On strict lower and upper sections of fuzzy orderings

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Abstract

Strict lower sections, strict upper sections and open order intervals (strict order intervals) are classical notions associated to crisp orderings (crisp total preorders). In this paper, we extend these notions to fuzzy orderings. We determine all the fuzzy orderings (they contain all the crisp orderings) which make it possible to compare two fuzzy strict lower sections and two fuzzy strict upper sections. We then deduce the properties (equality, inclusion and intersection) of fuzzy strict order intervals. In this way, we obtain fuzzy extensions of well-known properties of crisp strict sections and crisp strict order intervals, and we display one example of their applications.

Keywords: Fuzzy binary relation; Fuzzy strict section; Fuzzy strict order interval

1. Introduction

Individual preferences on a set $A$ could be ambiguous. In many works [1,2,6,7,10–12,14], such ambiguous preferences have been modelled by fuzzy binary relations. Thus, the fuzzification of fundamental notions and properties on crisp binary relations makes it possible to formulate and analyze problems (which are already solved when preferences are crisp) of classical utility theory and classical choice theory when preferences are fuzzy (ambiguous).

Amongst these particular notions, we have strict lower sections, strict upper sections and strict order intervals (see [5]).

The aim of this paper is to extend these three notions to fuzzy orderings and to determine the fuzzy orderings for which these extensions preserve their properties (inclusions, equality and intersections).
We also point out a field of application for these results. We assume throughout that the semantic concept underlying the binary relation (crisp or fuzzy) is preferred.

The paper is organized as follows:

Section 2 contains basic concepts on fuzzy sets and fuzzy binary relations. Here we distinguish three types of binary relations: crisp ordering, strong fuzzy ordering and fuzzy ordering. We also note that the set of strong fuzzy orderings on a set \( A \), which contains the set of crisps orderings on \( A \), is a classical, particular and important subset of the set of fuzzy orderings on \( A \).

Section 3 contains five subsections. In Section 3.1, we define the notions of fuzzy strict lower section, fuzzy strict upper section and fuzzy strict order interval which are associated to fuzzy orderings. We note that these fuzzy notions become classical and crisp ones when they are associated to crisp orderings and we recall the properties (equality and inclusions) of crisp strict sections. We show that fuzzy orderings exist for which the extensions of crisp strict sections do not preserve these properties. Thus, in Section 3.2, we introduce five conditions on fuzzy binary relations and show that they are satisfied by strong fuzzy orderings (therefore by crisp orderings), but not always satisfied by fuzzy orderings. With these conditions, we determine, in Section 3.3, all the fuzzy orderings for which the extensions of crisp strict sections preserve each of their (recalled) properties. With the properties of fuzzy strict sections, we deduce, in Section 3.4, the properties (equality, inclusions and intersection) of fuzzy strict order intervals. In Section 3.5, we give an example of the application of the results obtained on the “Theory of revealed preference” in Economics.

Section 4 contains some concluding remarks.

2. Preliminaries

Let \( A \) be a given set of alternatives. We assume that \( |A| \geq 3 \).

For each \( a \) and \( b \) on \([0, 1]\), we denote \( \max(a, b) = a \lor b \) and \( \min(a, b) = a \land b \).

The following definitions are classical (see [2,7–9,15]).

**Definition 1.** (i) A fuzzy set \( E \) of \( A \) is defined by its membership function \( \mu_E : A \rightarrow [0, 1] \) where \( \mu_E(x) \) is the degree to which \( x \) belongs to \( E \).

A t-norm is a function \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) satisfying \( i_1 \) \( x \ast 1 = x \), \( i_2 \) \( x \ast y \leq u \ast v \) if \( x \leq u \) and \( y \leq v \), \( i_3 \) \( x \ast y = y \ast x \) and \( i_4 \) \( (x \ast y) \ast u = x \ast (y \ast u) \) for all \( x, y, u, v \in [0, 1] \).

(ii) Let \( E \) and \( F \) be two fuzzy sets of \( A \) and \( * \) be a t-norm.

Strict inclusion is defined by \( E \subseteq F \iff (\forall x, \mu_E(x) \leq \mu_F(x)) \) and \( \exists x, \mu_E(t) < \mu_F(t) \).

Inclusion is defined by \( E \subseteq F \iff (\forall x, \mu_E(x) \leq \mu_F(x)) \).

Equality is defined by \( E = F \iff (\forall x, \mu_E(x) = \mu_F(x)) \).

The intersection of \( E \) and \( F \) associated to \( * \) is the fuzzy set of \( A \) denoted by \( E \cap F \) and defined by: \( (\forall x, \mu_{E \cap F}(x) = \mu_E(x) * \mu_F(x)) \).

**Definition 2.** A fuzzy weak preference relation (FWPR) is a function \( R : A \times A \rightarrow [0, 1] \).

- \( R \) is crisp iff \( (\forall x, y) R(x, y) \in \{0, 1\} \).
- \( R \) is reflexive iff \( (\forall x) R(x, x) = 1 \).
- \( R \) is antisymmetric iff \( (\forall x, y) R(x, y) \land R(y, x) = 0 \).
1. \( R \) is symmetric iff \( \forall x, y \in A, R(x, y) = R(y, x) \).
2. \( R \) is connected iff \( \forall x, y \in A, R(x, y) + R(y, x) \geq 1 \).
3. \( R \) is strongly connected iff \( \forall x, y \in A, R(x, y) \lor R(y, x) = 1 \).
4. \( R \) is transitive iff \( \forall x, y, z \in A, R(x, y) \land R(y, z) \geq R(x, z) \).

A reflexive and connected FWPR \( R \) satisfies condition \( T \) iff \( \forall x, y \in A, R(x, y) = R(y, x) = R(y, z) = R(z, y) \) implies \( R(x, z) = R(z, x) \) (see [13]).

We assume throughout that all FWPRs are defined in \( A \), reflexive and connected.

**Remark 1.** Let \( R \) be an FWPR.

(i) \( R(x, y) \) is interpreted as the degree to which \( x \) is “at least as good as” \( y \). If \( R \) is crisp, then we denote \( R(x, y) = 1 \) by \( xRy \) and \( R(x, y) = 0 \) by \( not(xRy) \).

(ii) Strong connectedness implies reflexivity and connectedness.

We shall need the following types of FWPRs.

**Definition 3.** A fuzzy ordering on \( A \) is an FWPR which satisfies reflexivity, connectedness and transitivity.

A strong fuzzy ordering on \( A \) is an FWPR which satisfies strong connectedness and transitivity.

A crisp ordering (a crisp total preorder) on \( A \) is a crisp weak preference relation which satisfies reflexivity, connectedness and transitivity.

