in case \( x_i y_i \) is a chord) or contain an induced house (in case \( x_i y_i \) is missing). So \( x_2 \neq u' \). Then, to avoid the 4-fan, \( y_1 x_3 \) cannot be a chord, so that, by Claims 3 and 1, \( x_2 y_2 \) is a chord. Now, to avoid a house on \( y_2, y_1, s, y_i, x_2 \), there is no chord between \( y_i \) and \( x_2 \). To avoid the 3-sun on \( x_2, x_1, y_1, s, x_3, y_1 \) there must be a chord between \( y_i \) and \( x_i \). But this creates an induced house on \( x_2, x_1, y_1, s, x_3, y_1 \). This final impossibility shows that our assumption that \( u' \neq z \) is incorrect. Hence we can find an induced \( u, z \)-path \( R \) contained in \((u \rightarrow \cdots P \cdots \rightarrow z) \cup (u \rightarrow \cdots Q \cdots \rightarrow z)\) such that \( u \to \cdots R \cdots \to z \to \cdots P \cdots v \) and \( u \to \cdots R \cdots \to z \to \cdots Q \cdots \to w \) are induced paths in \( G \). \( \square \)

As a consequence of Lemma 5, we may replace an induced \( u, v \)-path and an induced \( u, w \)-path having a common vertex \( z \), by an induced \( u, v \)-path and an induced \( u, w \)-path having a common subpath from \( u \) to \( z \). We will use this trick extensively in the proof of Theorem 6.

**Theorem 6.** Let \( G = (V,E) \) be a connected graph. Then \( |J(u,v,w)| \leq 1 \), for any triple of vertices \( u,v,w \), if and only if \( G \) does not contain any of the graphs of Figs. 1–3 as induced subgraph.

**Proof.** It is easy to check that each of the graphs in Figs. 1–3 contains a triple of vertices \( u, v, w \) such that \( |J(u,v,w)| \geq 2 \). Hence none of these graphs may occur as induced subgraph in a svelte graph.

Conversely, assume that \( G \) is a connected graph without any of the graphs of Figs. 1, 2, and 3 as induced subgraph. Assume that \( G \) is not a svelte graph, and let \( u, v, w \) be a triple of vertices such that \( J(u,v,w) \) contains two distinct vertices \( a \) and \( b \).
Suppose that from one of the three vertices, say $u$, there are induced paths to the other two vertices containing both $a$ and $b$ but in different order. Say, $u \rightarrow \cdots a \rightarrow \cdots b \rightarrow \cdots v$ and $u \rightarrow \cdots a \rightarrow \cdots b \rightarrow \cdots w$ are such paths. Then we have $a \in J(u,b)$ and $b \in J(u,a)$, which is impossible by Lemma 1. So, if there exist induced paths from one vertex to the other two both containing $a$ as well as $b$, then $a$ and $b$ must be traversed in the same order. This implies, in particular, that there cannot be induced paths from each of the three to the other two all containing $a$ as well as $b$.

The structure of the proof is as follows. We distinguish a number of cases. By successive applications of Lemma 5, we find in each case a more convenient triple of vertices and a more convenient set of induced paths connecting them that still fail the condition. For the simplified situation we are able, using Claims 1–5 and Lemma 1, either to get a straightforward contradiction or to produce one of the forbidden subgraphs.

**Case 1:** There exists an induced $u,v$-path $P$ and there exists an induced $u,w$-path $Q$ such that both $P$ and $Q$ contain $a$ and $b$.