We denote by: \( CO \) the set of all crisp orderings on \( A \), \( SFO \) the set of all strong fuzzy orderings on \( A \) and \( FO \) the set of all fuzzy orderings on \( A \). We have: \( CO \subset SFO \subset FO \).

**Remark 2.** (i) If \( R \in SFO \), then \( R \) satisfies condition \( T \).

(ii) In [16], Zadeh defined a “Fuzzy weak ordering” as an FWPR which is reflexive, transitive and weakly connected, that is, \( \forall x, y \in A, R(x, y) > 0 \) or \( R(y, x) > 0 \). “Connectedness” as defined in Definition 2 implies “weakly connectedness” and they are two fuzzy extensions of “crisp connectedness”. Even though it is general to work with weak connectedness, in this paper, we use connectedness because it is a natural fuzzification of crisp connectedness (As it is shown in [12, p. 362]) and there is an important literature on it (see [1,6,12–14]).

**Remark 3.** The defined notions of Definitions 2 and 3 are classical (see [1,2,6,12–14]). However, some of these notions exist in other references in various names, e.g., (i) “Strong connected” is called “complete” and “Strong linear” in [2,4], respectively, (ii) for the Zadeh’s min t-norm, in [4], “A preordering is an FWPR which is reflexive and transitive” and in [7], “An ordering is an FWPR which is transitive”. Thus, in our paper, “A fuzzy ordering is a preordering, in the sense of [4], which is connected and it is an ordering, in the sense of [7], which is reflexive and connected”.

Whichever the terminology is used for these notions, the reader shall always retain that we extend crisp strict sections to reflexive, connected and transitive FWPRs.

In the literature (see [2,3,5,11,12]), there are many factorizations of FWPR into symmetric component \( I \) and antisymmetric component \( P \). In this paper, we use the following antisymmetric component:
Let $R$ be an FWPR. Its antisymmetric component $P$ is defined by

$$\forall_{A}^{\subset^{c}}, P(x, y) = \max [R(x, y) - R(y, x), 0].$$

When $P(x, y) > 0$, $P(x, y)$ is the degree to which $x$ is strictly preferred to $y$. $P$ is called the fuzzy strict preference of $R$. In [11], it is proved that if $R$ is transitive, then $P$ is transitive.

Let us recall that, if $R$ is a crisp weak preference relation, then its strict preference $P$ is uniquely defined by: $\forall_{A}^{\subset^{c}}, xPy \Leftrightarrow (xRy$ and not $(yRx))$.

3. Fuzzy strict sections associated to fuzzy ordering

3.1. Definitions

In the literature of crisp partial orders (see [5]), we have the following definitions: Let $R \in CO$ and $P$ its strict preference. With each $x \in A$, we associate the following sets:

- the strict lower section $\leftarrow;x[ = \{t \in A; xPt\}$,
- the strict upper section $\rightarrow;x[ = \{t \in A; tPx\}$, and
- the strict order interval with end points $x, y, \rightarrow;x;y[ = \{t \in A; yPt$ and $tPx\}$.

In the following definition, we extend these notions to fuzzy orderings. Thus, to denote the fuzzy extensions of $\leftarrow;x[, \rightarrow;x[ and $\rightarrow;x;y[$, we have added the symbol $\sim$ above each of them. This gives respectively $\leftarrow;x[, \rightarrow;x[ and $\rightarrow;x;y[$. Let $R \in FO$, $P$ its fuzzy strict preference and $x, y \in A$.

**Definition 4.** (i) The fuzzy strict lower section of $x$ is the fuzzy set of $A$ denoted by $\leftarrow;x[ and defined by $\forall_{A}^{\sim}, \mu_{\leftarrow;x}[t] = P(x, t)$.

(ii) The fuzzy strict upper section of $x$ is the fuzzy set of $A$ denoted by $\rightarrow;x[ and defined by $\forall_{A}^{\sim}, \mu_{\rightarrow;x}[t] = P(t, x)$.

(iii) Let $\cap$ be the intersection associated to a t-norm $\ast$ and $x, y \in A$ such that $P(y, x) > 0$.

The fuzzy strict (open) order interval with end points $x, y \in A$, is the fuzzy set of $A$ denoted by $\rightarrow;x;y[ and defined by $\rightarrow;x;y[ = \rightarrow;x[ \cap \rightarrow;y[.

Let us give an intuitive meaning of these three fuzzy notions.

**Interpretation 1.** For each $x \in A$,

(i) $\leftarrow;x[ = \{(t, P(x, t)), t \in A\}$ gives the degree to which $x$ is strictly preferred to every other alternative.

(ii) $\rightarrow;x[ = \{(t, P(t, x)), t \in A\}$ gives the degree to which every other alternative is strictly preferred to $x$.

(iii) For $x, y \in A$ such that $P(y, x) > 0$, $\rightarrow;x;y[ = \{(t, P(t, x) \ast P(y, t)), t \in A\}$ gives the degree to which every other alternative is strictly preferred to $x$ and $y$ is strictly preferred to it.
In the rest of this paper, we establish properties of fuzzy strict sections and fuzzy strict order intervals and, then, point out a field of application of these results. We will proceed as follows:

- We recall the properties of inclusions and equality of crisp strict sections and show that these properties are not always preserved when we extend crisp strict sections to fuzzy orderings.
- This leads us to new conditions on FWPR which permit us to characterize the subset of all the fuzzy orderings for which the extensions of crisp strict sections preserve each of these properties.
- We show that, each of the subsets of fuzzy orderings obtained, contains the set SFO, i.e., our new conditions are satisfied by strong fuzzy orderings and crisp orderings.
- We deduce the properties of fuzzy strict order intervals (equality, inclusions and intersections) and give an example of the application of some of these results in Economics.

When fuzzy strict sections are associated to crisp orderings, they become crisp strict sections. Thus, in the literature of crisp partial orders (see [5]), we have the following properties: Let \(x, y \in A\) and \(R \subseteq CO\).

\[
\begin{align*}
(i) & \quad xRy \land yRx \Rightarrow \, [\sim], \, y[\subseteq] \sim, x[\setminus], \\
(ii) & \quad xRy \land \text{not}(yRx) \Rightarrow \, [\sim], \, y[\subseteq] \sim, x[\setminus], \\
(iii) & \quad xRy \Rightarrow \, [\sim], \, y[\subseteq] \sim, x[\setminus], \\
(iv) & \quad xRy \land yRx \Rightarrow \, [\sim], \, y[\subseteq] \sim, x[\setminus], \\
(v) & \quad xRy \land \text{not}(yRx) \Rightarrow \, [\sim], \, y[\subseteq] \sim, x[\setminus], \\
(vi) & \quad xRy \Rightarrow \, [\sim], \, y[\subseteq] \sim, x[\setminus],
\end{align*}
\]

When fuzzy strict sections are associated to fuzzy orderings, we do not always have similar properties as shown by the following example.

**Example 1.** Let \(A = \{a, b, c\}\).

(i) Consider \(R_1 \subseteq FO\) and satisfying \(T\) defined by: \(\forall x, R_1(x, x) = 1; R_1(c, a) = R_1(b, a) = 0, 65; R_1(c, b) = 0, 75; R_1(a, b) = 0, 8; R_1(a, c) = 0, 85; R_1(b, c) = 1\).

Then
\[
[\sim], a = \{(a; 0), (b; 0, 15), (c; 2, 2)\} \text{ and } [\sim], b = \{(a; 0), (b; 0), (c; 0, 25)\}.
\]

Thus, we have
\[
R_1(a, b) > R_1(b, a) \text{ and } \text{not}([\sim], b[\subseteq] \sim, a[\setminus]).
\]

(ii) Consider \(R_2 \subseteq FO\) and satisfying \(T\) defined by: \(\forall x, R_2(x, x) = 1; R_2(a, c) = R_2(b, c) = 0, 60; R_2(b, a) = R_2(a, b) = 0, 75; R_2(c, a) = 0, 85; R_2(c, b) = 0, 90\).

Then
\[
[\sim], b = \{(a; 0), (b; 0), (c; 0, 3)\} \text{ and } [\sim], a = \{(a; 0), (b; 0), (c; 0, 25)\}.
\]

Thus, we have
\[
R_2(a, b) = R_2(b, a) \text{ and } \text{not}([\sim], b[\subseteq] \sim, a[\setminus]).
\]

(iii) Consider \(R_3 \subseteq FO\) and satisfying \(T\) defined by: \(\forall x, R_3(x, x) = 1; R_3(c, a) = R_3(c, b) = 0, 3; R_3(b, a) = R_3(a, b) = 0, 5; R_3(c, b) = 0, 7; R_3(a, c) = 0, 8\).
Then
\[ \sim a = \{ (a; 0), (b; 0), (c; 0.5) \} \text{ and } \sim b = \{ (a; 0), (b; 0), (c; 0.4) \}. \]

Thus, we have
\[ R_3(a, b) = R_3(b, a) \text{ and } \text{not}(a \sim b = \sim a). \]

(iii) Consider the third fuzzy ordering
\[ R \text{ defined by: } \forall \}_{a,b}^c, \quad R_4(x, x) = 1; \quad R_4(a, b) = R_4(c, b) = 0.6; \]
\[ R_4(b, a) = R_4(b, c) = 0.5; \quad R_4(a, c) = 0.7; \quad R_4(c, a) = 0.85. \]

Then
\[ \sim a = \{ (a; 0), (b; 0), (c; 0.15) \} \text{ and } \sim b = \{ (a; 0.1), (b; 0), (c; 0.1) \}. \]

Thus, we have
\[ R_4(a, b) > R_4(b, a) \text{ and } \text{not}(a \sim [c] b \sim []). \]

This leads us to introduce new conditions on FWPR, which are satisfied by crisp orderings and by strong fuzzy orderings. Thus, we use these conditions to establish the properties of fuzzy strict sections and fuzzy strict order intervals.

3.2. New conditions on FWPR

We introduce five conditions:

Definition 5. Let \( R \) be an FWPR and \( P \) its fuzzy strict preference.

(i) \( R \) satisfies \( T_1 \) iff \[ R(z, y) = R(y, x) \] implies \[ R(z, x) = R(z, y) \iff R(y, z) = R(x, z) \],

(ii) \( R \) satisfies \( T_2 \) iff \[ R(x, y) > R(y, x) \] implies \[ R(z, x) = R(z, y) \Rightarrow R(y, z) \leq R(x, z) \],

(iii) \( R \) satisfies \( T_3 \) iff \[ R(x, y) > R(y, x) \] implies \[ R(z, x) = R(y, x) \Rightarrow P(y, z) \leq P(x, z) \],

(iv) \( R \) satisfies \( T_4 \) iff \[ R(x, y) > R(y, x) \] implies \[ R(z, x) = R(y, z) \Rightarrow R(x, z) \leq R(y, z) \],

(v) \( R \) satisfies \( T_5 \) iff \[ R(x, y) > R(y, x) \] implies \[ R(y, z) = R(y, x) \Rightarrow P(z, x) \leq P(z, y) \]

for all \( x, y, z \in A \).

The following example shows that there exist fuzzy orderings satisfying condition \( T \) which do not satisfy each of the five conditions. Also, the three first examples show that \( (T_i) \in \{1, 2, 3\} \) are independent while the three last ones show that \( T_1, T_2, \) and \( T_3 \) are independent.

Example 2. Let \( A = \{ a, b, c \} \),

(i) Consider the first fuzzy ordering \( R_1 \) defined in Example 1. \( R_1 \) satisfies \( T_1, T_2, \) but not \( T_3 \).

(ii) Consider \( R_5 \in FO \) and satisfying \( T \) defined by: \( \forall \}_{a,b}^c, \quad R_5(x, x) = 1; \quad R_5(c, a) = R_5(c, b) = 0.5; \]
\[ R_5(b, a) = 0.55; \quad R_5(a, b) = 0.6; \quad R_5(a, c) = 0.7; \quad R_5(b, c) = 0.8. \]
\( R_5 \) satisfies \( T_1, T_3, \) but not \( T_2 \).

(iii) Consider the third fuzzy ordering \( R_3 \) defined in Example 1. \( R_3 \) satisfies \( T_2, T_3, \) but not \( T_1 \). Also, \( R_3 \) satisfies \( T'_2, T'_3, \) but not \( T'_1 \).

(iv) Consider the fourth fuzzy ordering \( R_4 \) defined in Example 1. \( R_4 \) satisfies \( T_1, T'_2, \) but not \( T'_3 \).
(v) Consider $R_6 \in FO$ and satisfying $T$ defined by: $\forall x$, $R_6(x,x) = 1$; $R_6(c, a) = R_6(b, a) = 0.65$; $R_6(c, b) = 0.75$; $R_6(a, c) = R_6(b, c) = 0.8$; $R_6(a, b) = 0.85$. $R_6$ satisfies $T_1, T_2$, but not $(T'_2)$.

The following proposition shows that our five conditions are satisfied by strong fuzzy orderings (therefore by crisp orderings).

**Proposition 1.** Let $R \in FO$ and $P$ its fuzzy strict preference. If $R \in SFO$, then $R$ satisfies conditions $T_1, T_2, T_3, T'_2$ and $T'_3$.