We may assume that in going along $P$ and $Q$ from $u$ we encounter $a$ and $b$ in alphabetical order. Let $S$ and $T$ be induced $v,w$-paths such that $S$ contains $a$ but not $b$, and $T$ contains $b$ but not $a$. Then we have $a,b \in J(a,v,w)$, so that we may replace $u$ by $a$ and adapt $P$ and $Q$ accordingly. By Lemma 5, we may assume that $P$ and $Q$ have a common induced $a,b$-path, say, $a \rightarrow \cdots P \cdots \rightarrow b$. Similarly, by Lemma 5, we may assume that $P$ and $T$ have a common induced $v,b$-path, say, $v \rightarrow \cdots T \cdots \rightarrow b$, and that $Q$ and $T$ have a common induced $w,b$-path, say $w \rightarrow \cdots T \cdots \rightarrow b$. If not already so, let $v$ be the common vertex of $v \rightarrow \cdots S \cdots \rightarrow a$ and $v \rightarrow \cdots T \cdots \rightarrow b$ closest to $a$ and $b$, and let $w$ be the common vertex of $w \rightarrow \cdots S \cdots \rightarrow a$ and $w \rightarrow \cdots T \cdots \rightarrow b$ closest to $a$ and $b$. Now $v$ is the unique common vertex of the paths $v \rightarrow \cdots S \cdots \rightarrow a$ and $v \rightarrow \cdots T \cdots \rightarrow b$, and $w$ is the unique common vertex of the paths $w \rightarrow \cdots S \cdots \rightarrow a$ and $w \rightarrow \cdots T \cdots \rightarrow b$. Note that, $P$ being an induced $a,v$-path with $b$ as internal vertex, the subpath $a \rightarrow \cdots S \cdots \rightarrow v$ must be of length at least 2. Similarly the subpath $a \rightarrow \cdots S \cdots \rightarrow w$ is of length at least 2. Hence $a$ is an internal vertex of $S$ not adjacent to $v$ or $w$.

Suppose that $v \rightarrow \cdots S \cdots \rightarrow a$ and $w \rightarrow \cdots T \cdots \rightarrow b$ have some vertex $z$ in common. Then the subpath of $S$ from $w$ via $a$ to $z$ implies that $a \in J(w,z)$, whereas, $z$ being on the induced $w,a$-path $w \rightarrow \cdots T \cdots \rightarrow b \rightarrow \cdots P \cdots \rightarrow a$, we have $z \in J(w,a)$. This is impossible, by Lemma 1. Similarly, the paths $v \rightarrow \cdots T \cdots \rightarrow b$ and $w \rightarrow \cdots S \cdots \rightarrow a$ must be disjoint. Thus, we have produced two internally disjoint $v,w$-paths $S$ and $T$, where $a$ is an internal vertex of $S$ at distance at least 2 from $v$ as well as $w$. So, by Claim 4, there are at least two chords from $a$ to $T$. One of these could be the edge $ab$, but the other one creates an impossibility, the paths $a \rightarrow \cdots P \cdots \rightarrow b \cdots \rightarrow \cdots T \cdots \rightarrow v$ and $a \rightarrow \cdots P \cdots \rightarrow b \rightarrow \cdots T \cdots \rightarrow w$ being induced. This settles Case 1.
Case 2: There is an induced $v,w$-path containing both $a$ and $b$.

We may assume that, in going from $v$ to $w$ along the induced path, we encounter $a$ and $b$ in alphabetical order. By Case 1, there is no induced $u,v$-path containing both $a$ and $b$ and no induced $u,w$-path containing both $a$ and $b$. By successive applications of Lemma 5, we may conclude the existence of the following induced paths: an induced $u,a$-path $P$, an induced $u,b$-path $Q$, an induced $a,w$-path $P^*$, an induced $b,v$-path $Q^*$, an induced $a,v$-path $S$, an induced $b,w$-path $T$, and an induced $a,b$-path $R$, such that the path $P \to P^*$ is an induced $u,w$-path containing $a$, the path $Q \to Q^*$ is an induced $u,v$-path containing $b$, the path $P \to S$ is an induced $u,v$-path containing $a$, the path $Q \to T$ is an induced $u,w$-path containing $b$, and the path $S \to R \to T$ is an induced $v,w$-path containing $a$ and $b$ in this order. Furthermore, the paths $P \to R \to T$ and $Q \to R \to S$ are not induced, since both contain $a$ as well as $b$. If not already so, we may take $u$ to be the common vertex of $P$ and $Q$ closest to $a$ and $b$, and $v$ the common vertex of $S$ and $P^*$ closest to $a$ and $b$, and $w$ the common vertex of $T$ and $Q^*$ closest to $a$ and $b$. Note that now $u$ is the unique common vertex of the paths $P$ and $Q$, and $v$ is the unique common vertex of the paths $S$ and $P^*$, and $w$ is the unique common vertex of $T$ and $Q^*$. Since $S \to R$ is an induced $v,b$-path, it follows that $Q^*$ is of length at least two. Let $x$ be the neighbor of $b$ on $Q^*$. Similarly, let $y$ be the neighbor of $a$ on $P^*$, which is distinct from $w$. Let $p$ be the neighbor of $a$ on $P$, and let $q$ be the neighbor of $b$ on $Q$.