**Proof.** The proof of this proposition is obvious. $\square$

In the next subsection, we establish the properties of fuzzy strict sections.

### 3.3. Properties of fuzzy strict sections

To establish these properties, we shall need the following lemma which determines all the fuzzy orderings whose fuzzy strict preferences $P$ satisfy the following property:

$$\forall x, y, z \in A, \ (P(x, y) = 0 \text{ and } P(y, z) = 0) \Rightarrow P(x, z) = 0. \quad (1)$$

**Lemma 2.** Let $R$ be an FWPR and $P$ its fuzzy strict preference. If $R \in FO$, then $R$ satisfies condition $T \Leftrightarrow P$ satisfies (1).

**Proof.** Suppose that $R \in FO$.

($\Rightarrow$) It is obvious to show that if $P$ satisfies (1) then, $R$ satisfies condition $T$. ($\Rightarrow$) Suppose that $R$ satisfies condition $T$ and show that $P$ satisfies (1). Let $x, y, z \in A$. Suppose that $P(x, y) = 0$, i.e., $R(y, x) \leq R(x, y)$ and $P(y, z) = 0$, i.e., $R(z, y) \leq R(y, z)$ (2)

and show that $P(x, z) = 0$.

As $P$ is transitive, then $P(x, y) \geq P(x, z) \wedge P(z, y)$ and $P(y, z) \geq P(y, x) \wedge P(x, z)$. We have two cases:

First case: If $P(z, y) > 0$ or $P(y, x) > 0$, then (2) implies that $P(x, z) = 0$.

Second case: If $P(z, y) = 0$ and $P(y, x) = 0$, then (2) implies that $R(x, y) = R(y, x)$ and $R(y, z) = R(z, y)$. Let us distinguish that: $R(x, y) = R(y, z)$, $R(y, z) < R(x, y)$ and $R(y, z) > R(x, y)$.

Suppose that $R(x, y) = R(y, z)$, then $R(x, y) = R(y, x) = R(y, z) = R(z, y)$. And condition $T$ implies that $R(x, z) = R(z, x)$. Hence $P(x, z) = 0$.

Suppose that $R(y, z) < R(x, y)$, then $R(y, z) = R(z, y) < R(x, y) = R(y, x)$. And the transitivity of $R$ implies that $R(x, y) = R(x, z)$ and $R(z, y) = R(z, x)$. Thus, $R(x, z) = R(z, x)$. Hence $P(x, z) = 0$.

Suppose that $R(y, z) > R(x, y)$, then $R(y, z) = R(z, y) > R(x, y) = R(y, x)$. And the transitivity of $R$ implies that $R(x, y) = R(x, z)$ and $R(y, x) = R(z, x)$. Thus, $R(x, z) = R(z, x)$. Hence $P(x, z) = 0$. $\square$

Also, to give these properties, we make the following remark.
Remark 4. Condition $T_1$ is equivalent to the conjunction of the two following conditions $T'_1$ and $T''_1$: $R$ satisfies $T'_1$ iff $[R(x, y) = R(y, x)]$ implies $[R(z, y) = R(y, z)]$, and $R$ satisfies $T''_1$ iff $[R(x, y) = R(y, x)]$ implies $[R(y, z) = R(x, z)]$. For all $x, y, z \in A$. 

The following proposition gives the properties of fuzzy strict lower sections. It determines all the fuzzy orderings under which it is possible to compare two fuzzy strict lower sections.

Remark 4. Condition $T_1$ is equivalent to the conjunction of the two following conditions $T'_1$ and $T''_1$: $R$ satisfies $T'_1$ iff $[R(x, y) = R(y, x)]$ implies $[R(z, y) = R(y, z)]$, and $R$ satisfies $T''_1$ iff $[R(x, y) = R(y, x)]$ implies $[R(y, z) = R(x, z)]$. For all $x, y, z \in A$.

Proposition 3. Let $R \in FO$:

(i) $[R$ satisfies $T$ and $T'_1] \iff [\forall x, y \in A, R(x, y) = R(y, x) \Rightarrow \approx x, y]$. 

(ii) $[R$ satisfies $T_1$ and $T'_1] \iff [\forall x, y \in A, R(x, y) > R(y, x) \Rightarrow \approx x, y]$. 

(iii) $[R$ satisfies $T, T'_1, T_1$ and $T'_1] \iff [\forall x, y \in A, R(x, y) > R(y, x) \Rightarrow \approx x, y]$. 

Proof (Proposition 3, first result). $(\Rightarrow)$ Suppose that $R$ satisfies $T$ and $T'_1$. By Lemma 2, we have (1). Let $x, y \in A$. Suppose that $R(x, y) = R(y, x)$, i.e., $P(x, y) = P(y, x) = 0$ and show that $\forall t, P(x, t) = P(y, t)$. Let $t \in A$.

(i) If $P(x, t) = 0$ or $P(y, t) = 0$, then (1) implies that $P(x, t) = P(y, t) = 0$.

(ii) If $P(x, t) > 0$ and $P(y, t) > 0$, i.e., $R(x, t) > R(t, x)$ and $R(y, t) > R(t, y)$, then $P(x, t) = P(y, t)$.

Suppose (ii) of (4), then (3) implies that $R(t, y) < R(y, t) < R(x, y) = R(x, y)$. And the transitivity of $R$ implies that $R(t, y) = R(t, x)$ and $R(y, t) = R(x, t)$. Hence $P(x, t) = P(y, t)$.

Suppose (ii) of (4), then (3) implies that $R(t, y) < R(y, t) < R(x, y) = R(x, y)$. And the transitivity of $R$ implies that $R(t, y) = R(t, x)$ and $R(y, t) = R(x, t)$. Hence $P(x, t) = P(y, t)$.

Suppose (iii) of (4), then (3) implies that

(i) $R(t, y) \leq R(x, y) = R(y, x) < R(y, t)$ or (ii) $R(x, y) = R(y, x) < R(t, y) < R(y, t)$.

Suppose (i) of (5), then the transitivity of $R$ implies that $R(t, y) = R(t, x)$. Thus, condition $T'_1$ implies that $R(y, t) = R(x, t)$. Hence $P(x, t) = P(y, t)$.

Suppose (ii) of (5), then the transitivity of $R$ implies that $R(y, x) = R(t, x)$. Since $R(x, y) = R(y, x) < R(x, t)$, we have $R(x, y) < R(t, x) \land R(t, y)$. This contradicts the transitivity of $R$.

Suppose (i) of (5), then the transitivity of $R$ implies that $R(t, y) = R(t, x)$. Thus, condition $T'_1$ implies that $R(y, t) = R(x, t)$. Hence $P(x, t) = P(y, t)$.

Suppose (ii) of (5), then the transitivity of $R$ implies that $R(y, x) = R(t, x)$. Since $R(x, y) = R(y, x) < R(x, t)$, we have $R(x, y) < R(t, x) \land R(t, y)$. This contradicts the transitivity of $R$.

(\Leftarrow) Suppose that

$[\forall x, y \in A, R(x, y) = R(y, x) \Rightarrow \approx x, y]$. 

and show that $R$ satisfies conditions $T$ and $T'_1$. 

Let us show that $R$ satisfies condition $T$. Let $x, y, z \in A$. Suppose that $R(x, y) = R(y, x) = R(y, z) = R(z, y)$ and show that $R(x, z) = R(z, x)$, i.e., $R(x, z) \leq R(z, x)$ and $R(z, y) \leq R(x, z)$. 

...
As $R(x,y) = R(y,x)$, then (6) implies that $P(y,z) = P(x,z)$. Thus, $P(x,z) = 0$ (since $R(y,z) = R(y,x)$, i.e., $P(y,z) = P(y,x) = 0$). Hence $R(x,z) \leq R(z,x)$.

3. As $R(y,z) = R(y,x)$, then (6) implies that $P(x,z) = P(y,x)$. Thus, $P(z,x) = 0$ (since $R(y,x) = R(x,y)$, i.e., $P(y,x) = P(y,y) = 0$). Hence $R(x,z) \leq R(z,x)$.

Let us show that $R$ satisfies condition $T'_1$. Let $x, y, z \in A$. Suppose that $R(x, y) = R(y, x)$ and $R(z, x) = R(z, y)$ and show that $R(y, z) = R(x, z)$.

7. Suppose that $R(x, z) < R(x, y) = R(y, x) < R(z, x) = R(z, y)$, then the transitivity of $R$ implies that $R(z, y) = R(x, z)$.

9. Suppose that $R(x, z) = R(x, y) = R(y, x) < R(z, x) = R(z, y)$, then the transitivity of $R$ implies that $R(y, x) = R(y, z)$. Hence $R(x, z) = R(y, z)$.

11. Suppose that $R(x, y) = R(y, x) < R(z, x) = R(z, y) \leq R(x, z)$ or $R(x, y) = R(y, x) < R(x, z) = R(z, y)$, then $R(x, y) < R(z, x) \land R(y, z)$. This contradicts the transitivity of $R$.

15. Suppose that $R(z, x) = R(z, y) \leq R(y, x) = R(y, z) \leq R(z, y)$, then the transitivity of $R$ implies that $R(z, x) \geq R(z, z)$, and $R(y, z) \leq R(y, z)$, then $R(z, y) = R(z, x) = R(z, x) = R(z, x) = R(z, x) = R(z, x)$. Hence $R(x, z) = R(y, z)$. □

Proof (Proposition 3, second result).

$(\Rightarrow)$ Suppose that $R$ satisfies $T_2$ and $T_3$. Let $x, y, z \in A$. Suppose that

$$R(x, y) > R(y, x)$$

and show that: $\exists t, P(y, t) = P(x, t)$ and $\forall t, P(y, t) = P(x, t)$. (a) For $t_0 = y$, we have $P(y, t_0) < P(x, t_0)$.

(b) Let $t \in A$. Let us show that $P(y, t) = P(x, t)$.

If $P(y, t) = 0$, then we have the result.

If $P(y, t) > 0$, i.e., (3), then $P(y, t) = R(y, t) - R(t, y)$.

27. Suppose that $R(y, t) < R(x, y)$, then (3) and the transitivity of $R$ imply that $R(y, t) < R(x, t)$ and $R(x, t) < R(t, y)$. Hence $P(x, t) = R(x, t) - R(t, x)$ and $P(y, t) = P(x, t)$.

29. Suppose that $R(y, t) > R(x, y)$, then (7) and the transitivity of $R$ imply that $R(x, y) \leq R(x, t)$. Thus, we have two cases:

(i) $R(y, x) < R(x, y) \leq R(x, t) < R(y, t)$ and

(ii) $R(y, x) < R(x, y) < R(y, t) \leq R(x, t)$. (8)

31. Suppose (i) of (8), then the transitivity of $R$ implies that $R(t, x) \leq R(y, x)$. If $R(t, x) < R(y, x)$, then the transitivity of $R$ implies that $R(t, x) = R(y, x)$. This contradicts condition $T_2$ since $R(t, x) < R(y, x)$.

Thus, $R(t, x) = R(y, x) < R(x, y) \leq R(x, t) < R(y, t)$ and $P(x, t) = R(x, t) - R(t, x)$. And condition $T_3$ implies that $P(y, t) \leq P(x, t)$.

35. Suppose (ii) of (8), then the transitivity of $R$ implies that $R(t, x) \leq R(y, x)$.

If $R(t, x) < R(y, x)$, then the transitivity of $R$ implies that $R(t, x) = R(y, x)$ and $P(x, t) = R(x, t) - R(t, x)$. Hence $P(x, t) \geq P(y, t)$.

If $R(t, x) = R(y, x)$, then the transitivity of $R$ implies that $R(t, x) \leq R(t, y) \leq R(x, y)$ and $P(x, t) = R(x, t) - R(t, x)$. Hence $P(x, t) \geq P(y, t)$. 

39.
Proof. With Remarks 2 and 4 and Propositions 1 and 3, we deduce the proof of this corollary. □

The following corollary shows that when fuzzy strict lower sections are associated to strong fuzzy orderings then, the properties of inclusion and equality of crisp strict lower sections are preserved.

Corollary 4. Let \( R \in SFO \).

\[
\begin{align*}
(\forall x, y \in A, \ R(x, y) &= R(y, x) \Rightarrow \tilde{x} \subseteq \tilde{y} \subseteq \tilde{x}]. \\
R(x, y) &> R(y, x) \Rightarrow \tilde{x} \subseteq \tilde{y} \subseteq \tilde{x}]. \\
R(x, y) &> R(y, x) \Rightarrow \tilde{x} \subseteq \tilde{y} \subseteq \tilde{x}. 
\end{align*}
\]

Proof. The proof of this proposition is analogous to the previous one. □

The following proposition gives the properties of fuzzy strict upper sections. It determines all the fuzzy orderings under which it is possible to compare two fuzzy strict upper sections.