First we establish some facts.

Fact (i): $P \cap T = \emptyset$, and $Q \cap S = \emptyset$.

If $P$ and $T$ would share a vertex $z$, then, $z$ being on $R \to T$, we would have $z \in J(a,w)$, and $a$ being on $z \to \cdots P \cdots \to a \to P^*$, we would have $a \in J(z,w)$, which is impossible by Lemma 1. So the paths $P$ and $T$ are disjoint. Similarly, $Q$ and $S$ are disjoint.

Fact (ii): $ab$ is an edge.

Assume the contrary. First, we prove that there is a chord from $a$ to an internal vertex of $Q$ and a chord from $b$ to an internal vertex of $P$. If $P \to P^*$ and $Q \to T$ are internally disjoint $u,w$-paths, then there must be a chord from $a$ to $Q \to T$ but not to $T$, whence to an internal vertex of $Q$. If $P \to P^*$ and $Q \to T$ are not disjoint, then $Q$ and $P^*$ are not disjoint. In this case, let $z$ be the first vertex from $u$ on $Q$ also on $P^*$. Since $P$ and $Q$ share only $u$, it follows that $z$ is distinct from $a$. Then $u \to \cdots P \cdots \to a \to \cdots P^* \cdots \to z$ and $u \to \cdots Q \cdots \to z$ are two internally disjoint $u,z$-paths. If they induce a 4-cycle, then $a$ must be adjacent to $z$, whence this is a chord from $a$ to an internal vertex of $Q$. Otherwise, by Claim 2, there must be a chord from $a$ to an internal vertex of $Q$ anyway. Similarly, we find a chord from $b$ to an internal vertex of $P$. Now we show that $P \to P^*$ and $Q \to T$ are indeed internally disjoint.
u, w-paths. For, if not, then Q and P* must share a vertex. Let z be the last vertex from u on Q also on P*. Then z \rightarrow \cdots P* \cdots \rightarrow w and z \rightarrow \cdots Q \cdots \rightarrow b \rightarrow T are two internally disjoint z, w-paths. Suppose that there is a chord from b to an internal vertex of z \rightarrow \cdots P* \cdots \rightarrow w. Now the chords from b to P \rightarrow P* are non-consecutive, since ab is missing. The subpath of P \rightarrow P* between the first and the last chord from b contains a and z as internal vertices. Hence this subpath together with b either contains a long cycle or a house, which is impossible. So there cannot be a chord from a;w-path. For, if not, then u;w \rightarrow \cdots \rightarrow \text{a chord from a;w-path.}

Assume that there is a chord between P–a and T–b and there is no chord between Q–b and S–a.

Fact (iii): There is no chord between P–a and T–b and there is no chord between Q–b and S–a.

Assume that there is a chord between P–a and T–b. Let rz be such a chord with r on P as close as possible to a and then z on T as close as possible to w. Then r \rightarrow \cdots P* \rightarrow a \rightarrow P* and r \rightarrow z \rightarrow \cdots T \rightarrow w are two internally disjoint r, w-paths, the first of which is of length at least 3. So, by Claim 1, the neighbors of r on the two paths must be adjacent. Because of the choice of rz, the neighbor of r on P must be a, by which we would have the edge az, contradicting the fact that a \rightarrow T is induced. Similarly, there is no chord between Q–b and S–a.

Fact (iv): ax and by are chords.

Assume that by is not a chord. Note that a \rightarrow b \rightarrow T and P* are internally disjoint induced a, w-paths. Since by is missing, they must induce a 4-cycle. Hence y and b are common neighbors of a and w, and P* = a \rightarrow y \rightarrow w. This implies that y is not on Q, so that Q and P* are disjoint. By Fact (i), it follows that P \rightarrow P* and Q \rightarrow T are internally disjoint, where the first of the two has length at least 3. But now by being a missing chord is in conflict with Claim 1, so that by must be a chord after all. Similarly, ax is a chord.