Proposition 5. Let \( R \in FO \).

\[
\begin{align*}
(\forall x, y \in A, \ R(x, y) &= R(y, x) \Rightarrow \tilde{x} \subseteq \tilde{y} \subseteq \tilde{x}]. \\
R(x, y) &> R(y, x) \Rightarrow \tilde{x} \subseteq \tilde{y} \subseteq \tilde{x}. \\
R(x, y) &> R(y, x) \Rightarrow \tilde{x} \subseteq \tilde{y} \subseteq \tilde{x}. 
\end{align*}
\]

Proof. The proof of this proposition is analogous to the previous one. □
Corollary 6. Let $R \in SFO$.

(i) $R(x, y) = R(y, x)$ \iff $\uparrow y \rightarrow \downarrow x \equiv \downarrow x \rightarrow \uparrow y$.

(ii) $R(x, y) > R(y, x)$ \iff $\uparrow y \rightarrow [\subseteq] \downarrow x \equiv \downarrow x \rightarrow [\subseteq] \uparrow y$.

(iii) $R(x, y) \geq R(y, x)$ \iff $\uparrow y \rightarrow [\subseteq] \downarrow x \equiv \downarrow x \rightarrow [\subseteq] \uparrow y$.

Proof. With Remarks 2 and 4 and Propositions 1 and 5, we deduce the proof of this corollary.

The following result gives the link between fuzzy strict lower section and fuzzy strict upper
section. It extends to fuzzy orderings the following property of crisp strict sections:

Let $R \in CO$,

for all $x, y \in A$, $xRy \Rightarrow x \rightarrow \uparrow \cap \downarrow y = \emptyset$.

Proposition 7. Let $R \in FO$ and $\cap$ the intersection associated to a t-norm $\ast$.

For all $x, y \in A$, $R(x, y) \geq R(y, x)$ \iff $\uparrow y \rightarrow [\cap] \downarrow x \equiv \downarrow x \rightarrow [\cap] \uparrow y = \emptyset$.

Proof. In [2, 8], it is proved that $\forall_{[0,1]}^{a,b}$, $a \ast b \leq a \wedge b$. With this inequality and the transitivity of $P$, we deduce the proof of this result.

We now establish some properties of fuzzy strict order intervals.

3.4. Properties of fuzzy strict order intervals

The following proposition gives sufficient conditions under which two fuzzy strict order intervals are equal.

Proposition 8. Let $R \in FO$ and satisfying conditions $T$ and $T_1$. Let $P$ be its fuzzy strict preference.

$\forall_{A}^{x,y,z,w}$ such that $P(x, y) > 0$.

(i) $R(x, z) = R(z, x)$ \iff $\uparrow y \rightarrow \downarrow x, \uparrow \equiv \downarrow z$.

(ii) $R(y, z) = R(z, y)$ \iff $\uparrow y \rightarrow \downarrow x, \downarrow \equiv \uparrow z$.

(iii) $(R(x, z) = R(z, x)$ and $R(y, w) = R(w, y))$ \iff $\uparrow y \rightarrow \downarrow x, \uparrow \equiv \downarrow z$.

Proof. As $R \in FO$ and satisfying condition $T$, then Lemma 2 implies (1). Consider $x, y, z, w \in A$ such that $P(x, y) > 0$.

(i) First result: Suppose that

$R(x, z) = R(z, x), i.e., P(x, z) = P(z, x) = 0$ (11)

and show that $\uparrow y \rightarrow \downarrow x, \downarrow \equiv \uparrow z$.

Eqs. (11) and (1) imply that $P(z, y) > 0$. Also, (11) and the first result of Proposition 3 imply that $\downarrow \equiv \uparrow z \equiv \downarrow \equiv \uparrow x$. Hence $\uparrow y \rightarrow \downarrow x, \downarrow \equiv \uparrow z$.

(ii) The proofs of the two last results are analogous to the previous one.

The following proposition gives two comparisons of fuzzy strict order intervals.
Proposition 9. Let $R \in \text{FO}$ and $P$ its fuzzy strict preference. \( \forall_{A}^{x,y,z} \) such that \( P(x,z) > 0 \) and \( P(z,y) > 0 \).

(i) \( R \) satisfies conditions $T_2$ and $T_5$ then the transitivity of $P$ implies that $P(x,y) > 0$. 

To prove our results, we need the following classical property of the fuzzy sets (see [8,15]): Let $E,F$ and $G$ be three fuzzy sets of $A$.

If $E \subset F$, then $E \cap G \subset F \cap G$. 

First result: Suppose that $R$ satisfies $T_2$ and $T_5$ and show that $\lceil \bar{y},z[\subseteq] y, x \rfloor$. (i) of (12) and the second result of Proposition 3 imply that $\lceil \bar{z},x[\subseteq] x \rfloor$. Thus, (13) and the definition of fuzzy strict order intervals imply that $\lceil \bar{y},z[\subseteq] \bar{y}, x \rfloor$.

With (ii) of (12), Proposition 5 and (13), the proof of the second result is analogous to the previous one.

We end this section by establishing the following result which gives sufficient conditions for which fuzzy strict order intervals preserve the properties of intersection of crisp strict order intervals. We assume that the intersection is associated to Zadeh’s min t-norm, that is, for all $a,b \in [0,1]$, $a \ast b = a \wedge b$.

Proposition 10. Let $R \in \text{FO}$ and $P$ its fuzzy strict preference.

(i) $\forall_{A}^{x,y,z,w} \text{ such that } P(y,x) > 0 \text{ and } P(w,z) > 0$, 

\[
\begin{align*}
(R(x,w) \geq R(w,x)) & \Rightarrow \lceil \bar{x},y[\cap] z, w[=] 0, \\
(R(z,y) \geq R(y,z)) & \Rightarrow \lceil \bar{x}, y[\cap] z, w[=] 0.
\end{align*}
\]

(ii) If $R$ satisfies conditions $T_2$, $T_5$, $T'_2$ and $T'_5$, then $\forall_{A}^{x,y,z,w} \text{ such that } P(y,x) > 0 \text{, } P(w,z) > 0 \text{ and } P(y,z) > 0$, 

\[
\begin{align*}
(R(z,x) > R(x,z) \text{ and } R(w,y) > R(y,w)) & \Rightarrow \lceil \bar{x},y[\cap] z, w[=] z, \bar{y} \rfloor.
\end{align*}
\]

(iii) If $R$ satisfies conditions $T$, $T_1$, $T_2$, $T_5$, $T'_2$ and $T'_5$, then $\forall_{A}^{x,y,z,w} \text{ such that } P(y,x) > 0$, 

\[
\begin{align*}
(P(w,z) > 0 \text{ and } P(w,x) > 0, \\
(R(x,z) \geq R(z,x) \text{ and } R(y,w) \geq R(w,y)) & \Rightarrow \lceil \bar{x},y[\cap] z, w[=] \bar{x}, \bar{w} \rfloor.
\end{align*}
\]

(iv) If $R$ satisfies conditions $T$, $T_1$, $T_2$, $T_5$, $T'_2$ and $T'_5$, then $\forall_{A}^{x,y,z,w} \text{ such that } P(y,x) > 0 \text{ and } P(w,z) > 0$, 

\[
\begin{align*}
(R(x,z) \geq R(z,x) \text{ and } R(w,y) > R(y,w)) & \Rightarrow \lceil \bar{x}, y[\cap] z, w[=] \bar{x}, \bar{y} \rfloor, \\
(R(z,x) > R(x,z) \text{ and } R(y,w) \geq R(w,y)) & \Rightarrow \lceil \bar{x}, y[\cap] z, w[=] \bar{x}, \bar{w} \rfloor.
\end{align*}
\]
Proof. Let us recall that
\[ \tilde{x} \cap \tilde{y} = \tilde{[x \cap y]} \]
(14)
To prove our results, we need the following classical properties of the fuzzy sets (see [9,15]):

(i) If \( E \subset F \) or \( E \subseteq F \), then \( E \cap F = E \) and (ii) \( E \cap \emptyset = \emptyset \).
(15)
With (14), (ii) of (15) and Proposition 7, we deduce the proof of the first result.

Second result: Suppose that \( R \) satisfies conditions \( T_2, T_3, T'_2 \) and \( T'_3 \). Let \( x, y, z, w \in A \) such that
\[ P(y \cap z) > 0, P(w \cap \emptyset) > 0, \]
and show that \( \tilde{[x \cap y \cap z \cap w]} = \tilde{z} \cap \tilde{y} \).
As \( R(z, x) > R(x, z) \), the second result of Proposition 5 implies that \( \tilde{z} \cap \tilde{y} \subseteq \tilde{[x \cap z]} \). And (i) of (15) implies that \( \tilde{z} \cap \tilde{y} \subseteq \tilde{z} \cap \tilde{y} \).
Also, as \( R(w, y) > R(y, w) \), then the second result of Proposition 3 implies that \( \tilde{y} \supseteq \tilde{w} \).
And (i) of (15) implies that \( \tilde{y} \supseteq \tilde{y} \).
Thus, since \( P(y \cap z) > 0 \), (14) implies that \( \tilde{x} \cap \tilde{y} \cap \tilde{z} \cap \tilde{w} = \tilde{z} \cap \tilde{y} \).
With Propositions 3 and 5, (i) of (15) and (14), the proofs of the two last results are analogous to the previous one.

Remark 5. The results of Propositions 8–10 are fuzzy extensions of well-known properties of crisp strict order intervals in the classical theory of partial orders.

In the following subsection, we give an example of the application of some of these results on the “Theory of revealed preference” in Economics.

3.5. Example of application

In real life, the individual, even though his preference is fuzzy, will have to make a choice, necessarily exact. This raises, on the “Theory of revealed preference”, the question as how unambiguous or exact choice are generated by fuzzy preference and whether the exact choices induced by fuzzy preferences satisfy certain plausible rationality conditions (see, e.g., [1,13]).

In this subsection, we introduce, by means of fuzzy strict lower section, three alternatives rules for generating exact choices from FWPRs and determine, by means of Lemma 2 and Proposition 3, fuzzy orderings under which each of these three rules satisfies two classical and fairly weak rationality conditions.

We assume that: \( A \) is a finite set of alternatives and \( |A| \geq 4 \); \( A \) is the set of nonempty crisp subsets of \( A \) and \( G \) is the set of FWPRs on \( A \).

Definition 6 (see Barrett et al. [1, p. 198, Definition 2.4]). A preference based choice function (PCF) is a function \( \mathcal{C} : A \times C' \rightarrow A \) such that
\[ \emptyset \neq G' \subseteq G \text{ and } \forall B \in A, \forall R \in G', \emptyset \neq \mathcal{C}(B, R) \subseteq B. \]
Remark 6 (see Barrett et al. [1, p. 198, Remark 2.5]). Intuitively, $G'$ figuring in Definition 6 constitutes the set of “admissible” FWPRs. Given an admissible FWPR $R$, and given a crisp set $B$ of available alternatives, $\mathcal{C}(B, R)$ constitutes the exact set of alternatives chosen from $B$, on the basis of $R$.

The following notation will permit us to introduce three PCFs.

Notation. Let $B \in \mathcal{A}$, $R \in G'$ and $P$ its fuzzy strict preference and suppose $x \in B$. Then we denote

$$h(\leftarrow, x \cap B) = \bigvee_{t \in B - \{x\}} P(x, t) = hl_x(B, R),$$

$$\text{Card}(\leftarrow, x \cap B) = \sum_{t \in B - \{x\}} P(x, t) = cl_x(B, R),$$

$$S(\leftarrow, x \cap B) = \{t \in B, P(x, t) > 0\} = Sl_x(B, R) \text{ and } |Sl_x(B, R)| = sl_x(B, R).$$

Remark 7. Since in the literature [8,9,16], there exist the definitions of “height”, “cardinality” and “support” of a fuzzy set of $A$, it may be useful in remembering this notation if the reader interprets $hl_x(B, R)$, $cl_x(B, R)$ and $Sl_x(B, R)$ as “height”, “cardinality” and “support” of $\leftarrow, x \cap B$ (this intersection of $B$ and the strict lower section of $x$ is a fuzzy set of $A$), respectively. Also, $sl_x(B, R)$ is the cardinality of $Sl_x(B, R)$.

We now introduce three PCFs.

Definition 7. Let $\mathcal{C} : \mathcal{A} \times G' \to \mathcal{A}$ be a PCF.

(i) $\mathcal{C}$ is max-$hl$, iff $\forall B \in \mathcal{A}$, $\forall R \in G'$,

$$\mathcal{C}(B, R) = \{x \in B, hl_x(B, R) \geq hl_y(B, R), \forall y \in B - \{x\}\}.$$

(ii) $\mathcal{C}$ is max-$cl$, iff $\forall B \in \mathcal{A}$, $\forall R \in G'$,

$$\mathcal{C}(B, R) = \{x \in B, cl_x(B, R) \geq cl_y(B, R), \forall y \in B - \{x\}\}.$$

(iii) $\mathcal{C}$ is max-$sl$, iff $\forall B \in \mathcal{A}$, $\forall R \in G'$,

$$\mathcal{C}(B, R) = \{x \in B, sl_x(B, R) \geq sl_y(B, R), \forall y \in B - \{x\}\}.$$

Remark 8. (1) Each of these PCFs generates exact choices from fuzzy preference.