To settle Case 2, we have to distinguish two subcases.

Subcase 2.1: u is adjacent to a or b.

Say that u is adjacent to a, that is u = p. Note that now u \rightarrow a \rightarrow S and Q \rightarrow Q* are two internally disjoint u, v-paths. By Claim 4, vertex a has a chord to all internal vertices of Q. By Claims 1 or 4, we must find a common neighbor s of a and x on
$S \cup Q^*$. Note that, by Fact (iii) or by $Q \to Q^*$ being induced, $s$ is not adjacent to any of the vertices on $Q$. So $a$ together with $Q \to x \to s$ form a fan. To avoid the 4-fan, $Q$ must be of length 1, whence we have $p = u = q$. If $s$ were not on $S$, then $s$ would be an internal vertex of $Q^*$. In that case, to avoid the 4-fan, there are no other chords from $a$ to $Q^*$, so that by Claim 1, there should be a chord from $s$ to the neighbor of $a$ on $S$. But then again we would have a 4-fan. So $s$ is on $S$. Similarly, we find a common neighbor $t$ of $b$ and $y$ on $T$. Note that $s \to a \to b \to t$ is an induced path. If $x = y$, then we would have an induced 3-sun on $u, a, b, s, x, t$. So we have $x \neq y$. If $x$ and $y$ are not adjacent, then $u, a, b, x, y$ induce a $K_{1,1,3}$. So $x$ and $y$ must be adjacent. But now we have produced a bonnet on the vertices $u, a, b, x, y, s, t$. This impossibility settles Subcase 2.1.

**Subcase 2.2:** $u$ is not adjacent to $a$ or $b$.

Since $S \to a \to b \to Q$ is not induced, there must be a second chord from $a$ to $Q$ by Fact (iii). To avoid an induced house or long cycle on $a$ together with the path $x \to b \to Q$, it follows that $aq$ is a chord. Similarly, $bp$ is a chord.

Assume that $pq$ is missing. To avoid a $K_{1,1,3}$ on $p, q, a, b, y$, there is an edge between $y$ and $q$. Hence $u \to \cdots P \cdots \to p \to a \to q \to \cdots Q \cdots \to u$ cannot induce a 4-cycle. Then, by Claim 1, $a \to q \to \cdots Q \cdots \to u$ cannot be induced. Hence $a$ must be adjacent to the neighbor $r$ of $q$ on $q \to \cdots Q \cdots \to u$, where $r$ must be distinct from $u$. To avoid the 4-fan, $a$ has no other neighbors than $x, b, q, r$ on $Q \to Q^*$. So $a \to x \to \cdots Q^* \to v$ is induced. Let $s$ be the neighbor of $a$ on $S$. By Fact (iii) there are no chords from $s$ to $q$ or $r$. Hence, to avoid the 4-fan, $xs$ cannot be a chord. But then, by Claim 1, $a \to x \to \cdots Q^* \to v$ and $S$ induce a 4-cycle, which together with $b$ induces a house. So $pq$ is a chord.

As observed, $p \to b \to T$ is an induced $p, w$-path. If $p \to b \to Q^*$ is induced, then we may replace $u$ by $p$ and we are in Subcase 2.1. So we may assume that $p \to b \to Q^*$ is not induced. To avoid long cycles and a house in $p$ together with $q \to b \to Q^*$, vertex $p$ must be adjacent to $x$. Similarly, $q$ must be adjacent to $y$. Since $p \to a \to P^*$ is induced, it follows that $x \neq y$.

Let $s$ be the neighbor of $a$ on $S$, and let $t$ be the neighbor of $b$ on $T$. Note that, by Fact (iii), $p$ and $t$ are not adjacent, and $q$ and $s$ are not adjacent. If $y$ and $t$ are not adjacent, then $b$ is adjacent to the neighbor $z$ of $y$ on $P^*$. In this case, to avoid the 4-fan on $b$ and $p \to a \to y \to z \to Q^*$, the path $b \to z \to \cdots P^* \cdots \to w$ is induced, whence $zt$ is a chord. But now $b, q, a, y, z, t$ induce a 4-fan. So $y$ and $t$ are necessarily adjacent. Similarly, $x$ and $s$ are adjacent. If $x$ and $t$ were adjacent, then we would have a 3-sun on $q, a, s, x, t, b$. So $x$ and $t$ are not adjacent. Similarly, $y$ and $s$ are not adjacent. If $x$ and $y$ were adjacent, then $x, y, b, q, t$ would induce a $K_{1,1,3}$, so $x$ and $y$ are not adjacent. But now $b, x, p, q, y, t$ induce a 4-fan. This final impossibility settles Subcase 2.2 as well as Case 2.

**Case 3:** There is no induced path containing both $a$ and $b$ between any two of $u, v, w$. 
By successive applications of Lemma 5, we find induced paths $P, Q, R$ connecting $a$ to $u, v, w$, respectively, and induced paths $X, Y, Z$ connecting $b$ to $u, v, w$, respectively, such that $P \rightarrow Q$, $P \rightarrow R$, $Q \rightarrow R$ are induced paths containing $a$ between the respective pairs of $u, v, w$, and $X \rightarrow Y$, $X \rightarrow Z$, $Y \rightarrow Z$ are induced paths containing $b$ between the respective pairs of $u, v, w$. If not already so, we choose $u$ to be the common vertex on $P$ and $X$ closest to $a$ and $b$, and $v$ the common vertex on $Q$ and $Y$ closest to $a$ and $b$, and $w$ the common vertex on $R$ and $Z$ closest to $a$ and $b$. Then we have $P \cap X = \{u\}$, $Q \cap Y = \{v\}$, and $R \cap Z = \{w\}$. Let $p, q, r$ be the neighbors of $a$ on $P, Q, R$, respectively, and let $x, y, z$ be the neighbors of $b$ on $X, Y, Z$, respectively. Note that $p$ and $q$ are not adjacent, being vertices on the induced path $P \rightarrow Q$. Similarly, each pair of $p, q, r$ is mutually non-adjacent and each pair of $x, y, z$ is mutually non-adjacent. We consider two subcases.

Subcase 3.1: $a$ and $b$ are not adjacent.

If there are chords from $a$ to $X$ as well as $Y$ as well as $Z$, then, to avoid long cycles, these must be to $x$, $y$ and $z$, by which we have an induced $K_{2,3}$. So we may assume that there are no chords from $a$ to $Z$. In particular $r \neq w$. This implies that $P \cap Z = \emptyset$. For, otherwise, let $t$ be the last vertex of $P$ on $Z$, going from $u$ to $a$. Now $t \rightarrow \cdots P \rightarrow a \rightarrow R$ and $t \rightarrow \cdots X \rightarrow Y \rightarrow Z \rightarrow w$ are two internally disjoint induced $t, w$-paths, where the first path is of length at least 3. But then, by Claim 1, there should be a chord from $a$ to $Z$.

Next we deduce the existence of a chord from $a$ to an internal vertex of $X$. If $X \cap R = \emptyset$, then $P \rightarrow R$ and $X \rightarrow Z$ are two internally disjoint $u, w$-paths, and we have a chord from $a$ to $X \rightarrow Z$, whence to $X$. If $X \cap R \neq \emptyset$, then let $s$ be the first vertex of $X$ on $R$, going from $u$ to $b$. Now $P \rightarrow a \rightarrow \cdots X \rightarrow s$ and $u \rightarrow \cdots X \rightarrow s$ are two internally disjoint $u, s$-paths. Assume that they induce a 4-cycle.