(2) It is easy to see that $\mathcal{C}$ is max-$hl$ if, and only if, $\mathcal{C}$ is max-$MD$ (cf. Barrett et al. in [1]).

We now recall two classical and fairly weak rationality conditions of a PCF. Each of them lays down sufficient conditions for choosing alternative, given an admissible FWPR $R$, and given a feasible set of alternatives.

Definition 8 (see [1, p. 198, Definition 4.1]). Let $\mathcal{C} : \mathcal{A} \times G' \to \mathcal{A}$ be a PCF.

(1) $\mathcal{C}$ satisfies reward for pairwise weak dominance (RPWD) iff $\forall B \in \mathcal{A}$, $\forall R \in G'$, $\forall x \in B$,

$$(\forall y \in B - \{x\}, R(x, y) \geq R(y, x)) \Rightarrow x \in \mathcal{C}(B, R).$$
(2) \( \mathcal{G} \) satisfies reward for pairwise strict dominance (RPSD) iff \( \forall B \in \mathcal{G}, \forall R \in G', \forall x \in B, \)
\[
(\forall y \in B - \{x\}, \ R(x,y) > R(y,x)) \Rightarrow x \in \mathcal{G}(B,R).
\]

**Remark 9** (see [1, p. 198, Remarks 4.2 and 4.3]). (1) It is clear that RPWD implies RPSD, but the converse is not necessarily true.
(2) RPSD requires that, if an available alternative, say \( x \), “strictly prefers” every other available alternative in a pairwise comparison, then \( x \) should be chosen. In contrast, RPWD requires \( x \) to be chosen if it “weakly prefers” every other available alternative in a pairwise comparison.

Given one of our three PCFs, and given one of the two rationality conditions, the main question is now to determine a set of FWPRs under which that PCF satisfies the given condition.

The two following results show that when admissibles FWPRs are fuzzy orderings then, each of our three PCFs violates at least one of the rationality conditions.

(1) It is clear that RPWD implies RPSD, but the converse is not necessarily true.

(2) RPSD requires that, if an available alternative, say \( x \), “strictly prefers” every other available alternative in a pairwise comparison, then \( x \) should be chosen. In contrast, RPWD requires \( x \) to be chosen if it “weakly prefers” every other available alternative in a pairwise comparison.

**Proposition 11.** (1) If \( G' \subseteq FO \) and \( \mathcal{G} \) is max-cl PCF, then \( \mathcal{G} \) violates both RPWD and RPWD.
(2) If \( G' \subseteq FO \) and \( \mathcal{G} \) is max-sl PCF, then \( \mathcal{G} \) violates RPWD.

**Proof.** The proof consists of these two following counterexamples: Let \( G' \subseteq FO \) and suppose \( B = \{x, y, z\} \).

Consider \( R_7 \in FO \) such that \( \forall a, R_7(a,a) = 1; R_7(z,x) = R_7(y,x) = R_7(z,y) = 0.5; R_7(x,y) = R_7(x,z) = 0.6 \) and \( R_7(y,z) = 0.9 \).
Consider \( R_8 \in FO \) such that \( \forall a, R_8(a,a) = 1; R_8(x,y) = R_8(y,x) = R_8(z,x) = R_8(z,y) = 0.7; R_8(y,z) = 0.8 \) and \( R_8(z,y) = 0.9 \).

\[
sl_x(B,R_7) = 0; \ cl_x(B,R_7) = 0.2 \quad \text{and} \quad \ cl_y(B,R_7) = 0.4,
\]
\[
sl_y(B,R_8) = sl_y(B,R_8) = 0 \quad \text{and} \quad sl_x(B,R) = 1.
\]

Then, \( R_7(x, y) > R_7(y, x) \) and \( R_7(x, z) > R_7(z, x) \). Also, \( R_8(x, y) \geq R_8(y, x) \) and \( R_8(z, x) \geq R_8(z, y) \).

However, if \( \mathcal{G} \) is max-cl, then \( x \notin \mathcal{G}(B,R_7) \); this violates both RPWD and RPSD. Also, if \( \mathcal{G} \) is max-sl, then \( x \notin \mathcal{G}(B,R_7) \); this violates RPWD.

The last results (Barrett’s result and Proposition 11) lead us to determine, in the following proposition, a subset of fuzzy orderings under which the max-hl PCF or the max-cl PCF satisfies both RPWD and RPSD and, the subset of all the fuzzy orderings under which the max-sl PCF satisfies these rationality conditions.

**Proposition 12.** Let \( \mathcal{G} : \mathcal{A} \times G' \rightarrow \mathcal{A} \) be a PCF.

(1) If \( G' \subseteq G_1 = \{R \in FO, \ R \text{ satisfies } T, \ T'_1, \ T_2 \text{ and } T_3\} \) and \( \mathcal{G} \) is max-hl PCF or max-cl PCF, then \( \mathcal{G} \) satisfies both RPWD and RPSD.
(2) If $G' \subseteq FO$ and $\mathcal{C}$ is max-sl PCF, then

$$G' \subseteq G_2 = \{ R \in FO, R \text{ satisfies } T \} \iff \mathcal{C} \text{ satisfies both RPWD and RPSD.}$$

**Proof.** With Lemma 2 and Proposition 3, we deduce the proof of this result. □

**Remark 10.** (1) The results of Proposition 12 specify whether the exact choices induced by fuzzy preferences and by means of each of our three PCFs satisfy the two plausible rationality conditions RPWD and RPSD.

(2) $R_7 \notin G_1$ since $R_7$ violates conditions $T_2$ and $T_3$. Furthermore, $R_7$ is the counterexample proposed by Barret et al. to show their result. Also, $R_8 \notin G_2$ since $R_8$ violates condition $T$.

(3) With Proposition 1, it is easy to see that when admissibles FWPRs are strong fuzzy orderings then, our three PCFs satisfy both rationality conditions.

**4. Conclusion**

In this paper, we have extended crisp strict sections and crisp strict order intervals to fuzzy orderings. We have determined all the fuzzy orderings under which it is possible to compare two fuzzy strict sections and we have determined some properties (equality, inclusions and intersections) of fuzzy strict order intervals.

These introduced fuzzy notions have permitted us to restructure a set of alternatives $A$ given the fuzzy individual preferences on $A$. As we have illustrated in Section 3–5, this reconstruction could be utilized, when preferences are fuzzy, in the following areas: theory of revealed preference, Arrovian social choice theory and utility theory.

However, we believe that, contrary to classical and crisp transitivity on $\{0,1\}$, fuzzy transitivity on $[0,1]$ does not handle certain problems relating to rationality on triplets. It is therefore in the view of covering this shortcoming that we have imposed five conditions and a classical condition $T$.

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**References**