Then it follows that $u = p$ and $r = s$. Moreover, $u$ and $r$ have a common neighbor $t$ such that $u \rightarrow a \rightarrow r \rightarrow t \rightarrow u$ is the induced 4-cycle. Now $u \rightarrow a \rightarrow Q$ and $u \rightarrow t \rightarrow r \rightarrow X \cdots \rightarrow b \rightarrow Y$ are two induced $u, v$-paths, of which the last one is of length at least 4. Since the chord at is missing, we get a contradiction with Claim 1, even in the case that the paths $r \rightarrow \cdots X \rightarrow b$ and $Q$ are not internally disjoint. So $P \rightarrow a \rightarrow \cdots X \rightarrow s$ and $u \rightarrow \cdots X \rightarrow s$ do not induce a 4-cycle. But this implies, by Claim 1, that $a$ has a chord to an internal vertex of $X$ after all. By replacing $u$ and $X$ by $v$ and $Y$, respectively, we also deduce the existence of a chord from $a$ to an internal vertex of $Y$. Now we have chords from $a$ to $X \rightarrow Y$, where in between the chord $ab$ is missing. This is only possible if these chords are precisely $ax$ and $ay$.

If we would have $u = x$, then $x \rightarrow a \rightarrow r \rightarrow \cdots R \rightarrow \cdots w$ and $x \rightarrow b \rightarrow Z$ would be two internally disjoint $x, w$-paths, the first of which is of length at least 3. This would imply the existence of chord $ab$. So we have $x \neq u$, and, similarly, $y \neq v$. If $p = u$, then $a \rightarrow u \rightarrow \cdots X \rightarrow x \rightarrow a$ is an induced cycle. So it is of length at most 4. But then this cycle, together with $b$ and $y$ would contain an induced house or domino. So $p \neq u$. Now $P$ and $a \rightarrow x \cdots X \rightarrow u$ are two internally disjoint $a, u$-paths. If they
induce a 4-cycle, then together with $b$ and $y$ they would induce a domino, unless $yp$ is a chord, in which case $p,a,x,b,y$ would induce a house. Hence, by Claim 1, we deduce the existence of the chord $px$.

Since $ab$ is missing and $x \rightarrow b \rightarrow Y$ is an induced $x,v$-path, $x \rightarrow a \rightarrow Q$ cannot be induced. Hence there are chords from $x$ to internal vertices of $Q$. To avoid an induced long cycle or house in $x,p,Q$, we must have the chord $xq$. Similarly, we deduce the existence of the chord $xr$. But now $a,x,p,q,r$ induce a $K_{1,1,3}$, which is not allowed.

This settles Subcase 3.1.

**Subcase 3.2:** $a$ and $b$ are adjacent.

If $a$ were adjacent to all of $x,y,z$, then we would have an induced $K_{1,1,3}$. So, without loss of generality, there is no chord between $a$ and $z$.

First assume that there is yet another chord from $a$ to $Z$. To avoid long cycles, this must be the chord $at$, where $t$ is the neighbor of $z$ on $Z$ different from $b$. Now $a$ cannot have chords to $X$ or $Y$ (otherwise we would get a house, a domino or a long cycle). Therefore, $a \rightarrow b \rightarrow X$ is an induced $a,u$-path, as is $P$. Hence, if the chord $bp$ would be missing, then these two paths must induce a 4-cycle. This would imply that $u= x \neq p$. Now the chord $pz$ would produce an induced $K_{2,3}$ on $x,p,a,b,z$, so that it cannot be there. But then the chord $pt$ would produce an induced house on $p,a,b,z,t$, hence it is also missing. Now we have an induced domino on $x,p,a,b,z,t$.

This impossibility forces the existence of chord $bp$. Similarly, $bq$ is a chord. To avoid $K_{1,1,3}$, there is no chord between $b$ and $r$. Now $b$ has two chords $hp$ and $ba$ to the induced path $P \rightarrow R$, whence, to avoid the house and long cycles, there is no chord from $b$ to an internal vertex of $R$. So $b \rightarrow R$ is induced. If $b \rightarrow R$ and $Z$ would induce a 4-cycle, then we would have $r = w = t$. But then we would get a contradiction when we apply Claim 1 on $R \rightarrow P$ and $Z \rightarrow X, az$ being a missing chord. So $b \rightarrow R$ and $Z$ do not induce a 4-cycle. Hence, by Claim 1, we again should have the chord $az$. This absurdity shows that there can be no chord from $a$ to an internal vertex of $Z$.

Since there is no chord from $a$ to an internal vertex of $Z$, the path $a \rightarrow Z$ is induced. If $br$ were not a chord, then, by Claim 1, $R$ and $a \rightarrow Z$ should induce a 4-cycle, so that we would have $z = w$. But now $w \rightarrow X$ and $w \rightarrow r \rightarrow a \rightarrow P$ are two internally disjoint induced $w,u$-paths, of which the first has length at least 3, with $br$ a missing chord. This is in contradiction with Claim 1. So $br$ must be a chord.

To avoid a $K_{1,1,3}$, one of the chords $hp$ and $bq$ must be missing, say there is no chord between $b$ and $q$. Now we apply the above argument on the missing chord $bq$, and we deduce that there are no chords from $b$ to internal vertices of $Q$, so that $b \rightarrow Q$ is induced, and that $ay$ is a chord. Since $az$ is missing, it follows that $r \neq w$. Similarly, $y \neq v$.

If we would have $P \cap Z \neq \emptyset$, then choose $t$ to be the last vertex of $P$ on $Z$, going from $u$ to $a$. Now we have the induced paths $t \rightarrow \cdots P \cdots \rightarrow a \rightarrow R$ and $a \rightarrow b \rightarrow \cdots Z \cdots \rightarrow t \rightarrow \cdots Z \cdots \rightarrow w$, which produces a conflict with Lemma 1. So we have $P \cap Z = \emptyset$. Similarly, we have $X \cap Q = \emptyset$. 
Besides the chord $ab$, there must be a second chord from $a$ to $X \rightarrow Z$, whence to $X$. For, if $X \cap R = \emptyset$, then it follows from Claim 4 applied to the internally disjoint paths $P \rightarrow Q$ and $X \rightarrow Z$. And, if $X \cap R \neq \emptyset$, then let $s$ be the first vertex from $u$ on $X$ lying on $R$. Then $u \rightarrow \cdots P \cdots \rightarrow a \rightarrow \cdots R \cdots \rightarrow s$ and $u \rightarrow \cdots X \cdots \rightarrow s$ are two internally disjoint induced $u,s$-paths. If they would induce a 4-cycle, then we have $r = s$ and $u = p$ and $u \rightarrow \cdots X \cdots \rightarrow s = u \rightarrow t \rightarrow s$, for some vertex $t$ not adjacent to $a$. But now $u \rightarrow a \rightarrow b$ and $u \rightarrow \cdots X \cdots \rightarrow b$ are two internally disjoint $u,b$-paths, of which the second one is of length at least 3, where the necessary chord $at$ is missing. So they do not induce a 4-cycle. Hence, by Claim 4, there must be a chord from $a$ to $u \rightarrow \cdots X \cdots \rightarrow s$ after all. Since $ab$ and $ay$ are already two chords from $a$ to the induced path $X \rightarrow Y$, the edge $ax$ must be a chord.

If there were a chord from an internal vertex $s$ of $X$ to an internal vertex $t$ of $Q$, then take the chord $st$ with $s$ as close as possible to $b$ and $t$ close as possible to $v$. Let $s'$ be the neighbor of $s$ on $X$ going to $b$. Possibly $s' = b$, but in any case there is no chord from $s'$ to an internal vertex of $Q$. Then $s \rightarrow t \rightarrow \cdots Q \cdots \rightarrow v$ is induced and internally disjoint from the induced path $s \rightarrow \cdots X \cdots \rightarrow b \rightarrow t \rightarrow \cdots Y \cdots \rightarrow v$. Since this last path is of length at least 3, $s't$ must be a chord, by Claim 1, contradicting the fact that there are no chords from $s'$ to internal vertices of $Q$. So there is no chord from an internal vertex of $X$ to an internal vertex of $Q$, in particular, there is no chord form $x$ to $q$. To avoid a house on $a,q,z,b,x$, we have no chord between $q$ and $z$. Thus we have shown that the path $q \rightarrow a \rightarrow b \rightarrow z$ is induced.

Now we will show that there is an induced $u,q$-path containing $b$. Above we already showed that $b \rightarrow a \rightarrow Q$ is induced. So, by Claim 2, there must be chords from $q$ to $Y$. Let $qt$ be the chord with $t$ as close as possible to $b$. Then $q$ has no chords to internal vertices of $t \rightarrow \cdots Y \cdots \rightarrow b$. Since $q$ also has no chords to vertices of $X$, the path $q \rightarrow t \rightarrow \cdots Y \cdots \rightarrow b \rightarrow X$ is induced. This implies that $b \in J(u,q)$. Similarly, we show that $a \in J(u,z)$. Now $u,q,z,a,b$ are vertices satisfying the conditions of Case 2. This settles Case 3, and completes the proof of the theorem.

7. Concluding remarks

Two classes of graphs are closely connected to the class of svelte graphs. A distance-hereditary graph is a connected graph, in which every induced path is a geodesic, i.e. a shortest path. Otherwise stated, the distance-hereditary graphs are precisely the graphs in which the geodesic convexity and the induced-path convexity coincide. These graphs were characterized by Howorka [5] as the connected graphs without the long cycles, the house, the domino and the 3-fan as induced subgraph, see Fig. 4.

These graphs were further studied and characterized by Bandelt and Mulder [1] and D’Atri and Moscarini [3]. An interesting subclass consists of the so-called Ptolemaic graphs, these are the distance-hereditary graphs without induced 4-cycles, see Howorka.
[5]. Recall that a graph is chordal if it does not contain induced cycles of length at least 4. So the Ptolemaic graphs are the chordal distance-hereditary graphs. In view of our theorem, we have the following two simple corollaries.

**Corollary 7.** A connected graph $G$ is a svelte distance-hereditary graph if and only if it does not contain the long cycles, the house, the domino, the 3-fan, or the complete multipartite graphs $K_{2,3}$ and $K_{1,1,3}$ as induced subgraph.

**Corollary 8.** A connected graph $G$ is a svelte Ptolemaic graph if and only if it is chordal and does not contain the 3-fan or the complete multipartite graph $K_{1,1,3}$ as induced subgraph.

Above we have given four natural axioms of a betweenness relation. Of course, we could impose extra axioms and obtain special types of betweenness. One such axiom, which is also quite natural, is the following:

$$\text{if } (u,w,v),(u,x,w) \in B, \text{ then } (x,w,v) \in B.$$  
(b5)

The betweenness of the geodesic interval satisfies this axiom, as we have that $w$ lies in $I(x,v)$, whenever $x$ lies in $I(u,w)$ and $w$ lies in $I(u,v)$. The betweenness relation of the induced-path interval, in general, does not satisfy (b5). Take the 3-fan, and let $u$ and $v$ be the vertices of degree 2, let $w$ be the vertex of degree 3 adjacent to $v$, and let $x$ be the vertex of degree 4. Then $w$ is between $u$ and $v$, and $x$ is between $u$ and $w$, but $w$ is not between $x$ and $v$. The 3-fan is precisely the extra forbidden subgraph we need, as is stated in the next theorem, which contains a new characterization of distance-hereditary graphs involving the notion of betweenness.

**Theorem 9.** Let $G=(V,E)$ be a connected graph. Then the following conditions are equivalent:

(i) $B_J$ is a betweenness and the 3-fan is a forbidden subgraph in $G$,
(ii) $B_J$ is a betweenness satisfying (b5),
(iii) $J(u,v) = I(u,v)$, for any two vertices $u$ and $v$ in $G$,
(iv) $G$ is a distance-hereditary graph.
Proof. The equivalence of (i), (iii) and (iv) follows from Lemma 1, the definition of distance-hereditary graphs and Howorka’s characterization of distance-hereditary graphs. Clearly, (iii) implies (ii), and (ii) implies (i).

In the case of median graphs with the geodesic convexity the corresponding notion of betweenness gives rise to interesting algebraic results. It would be interesting to study the betweenness of the induced path convexity for the graphs in Lemma 1, and for the svelte graphs. For instance, one may look for a characterization of svelte graphs purely in terms of betweenness, where the underlying graph $G_B = (V, E_B)$ of a betweenness relation $B$ on the set $V$ is defined as follows: $uv$ is an edge in $G_B$ if and only if $u$ and $v$ are distinct and there is no element strictly between $u$ and $v$.

References